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Continuous dependence on parameters of differential inclusion using new techniques of fixed point theory

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Abstract. In this paper, we establish the global existence and the continuous dependence on parameters for a set solutions to a class of time-fractional partial differential equation in the form

$$\begin{cases} \dfrac{\partial}{\partial t}u(t) + \mathcal{K}\mathcal{A}^{\sigma_1}\dfrac{\partial}{\partial t}u(t) + \mathcal{A}^{\sigma_2}u(t) \in F(t, u(t), \mu), & t \in \mathcal{I}, \\ u(T) = h, \ (\text{ resp. } u(0) = h) & \text{on } \Omega, \end{cases}$$

where $\sigma_1, \sigma_2 > 0$ and $\mathcal{I} = [0, T)$ (resp. $\mathcal{I} = (0, T]$). Precisely, first results are about the global existence of mild solutions and the compactness of the mild solutions set. These result are mainly based on some necessary estimates derived by considering the solution representation in Hilbert spaces. The remaining result is the continuous dependence of the solutions set on some special parameters. The main technique used in this study include the fixed point theory and some certain conditions of multivalued operators.

1. Introduction

Let T be a positive number, Ω be a bounded domain with sufficiently smooth boundary $\partial\Omega$ in Euclid space $\mathbb{R}^N$ and $\sigma_1, \sigma_2 > 0$. We first consider the final value problem which finds $u = u(t, x)$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) + \mathcal{K}\mathcal{A}^{\sigma_1}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{A}^{\sigma_2}u(t,x) \in F(t, u(t)), & (t,x) \in [0,T) \times \Omega \\ u(T,x) = h(x), & x \in \Omega, \end{cases} \tag{1.1}$$

and the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) + \mathcal{K}\mathcal{A}^{\sigma_1}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{A}^{\sigma_2}u(t,x) \in F(t, u(t)), & (t,x) \in (0,T] \times \Omega \\ u(0,x) = h(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $\frac{\partial}{\partial t}u$ is the symbol for the derivative with respect to the variable t of the function u, $\mathcal{K}$ is a positive constant and $\mathcal{A}$ is a self-adjoint operator with fractional order $\sigma \in \{\sigma_1, \sigma_2\}$ on the Hilbert space $\mathcal{H}$, this is, $\langle \mathcal{A}^\sigma u, w\rangle = \langle u, \mathcal{A}^\sigma w\rangle$ for all $\sigma \in \{\sigma_1, \sigma_2\}$.

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Recently, differential equations and inclusions have gain much attention according to wide applications in physics, economic, control theory, etc, see e.g. [4, 6, 8–10, 16–19, 38–41]. There have been many studies on the existence and the stability of the solution of the problem with the source single-valued function or with non-integer order derivatives, for example [1–3, 5, 7, 11–15, 21–24, 26–37]. In [12], Anh-Ke-Lan studied the following fractional differential inclusion

$$\partial_t^\alpha u(t) - Au(t) \in F(t, u, u_t), \quad t > 0,\ 0 < \alpha < 1, \tag{1.3}$$

involving impulsive effects. They proved the global solvability and weakly asymptotic stability for solutions by analyzing the behavior of its solutions on the half-line. This equation was also studied in [21]. In [27], Phong-Lan concerned with the retarded fractional evolution inclusion

$$\partial_t^\alpha u(t) - Au(t) \in F(t, u_t), \quad t > 0, 0 < \alpha < 1, \tag{1.4}$$

equipped with the history condition

$$u(s) = \varphi(s), \quad s \in [-h, 0], h > 0,$$

in a Banach space X, where A is a closed linear operator in X, F is a multimap, φ is the history of solutions. By assuming F superlinear, they established the existence of decay global solution. However, in control theories, a common problem with F is a multivalued function. In addition to considering the existence and continuity of the solution set, the compactness of the solution set is also often of interest. In particular, when the input data F is noisy by the parameter μ, we need to consider the continuous dependence of the solution set on this parameter.

In [25], Ngoc-Tri discussed the existence and compactness of the solutions set of following fractional pseudo-parabolic inclusion

$$\begin{cases} \partial_t^\alpha u + \kappa(-\Delta)^{\sigma_1}\partial_t^\alpha u + (-\Delta)^{\sigma_2} u & \in & F(t, u), & 0 < t < T,\ x \in \Omega, \\ u(t, x) & = & 0, & 0 < t < T,\ x \in \partial\Omega, \\ u(0, x) & = & \varphi(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where ∂_t^α signifies the Caputo time derivative of fractional order $\alpha \in (0, 1)$. In [25] we constructed useful bounds for solution operators by basing on asymptotic behaviors of the Mittag-Leffler functions to prove the compactness and continuous dependence on parameters of solutions set of Problem (1.5).

Our aim in this paper is devoted to study the final/initial value problem for differential inclusions (1.1)/(1.2). We establish the existence and the compactness of the solutions set and discuss on the dependence of the solutions of the following parameterized problems on the parameter μ in a metric space (E, d). It's more obvious that we consider the following problems which $u = u(t, x)$ satisfying one of the following system equations

$$\begin{cases} \dfrac{\partial}{\partial t}u(t, x) + \mathcal{K}\mathcal{A}^{\sigma_1}\dfrac{\partial}{\partial t}u(t, x) + \mathcal{A}^{\sigma_2}u(t, u) \in F(t, u, \mu), & (t, x) \in [0, T) \times \Omega \\ u(T, x) = h(x), & x \in \Omega \end{cases} \tag{1.6}$$

or

$$\begin{cases} \dfrac{\partial}{\partial t}u(t, x) + \mathcal{K}\mathcal{A}^{\sigma_1}\dfrac{\partial}{\partial t}u(t, x) + \mathcal{A}^{\sigma_2}u(t, u) \in F(t, u, \mu), & (t, x) \in (0, T] \times \Omega \\ u(0, x) = h(x), & x \in \Omega, \end{cases} \tag{1.7}$$

For a given $\mu_0 \in E$, the our main purpose is to study the continuous of respectively mild solutions set, namely, if μ near enough μ_0, the solution set corresponding to μ is contained in neighborhood of the solution sets corresponding to μ_0.

In addition to commonly used methods such as the evaluations by Fourier expansion of an element in the separable Hilbert space, using the Grönwall's inequality, we use a measure of compactness ν in the ordered space generated by a convex cone to consider the existence of fixed points of the ν-condensing multimap. To the best of my knowledge, there are not many studies on differential inclusions containing self-adjoint operators with fractional order and techniques using the measure of noncompactness that take values in cones.

Let $\mathcal{H}$ be a separable Hilbert, we denote by $Kv(\mathcal{H})$ (resp., $b(\mathcal{H})$) the all convex and compact (resp., bounded) subsets of $\mathcal{H}$ and consider problems (1.1) and (1.2) with the multifunction $F : [0,T]\times\mathcal{H} \to Kv(\mathcal{H})$ under the following condition (H):

(Ha) for every $v \in \mathcal{H}$, the multimap $t \mapsto F(t,v)$ has a strongly measurable selection, i.e., there is a measurable function $f_v(.) : [0,T] \to \mathcal{H}$ satisfying $f_v(t) \in F(t,v)$.

(Hb) the multimap $F(t,.) : \mathcal{H} \to Kv(\mathcal{H})$ is upper semicontinuous (u.s.c) for a.e. $t \in [0,T]$,

(Hc) there exists a function $\alpha \in L^1((0,T);\mathbb{R})$ such that

$$|||F(t,u)||| := \sup_{v\in F(t,u)} \|v\|_{\mathcal{H}} \le \alpha(t)(1+\|u\|_{\mathcal{H}}) \text{ for a.e. } t \in (0,T) \text{ and for all } u \in \mathcal{H}.$$

(Hd) There is $B \in L^1((0,T);\mathbb{R})$ satisfying

$$\chi(F(t,D)) \le B(t)\chi(D) \text{ for a.e. } t \in (0,T) \text{ for all } D \in b(\mathcal{H}),$$

here χ is MNC in $\mathcal{H}$ defined $\chi(D) = \inf\{\varepsilon > 0 : D \text{ has a finite } \varepsilon\text{-net}\}$.

We further assume that $0 < \sigma_2 \le \sigma_1$ for problems (1.1) and (1.6).

Our work shall be presented as follows. In the next section, we recall some properties of the multivalued operator that shall be used for the main results. Section 3 presents the global existence of mild solutions and compactness of the solution set of problems (1.1) and (1.2). Finally, we discuss on the continuous dependence on parameters μ of the solution set of problems (1.6) and (1.7).

2. Preliminaries

Throughout this paper, let $\dot{\mathbb{N}} = \mathbb{N}\backslash\{0\}$ and $\mathscr{P}(E)$ (resp., $b(E), K(E)$) be the all nonempty (resp., bounded, compact) subsets of E. Let $\mathcal{H}$ be a separable Hilbert space with an inner product denoted by $\langle,\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|_{\mathcal{H}}$. We denote by $\mathscr{C}([0,T];\mathcal{H})$ the space of all continuous functions from $[0,T]$ into $\mathcal{H}$ with norm $\|u\|_{\mathscr{C}([0,T],\mathcal{H})} = \sup_{t\in[0,T]} \|u(t,.)\|_{\mathcal{H}}$. The sequence $\{f_n\}$ in $\mathscr{C}([0,T];\mathcal{H})$ is said to be weakly convergent to f (resp., for almost every on $[0,T]$), written $f_n \rightharpoonup f$ (resp., a.e) on $[0,T]$), if $\langle f_n(t), f(t)\rangle$ tends 0 for all (resp., a.e) $t \in [0,T]$. Let (E,ρ) be a metric space and $A \subset E$. We denote the distance between a point $x \in X$ and A by $\text{dist}(x,A) := \inf\{\rho(x,y) : y \in A\}$, and the ϵ-neighbourhood of A by $\mathcal{N}_{\epsilon,\rho}(A) := \{y \in X : \text{dist}(y,A) < \epsilon\}$ (in short, $\mathcal{N}_\epsilon(A)$).

To establish our main results, we need some basic properties of multivalued analysis which can be found in [20]. Let us recall the concepts and these properties which shall use in the next sections.

Definition 2.1. [20, Definition 2.1.1] Let E be a Banach space and $(C,\le)$ a partially ordered set. A map $\varphi : \mathcal{Y} \subset \mathcal{P}(E) \to C$ is said to be a *measure noncompactness* (MNC) in $\mathcal{Y}$ if $\varphi(\overline{co}(D)) = \varphi(D)$ for all $D \in \mathcal{Y}$. A multi-mapping $F : E \to \mathcal{Y}$ is called *condensing* to φ (in short, φ-condensing) if $D \in \mathcal{Y}$ with $\varphi(D) \le \varphi(F(D))$ then D is relative compact in E.

Let G be a subset of a metric (E,d) and ϵ be a positive number. A subset A of E is said to be ϵ-net of G if $G \subset \bigcup_{x\in A}\{y \in E : d(x,y) < \epsilon\}$. If A is finite, A is called a *finite ϵ-net*. We need the Hausdorff measure χ which defined in [20, Definition 2.1.1], i.e., $\chi(G) = \inf\{\epsilon > 0 : G \text{ has a finite } \epsilon\text{-net}\}$.

Lemma 2.2. [20, Definition 2.1.1] *Let E be a Banach space and χ a Hausdorff MNC defined on family $\mathcal{F}$ of subsets of E. Then χ has the following properties:*

(a) *monotone: if $D_1 \subset D_2$ implies $\chi(D_1) \le \chi(D_2)$, for all $D_1, D_2 \in \mathcal{F}$.*

(b) *algebraically semiadditive: if $\chi(D_1 + D_2) \le \chi(D_1) + \chi(D_2)$ for all $D_1, D_2 \in \mathcal{F}$.*

(c) *nonsingular: if* $\chi(\{a\} \cup D) = \chi(D)$ *for all* $a \in E, D \in \mathcal{F}$.

(d) *regular:* $\chi(D) = 0$ *if and only if* D *is relatively compact,* $D \in \mathcal{F}$.

(e) *semi-homogeneity: that is* $\chi(\lambda D) = |\lambda|\chi(D)$ *for all* $\lambda \in \mathbb{R}, D \in \mathcal{F}$.

Definition 2.3. [20, Corollary 1.1.1] Let X and Y be topological spaces. A multimap $F : X \to \mathscr{P}(Y)$ is upper semicontinuous at the point $x \in X$ if, for every open set $W \subset Y$ such that $F(x) \subset W$, there exists a neighborhood $V(x)$ of x with property that $F(V(x)) \subset W$. A multimap is called *upper semicontinuous* (u.s.c) if it is upper continuous at every point $x \in X$.

When $(X, d), (Y, \rho)$ are metric spaces, it is clear that a multimap F form a metric space (X, d) into (Y, ρ) is u.s.c at point $x \in X$ iff for any $\epsilon > 0$, there exists $\delta > 0$ such that $F(w) \subset \mathcal{N}_{\epsilon,\rho}(F(x))$ for all $w \in \mathcal{N}_{\delta,d}(x)$.

For multimap $\mathcal{M} : E \to \mathscr{P}(E)$, we denote by $\mathrm{Fix}(\mathcal{M})$ the set of the all fixed points of $\mathcal{M}$, i.e., $\mathrm{Fix}(\mathcal{M}) = \{x \in E : x \in \mathcal{M}(x)\}$.

Lemma 2.4. [20, Corollary 3.3.1] *If* M *is a closed convex subset of Banach space* E *and* $\mathcal{M} : M \to Kv(M)$ *is a closed* φ*-condensing multimap, where* φ *is a nonsingular MNC defined on subsets of* M*, then* $\mathrm{Fix}(\mathcal{M}) \neq \emptyset$.

Lemma 2.5. [20, Propositions 3.5.1] *Let* M *be a closed subset of a Banach space* E *and* $\mathcal{M} : M \to K(M)$ *a closed multimap, which is* φ*-condensing on every bounded subset of* M*, where* φ *is a monotone MNC. If* $\mathrm{Fix}(\mathcal{M})$ *is bounded then it is compact.*

Lemma 2.6. [20, Propositions 3.5.2] *Let* X *be a closed subset of a Banach space* E*,* β *be a monotone MNC in* E*,* Y *be a metric space, and* $G : Y \times X \to K(E)$ *be a closed multimap which is* β*-condensing in the second variable and such that* $F(\lambda) := \mathrm{Fix}\, G(\lambda, \cdot) \neq \emptyset$*, for all* $\lambda \in Y$*. Then the multimap* $F : Y \to P(E)$ *is u.s.c.*

Definition 2.7. ([20, Definition 4.2.1]) Let E be a Banach space. A $\{f_n\}_{n\in\mathbb{N}} \subset L^1([0, d], E)$ is called

1. *integrably bounded* if there is $\alpha \in L^1([0, d], \mathbb{R})$ such that

$$\|f_n(t)\|_E \leq \alpha(t) \text{ for a.e } t \in [0, d] \text{ and for all } n \in \mathbb{N};$$

2. *semicompact* if it is integrably bounded and the set $\{f_n(t)\}_{n\in\mathbb{N}}$ is relatively compact for almost every $t \in [0, d]$.

In addition to the above mentioned basic properties of multivalued analysis, we also use the Grönwall's inequality presented in the following lemma.

Lemma 2.8. *(Grönwall) Let* $a \geq 0$*,* $0 < T \leq \infty$*, and continuous functions* $\beta, \mu : [0, T] \to \mathbb{R}_+$ *satisfying* $\int_0^T \beta(s)ds < \infty$*, and* $\sup_{t\in[0,T]} \mu(t) < \infty$*,* $0 \leq \gamma \leq \xi \leq T$ *and*

$$\mu(t) \leq a + \int_t^T \beta(s)\mu(s)ds \ \left(\text{resp. } \mu(t) \leq a + \int_0^t \beta(s)\mu(s)ds \right), \quad t \in [0, T].$$

Then $\mu(t) \leq ae^{\int_t^\xi \beta(s)ds} \left(\text{resp., } \mu(t) \leq ae^{\int_\gamma^t \beta(s)ds} \right)$ *for all* $t \in [0, T]$.

3. Main results

In the first part in this section, we present the mild solution for problems (1.1) and (1.2). In the next part, we establish the existence and compactness of the solutions set. In the final part, on the basis of these results, we discuss the continuous dependence of the solution set of the inclusions (1.6) and (1.7) on the parameter.

3.1. Representations of mild sulution

For $u \in \mathscr{C}([0,T];\mathcal{H})$, we denote

$$\mathcal{S}_F(u) = \left\{ f \in L^1((0,T);\mathcal{H}) | f(t,.) \in F(t,u), \ \text{ for a.e. } t \in (0,T) \right\}. \tag{3.1}$$

It is clear that $u = u(t,.)$ is a solution of Problem (1.1) if and only if there exists $f \in \mathcal{S}_F(u)$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) + \mathcal{K}\mathcal{A}^{\sigma_1}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{A}^{\sigma_2}u(t,x) = f(t,x), \ (t,x) \in [0,T) \times \Omega \\ u(T,x) = h(x), \qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega. \end{cases} \tag{3.2}$$

Assume that $\phi_\lambda \in \mathcal{H}$ is the eigen-function corresponding to the positive eigenvalue λ of the operator $\mathcal{A}$, i.e., $\mathcal{A}^\sigma(\phi_\lambda) = \lambda^\sigma \phi_\lambda$ for $\sigma \in \{\sigma_1, \sigma_2\}$. Taking the inner product of both sides of (3.2) with ϕ_λ, we obtain that

$$(1 + \mathcal{K}\lambda^{\sigma_1})\frac{\partial}{\partial t}\langle u(t), \phi_\lambda\rangle + \lambda^{\sigma_2}\langle u(t), \phi_\lambda\rangle = \langle f(t), \phi_\lambda\rangle. \tag{3.3}$$

Denote $y_\lambda(t) = \langle u(t), \phi_\lambda\rangle$ and $q_\lambda(t) = \langle f(t), \phi_\lambda\rangle$

$$y_\lambda(t) = h_\lambda e^{\int_t^T p_\lambda(s)ds} - \int_t^T q_\lambda(s) e^{\int_t^s p_\lambda(\tau)d\tau} ds, \tag{3.4}$$

where $p_\lambda(t) = \frac{\lambda^{\sigma_2}}{1+\mathcal{K}\lambda^{\sigma_1}}$, $h_\lambda = \langle h, \phi_\lambda\rangle$ and $q_\lambda(t) = \frac{\langle f(t),\phi_\lambda\rangle}{1+\mathcal{K}\lambda^{\sigma_1}}$.

For $s, t \in [0,T]$, $\int_t^s p_\lambda(\tau)d\tau = p_\lambda(t)(s-t)$, then (3.4) rewritten as

$$y_\lambda(t) = h_\lambda e^{p_\lambda(t)(T-t)} - \int_t^T q_\lambda(s) e^{p_\lambda(t)(s-t)} ds. \tag{3.5}$$

Throughout this paper, let $\phi_n, n \in \dot{\mathbb{N}}$, be the eigenfunction corresponding to the eigenvalues λ_n satisfying $0 < \lambda_1 < \lambda_2 < ...$ and $\lim_{n\to\infty} \lambda_n = \infty$. Furthermore, assume that $\{\phi_n\}_{n\in\dot{\mathbb{N}}}$ is an orthonormal basis of $\mathcal{H}$. If Problem (3.2) has a solution $u \in \mathscr{C}([0,T], \mathcal{H})$,

$$u(t) = \sum_{n=1}^{\infty} e^{\mu_n(T-t)}\langle h, \phi_n\rangle\phi_n - \sum_{n=1}^{\infty}\int_t^T \nu_n(s) e^{\mu_n(s-t)}\langle f(s), \phi_n\rangle\phi_n ds, \tag{3.6}$$

here

$$\rho_n = \frac{1}{1 + \mathcal{K}\lambda_n^{\sigma_1}}, \mu_n = \frac{\lambda_n^{\sigma_2}}{1 + \mathcal{K}\lambda_n^{\sigma_1}}, \ n = 1, 2, ...$$

This suggests to define the mild solution of the problem (1.1) as follows:

Definition 3.1. A function $u \in \mathscr{C}([0,T];\mathcal{H})$ is said to be a mild solution of Problem (1.1) if the following conditions are fulfilled

(i) $u(T,.) = h$, and

(ii) there exists $f \in \mathcal{S}_F(u)$ such that for every $t \in [0,T]$,

$$u(t,.) = \sum_{n=1}^{\infty} e^{\mu_n(T-t)}\langle h, \phi_n\rangle\phi_n(.) - \sum_{n=1}^{\infty}\int_t^T \rho_n e^{\mu_n(s-t)}\langle f(s), \phi_n\rangle\phi_n(.) ds, \tag{3.7}$$

where $\mu_n = \frac{\lambda_n^{\sigma_2}}{1+\mathcal{K}\lambda_n^{\sigma_1}}$ and $\rho_n = \frac{1}{1+\mathcal{K}\lambda_n^{\sigma_1}}$, $n \in \dot{\mathbb{N}}$.

With the same argument as above, we propose the following definition of a mild solution of (1.2).

Definition 3.2. An element $u \in \mathscr{C}([0,T];\mathcal{H})$ is called a mild solution of (1.2) if the following conditions are satisfied

(i) $u(0,.) = h$, and
(ii) there is $f \in \mathcal{T}_F(u)$ such that for any $t \in [0,T]$, we have

$$u(t,.) = \sum_{n=1}^{\infty} e^{-\mu_n t}\langle h, \phi_n\rangle \phi_n(.) + \sum_{n=1}^{\infty} \rho_n \left\{ \int_0^t \langle f(s), \phi_n\rangle e^{-\mu_n(t-s)} ds \right\} \phi_n(.), \tag{3.8}$$

where $\mu_n = \frac{\lambda_n^{\sigma_2}}{1+\mathcal{K}\lambda_n^{\sigma_1}}$ and $\rho_n = \frac{1}{1+\mathcal{K}\lambda_n^{\sigma_1}}$, $n \in \dot{\mathbb{N}}$.

By assumption $\sigma_1 \geq \sigma_2 > 0$, it is clear that $\{\rho_n\}$ and $\{\mu_n\}$ are bounded sequences in $\mathbb{R}$. Hence, if $h \in \mathcal{H}$ and $f \in L^1((0,T);\mathcal{H})$ then (3.7) and (3.8) are well defined and $u(t,.) \in \mathcal{H}$ for a.e $t \in [0,T]$.

3.2. Upper semicontinuous and condensing settings

For $f \in L^1((0,T);\mathcal{H})$, we define

$$\Phi(f)(t,.) = \sum_{n=1}^{\infty} \int_t^T \rho_n e^{\mu_n(s-t)} \langle f(s), \phi_n\rangle \phi_n(.) ds \tag{3.9}$$

and

$$\Psi(f)(t) = \sum_{n=1}^{\infty} \rho_n \int_0^t \langle f(s), \phi_n\rangle e^{-\mu_n(t-s)} \phi_n ds \text{ for } f \in L^1((0,T);\mathcal{H}). \tag{3.10}$$

It is clear that Φ and Ψ are well defined. In this subsection, our aim is to obtain the upper semicontinuous, χ-condensing properties of the multioperators $\Phi \circ \mathcal{S}_F$ and $\Psi \circ \mathcal{S}_F$. The following lemma helps us to obtain the above properties. The main tool for proving the lemma is the Arzela-Ascoli theorem.

Lemma 3.3. *Let $\{f_n\} \subset L^1((0,T);\mathcal{H})$ be a semicompact sequence. Then the following statements hold.*

a. *The set $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is equicontinuous.*
b. *The set $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is relatively compact in $\mathscr{C}([0,T];\mathcal{H})$.*
c. *$\Phi(f_n) \to \Phi(f_0)$ if $f_n \rightharpoonup f_0$.*

Proof. We first begin with proving the assertion a.. Assume that $t, t' \in [0,T]$ satisfying $0 \leq t < t' \leq T$. We write

$$\Phi(f_n)(t) - \Phi(f_n)(t') = \sum_{j=1}^{\infty} \mathcal{R}_j(n)(t) - \sum_{j=1}^{\infty} \mathcal{R}_j(n)(t'), \tag{3.11}$$

here

$$\mathcal{R}_j(n)(t) = \int_t^T \alpha_n(t,s,j) ds \text{ with } \alpha_n(t,s,j) = \rho_j e^{\mu_j(s-t)} \langle f_n(s), \phi_j\rangle \phi_j.$$

Then, we get

$$\mathcal{R}_j(n)(t) - \mathcal{R}_j(n)(t') = \int_t^{t'} \alpha_n(t,s,j) ds + \int_{t'}^T (\alpha_n(t,s,j) - \alpha_n(t',s,j)) ds. \tag{3.12}$$

By using mean value theorem for function $t \mapsto e^{\mu_j(s-t)}$, we obtain

$$e^{\mu_j(s-t)} - e^{\mu_j(s-t')} = \mu_j e^{\mu_j(s-\xi_j)}(t'-t) \text{ for some } \xi_j \in (t,t').$$

Therefore

$$\alpha_n(t,s,j) - \alpha_n(t',s,j) = \rho_j \mu_j e^{\mu_j(s-\xi_j)}(t'-t)\langle f_n(s), \phi_j\rangle \phi_j.$$

From the condition $\sigma_2 \le \sigma_1$, it implies that the set $\{\mu_j : j = 1, 2, ...\}$ is bounded, hence

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}(\alpha_n(t,s,j) - \alpha_n(t',s,j))\right\|_{\mathcal{H}}^2 &= \sum_{j=1}^{\infty} \rho_j^2 \mu_j^2 e^{2\mu_j(s-\xi_j)}\langle f_n(s), \phi_j\rangle^2 |t'-t|^2 \\ &\le C_1 \sum_{j=1}^{\infty}\langle f_n(s), \phi_j\rangle^2 |t'-t|^2 \\ &= C_1 \|f_n(s)\|_{\mathcal{H}}^2 |t'-t|^2. \end{aligned} \tag{3.13}$$

Further, we have

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}\alpha_n(t,s,j)\right\|_{\mathcal{H}}^2 &= \sum_{j=1}^{\infty} p_j^2 e^{2\mu_j(s-t)}\langle f_n(s), \phi_j\rangle^2 \\ &\le C_2 \sum_{j=1}^{\infty}\langle f_n(s), \phi_j\rangle^2 \\ &= C_2 \|f_n(s)\|_{\mathcal{H}}^2. \end{aligned} \tag{3.14}$$

Combination of (3.14), (3.13), (3.12) and (3.11) shows that

$$\|\Phi(f_n)(t) - \Phi(f_n)(t')\|_{\mathcal{H}} \le \sqrt{C_2}\int_t^{t'} \|f_n(s)\|_{\mathcal{H}} ds + \sqrt{C_1}\int_{t'}^{T} \|f_n(s)\|_{\mathcal{H}} ds |t'-t|. \tag{3.15}$$

From the semi-compactness assumption of the sequence $\{f_n\}$, it flows that it is integrably bounded, i.e, there exists $\alpha \in L^1([0,T], \mathbb{R})$ such that $\|f_n(s)\|_{\mathcal{H}} \le \alpha(s)$ for a.e $s \in [0,T]$ and for all $n \in \dot{\mathbb{N}}$. So from (3.15), we evaluate

$$\|\Phi(f_n)(t) - \Phi(f_n)(t')\|_{\mathcal{H}} \le C|t'-t| \quad \text{for all } n \in \dot{\mathbb{N}}. \tag{3.16}$$

This deduces the assertion a.

Next, we proceed to prove Part b. We shall prove the set $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is bounded at any point $t \in [0,T]$. Indeed, for every $t \in [0,T]$, since $\{f_n\}$ is integrally bounded, we get

$$\begin{aligned}\left\|\Phi(f_n)(t)\right\|_{\mathcal{H}} &\le C_0 \int_0^T \|f_n(s)\|_{\mathcal{H}} ds \\ &\le C_0 \int_0^T \alpha(s) ds = C \quad \forall n \in \dot{\mathbb{N}}. \end{aligned} \tag{3.17}$$

This implies that $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is relative compact in $\mathscr{C}([0,T], \mathcal{H})$ by Arzela-Ascoli theorem. Assertion c. is a consequence of b. with the note that Φ is bounded linear mapping from $L^1((0,T); \mathcal{H})$ to $\mathscr{C}([0,T]; \mathcal{H})$. □

By the same argument, we can also prove the following lemma.

Lemma 3.4. *Assume that the sequence $\{f_n\} \subset L^1((0,T); \mathcal{H})$ is semicompact. Then the following statements hold.*

a) *The set $\{\Psi(f_n) : n \in \dot{\mathbb{N}}\}$ is equicontinuous.*

b) *The $\{\Psi(f_n) : n \in \dot{\mathbb{N}}\}$ is relatively compact in $\mathscr{C}([0,T]; \mathcal{H})$.*

c) *If $f_n \rightharpoonup f_0$, then $\Psi(f_n) \to \Psi(f_0)$.*

Using the upper semicontinuous assumption (Hb) of F and applying Mazur's theorem, we obtain the following lemma.

Lemma 3.5. *Let $\{v_n\}_{n\in\dot{\mathbb{N}}} \subset \mathscr{C}([0,T];\mathcal{H})$ and $\{f_n\}_{\in\dot{\mathbb{N}}} \subset L^1((0,T);\mathcal{H})$ satisfying $f_n \in \mathcal{S}_F(v_n)$ for all $n \geq 1$. Then, if $v_n \to v$ and $f_n \rightharpoonup f$, $f \in \mathcal{S}_F(v)$.*

The closed property of the multioperator $\Phi \circ \mathcal{S}_F$ is consequence of the use Lemma 3.3 and Lemma 3.5.

Lemma 3.6. *Assume that the condition (H) is satisfied. Then $\Phi \circ \mathcal{S}_F$ and $\Psi \circ \mathcal{S}_F$ are closed multioperators from $L^1((0,T);\mathcal{H})$ into $\mathscr{C}([0,T],\mathcal{H})$.*

Proof. We prove the closed property of $\Phi\circ\mathcal{S}_F$, the one is argued similarly for $\Psi\circ\mathcal{S}_F$. Assume that sequences $\{v_n\}_{n\geq 1}$ and $\{z_n\}_{n\geq 1}$ in $\mathscr{C}([0,T];\mathcal{H})$ satisfying

$$v_n \to v, \;\; z_n \in \Phi \circ \mathcal{S}_F(v_n) \;\text{ và } z_n \to z.$$

We shall show that $z \in \Phi \circ \mathcal{S}_F(v)$. Indeed, let $\{f_n\}$ be a sequence in $L^1((0,T);\mathcal{H})$ satisfying $f_n \in \mathcal{S}_F(v_n)$ and $z_n = \Phi(f_n)$. It follows that $\{f_n\}$ is integrally bounded from the condition (Hc). Further, the condition (Hd) implies that $\{f_n\}$ is semicompact and also weakly compact in $L^1((0,T);\mathcal{H})$ (see [20, Theorem 5.1.2]). Without loss of generality, we may assume that $f_n \rightharpoonup f \in L^1((0,T);\mathcal{H})$. Using Lemma 3.3 gets $\Phi(f_n) \to \Phi(f) = z$ so we deduce $z \in \Phi \circ \mathcal{S}_F(v)$ by Lemma 3.5. □

The following result is a consequence of Lemma 3.3 (resp., 3.4) and Lemma 3.6.

Lemma 3.7. *Assume that the condition (H) is fulfilled. Then, the multioperator $\Phi \circ \mathcal{S}_F$ (resp., $\Psi \circ \mathcal{S}_F$) is u.s.c.*

Next, we present the condensing property of the multioperator $\Phi\circ\mathcal{S}_F$ associated with a suitable measure of noncompactness. Let $D \subset \mathscr{C}([0,T],\mathcal{H})$, we denote by $\Delta(D)$ the family of all denumerable subsets of D. Let L be a positive constant, we define

$$\nu_L(D) \triangleq \max_{Q\in\Delta(D)} \Big(\gamma_L(Q);\, \mathrm{mod}_C(Q)\Big),$$

where

$$\gamma_L(Q) \triangleq \sup_{t\in[0,T]} e^{Lt}\chi(Q(t)), \quad \mathrm{mod}_C(Q) \triangleq \lim_{\delta\to 0} \sup_{v\in D} \max_{|t'-t|\leq\delta} \|v(t') - v(t)\|,$$

$Q(t) = \{w(t) : w \in Q\}$. The MNC ν_L has the all properties which present in Lemma 2.2. The reader can find their proofs in [20, Example 2.1.4].

Lemma 3.8. *Assume that (H) is satisfied, $\mathcal{S}_F : \mathscr{C}([0,T];\mathcal{H}) \to \mathscr{P}(L^1(0,T);\mathcal{H})$ is defined by* (3.1) *and Φ is given by* (3.9)*. Then, there exists $L > 0$ such that $\Phi \circ \mathcal{S}_F$ is ν_L-condensing.*

Proof. Let D be a bounded subset of $\mathscr{C}([0,T];\mathcal{H})$ satisfying

$$\nu_L(D) \preceq \nu_L(\Phi \circ \mathcal{S}_F), \tag{3.18}$$

here the order $\preceq$ is taken in $\mathbb{R}^2$ induced by the positive cone $\mathbb{R}_+ \times \mathbb{R}_+$. We shall prove that D is relatively compact. Let $\{v_n\}$ be an any sequence in D, we set $g_n(t,.) = \Phi(f_n)(t,.)$ with $f_n \in \mathcal{S}_F(v_n)$ and

$$\nu_L(\{g_n : n \geq 1\}) = \left(\gamma_L(\{g_n : n \geq 1\}); \mathrm{mod}_C(\{g_n : n \geq 1\})\right),$$

number L shall show later. We have

$$
\begin{aligned}
e^{Lt}\chi(\{g_n(t,.) : n \geq 1\}) &= e^{Lt}\chi\left(\left\{\sum_{j=1}^{\infty}\left(\int_t^T \rho_n e^{\mu_j(s-t)}\langle f_n(s), \phi_j\rangle ds\right)\phi_j(.) : n \geq 1\right\}\right)\\
&\leq C_0 e^{Lt}\int_t^T \chi\left(\{f_n(s) : n \geq 1\}\right) ds\\
&\leq C_1 \sup_{s\in[0,T]}\left(e^{Ls}\chi(\{v_n(s,.) : n \geq 1\})\right)\int_t^T s^{\gamma_1}e^{-L(s-t)}ds, \qquad (3.19)
\end{aligned}
$$

where we have used χ-regularity condition (Hd) in the last estimate. From the above inequality, we obtain

$$
\gamma_L(\{g_n : n \geq 1\}) \leq C_1\left(\sup_{t\in[0,T]}\int_t^T s^{\gamma_1}e^{-L(s-t)}ds\right)\gamma_L(\{v_n : n \geq 1\}). \qquad (3.20)
$$

Since

$$
\lim_{L\to\infty}\left(\sup_{t\in[0,T]}\int_t^T s^{\gamma}e^{-L(s-t)}ds\right) = 0, \quad (\gamma > -1),
$$

there exists $L_0 > 0$ such that

$$
\sup_{t\in[0,T]}\int_t^T s^{\gamma_1}e^{-L(t-s)}ds < \frac{1}{4C_1} \quad \text{for all } L \geq L_0. \qquad (3.21)
$$

On the other hand, it implies $\gamma_{L_0}(\{g_n : n \geq 1\}) \geq \gamma_{L_0}(\{v_n : n \geq 1\})$ from (3.18). Hence, combining with (3.20) and (3.21), we get $\gamma_{L_0}(\{v_n : n \geq 1\}) = 0$. So $\chi(\{v_n(t,.)\}) = 0$ for all $t \in [0,T]$. From the conditions (Hc) and (Hd), it implies that $\{f_n\}$ is semicompact. Applying Lemma 3.3, we deduce that $\{g_n : n \geq 1\}$ is relatively compact, so $\nu_{L_0}(D) = (0,0)$. The proof is completed. □

Lemma 3.9. *Assume that the condition (H) is satisfied, $\mathcal{T}_F : \mathscr{C}([0,T];\mathcal{H}) \to \mathscr{P}(L^1(0,T);\mathcal{H})$ and Φ are defined by* (3.1) *and* (3.10)*, resp,. Then, there exists $L > 0$ such that $\Phi \circ \mathcal{T}_F$ is ν_L-condensing.*

Proof. This proof is based on the proof of Lemma 3.8. Assume that D is a bounded subset of $\mathscr{C}([0,T];\mathcal{H})$ satisfying

$$
\nu_L(D) \leq \nu_L(\Phi \circ \mathcal{T}_F). \qquad (3.22)
$$

Assume that $\{v_n\}$ is any sequence in D, we set $g_n(t,.) = \Phi(f_n)(t,.)$ with $f_n \in \mathcal{T}_F(v_n)$. We have

$$
\nu_L(\{g_n : n \geq 1\}) = (\gamma_L(\{g_n : n \geq 1\}); \mathrm{mod}_C(\{g_n : n \geq 1\}))
$$

and

$$
\begin{aligned}
e^{-Lt}\chi(\{g_n(t,.) : n \geq 1\}) &= e^{-Lt}\chi\left(\left\{\sum_{j=1}^{\infty}\rho_j\left(\int_0^t \langle f_n(s), \phi_j\rangle e^{-\mu_j(t-s)}ds\right)\phi_j(.) : n \geq 1\right\}\right)\\
&\leq C_0 e^{-Lt}\int_0^t \chi\left(\{f_n(s) : n \geq 1\}\right) ds\\
&\leq C_1 \sup_{s\in[0,T]}\left(e^{-Ls}\chi(\{v_n(s,.) : n \geq 1\})\right)\int_0^t s^{\gamma_1}e^{-L(t-s)}ds,
\end{aligned}
$$

where we have used χ-regularity condition (Hd) in the last estimate. From the above inequality, we get

$$\gamma_L(\{g_n : n \geq 1\}) \leq C_1 \left(\sup_{t\in[0,T]} \int_0^t s^{\gamma_1} e^{-L(t-s)} ds \right) \gamma_L(\{v_n : n \geq 1\}). \tag{3.23}$$

Since $\gamma > -1$,

$$\lim_{L\to\infty} \left(\sup_{t\in[0,T]} \int_0^t s^{\gamma} e^{-L(t-s)} ds \right) = 0,$$

it shows that there is $L_0 > 0$ such that

$$\sup_{t\in[0,T]} \int_0^t s^{\gamma_1} e^{-L(t-s)} ds < \frac{1}{4C_1} \quad \text{for all } L \geq L_0. \tag{3.24}$$

The rest of the proof is argued in the same way as above. We finish the proof. □

3.3. Existence and compactness

In this subsection, we shall establish the compact property of the mild solutions set, denoted by $\mathcal{G}_h^F[0,T]$ (resp., $\mathscr{S}_h^F[0,T]$), of the inclusion (1.1) (resp., (1.2)).

Theorem 3.10. *Assume that F satisfying the condition (H) and $h \in \mathcal{H}$. Then, the set $\mathcal{G}_h^F[0,T]$ is nonempty compact subset of $\mathscr{C}([0,T];\mathcal{H})$.*

Proof. Consider multioperator $\mathcal{M} : \mathscr{C}([0,T];\mathcal{H}) \to \mathscr{P}(\mathscr{C}([0,T];\mathcal{H}))$ defined

$$\mathcal{M}(u) := \left\{ v \in \mathscr{C}([0,T];\mathcal{H}) : v(t) = \sum_{n=1}^{\infty} e^{\mu_n(T-t)} \langle h, \phi_n\rangle \phi_n - \Phi(f)(t), f \in \mathcal{S}_F(u) \right\}.$$

It is clear that there exists $C_1 > 0$ such that for all $f \in L^1((0,T);\mathcal{H})$, we have

$$\left\| \sum_{n=1}^{\infty} e^{\mu_n(T-t)} \langle h, \phi_n\rangle \phi_n \right\|_{\mathcal{H}} + \|\Phi(f)(t)\|_{\mathcal{H}} \leq C_1 \left(\|h\|_{\mathcal{H}} + \int_t^T \|f(s)\|_{\mathcal{H}} ds \right). \tag{3.25}$$

Applying Lemma 3.7 and Lemma 3.8, we can choose $L_0 > 0$ largely enough satisfying (3.21) such that $\mathcal{M}$ is u.s.c and ν_{L_0}-condensing. Let us introduce the temporally weighted space

$$\mathscr{C}_{L_0}([0,T];\mathcal{H}) = \left\{ v \in \mathscr{C}([0,T];\mathcal{H}) : e^{L_0 t} \|v(t,.)\|_{\mathcal{H}} < \infty \ \forall t \in [0,T] \right\},$$

endowed with norm.

$$\|v\|_{\mathscr{C}_{L_0}([0,T];\mathcal{H})} = \sup_{t\in[0,T]} e^{L_0 t} \|v(t,\cdot)\|_{\mathcal{H}} \quad \forall v \in \mathscr{C}_{L_0}([0,T];\mathcal{H}).$$

In this space, we denote by $\overline{B}(r)$ the closed ball centered at the zero function with the radius r. We shall prove that there is $r > 0$ such that $\mathcal{M}$ maps the ball $\overline{B}(r)$ into itself. Indeed, choose r satisfying $r > C_1 e^{L_0 T} \|h\|_{\mathcal{H}} + \frac{e^{L_0 T} + r}{4}$. Let $u \in \overline{B}(r)$, $f \in \mathcal{S}_F(u)$, $v \in \mathcal{M}(u)$. From the condition (Hc), we get

$$\begin{aligned} e^{L_0 t} \|v(t)\|_{\mathcal{H}} &\leq e^{L_0 t} \left\| \sum_{n=1}^{\infty} e^{\mu_n(T-t)} \langle h, \phi_n\rangle \phi_n \right\|_{\mathcal{H}} + e^{L_0 t} \left\| \Phi(f)(t) \right\|_{\mathcal{H}} \\ &\leq C_1 \left(e^{L_0 t} \|h\|_{\mathcal{H}} + \int_t^T e^{-L_0(s-t)} e^{L_0 s} s^{\gamma_1} \left(1 + \|u(s,.)\|_{\mathcal{H}}\right) ds \right) \\ &\leq C_1 \left(e^{L_0 t} \|h\|_{\mathcal{H}} + \int_t^T s^{\gamma_1} (e^{L_0 s} + r) e^{-L_0(s-t)} ds \right) \\ &\leq C_1 \left(e^{L_0 T} \|h\|_{\mathcal{H}} + (e^{L_0 T} + r) \int_t^T s^{\gamma_1} e^{-L_0(s-t)} ds \right) < r. \end{aligned} \tag{3.26}$$

It implies $v \in \overline{B}(r)$. Hence, $\mathcal{G}_h^F[0,T] \neq \emptyset$ by applying Lemma 2.4. It is remain to prove that $\mathcal{G}_h^F[0,T]$ is a compact set. Assume that $u \in \mathcal{G}_h^F[0,T]$ and $t \in [0,T]$. Then $u \in \mathcal{M}(u)$. Applying Grönwall's inequality and using the condition (Hc), we obtain

$$\begin{aligned} \|u(t,.)\|_{\mathcal{H}} &\leq C_0\|h\|_{\mathcal{H}} + C_1 \int_t^T s^{\gamma_1}(1 + \|u(s,.)\|_{\mathcal{H}})ds \\ &\leq C_2\|h\|_{\mathcal{H}} e^{\int_t^T s^{\gamma_1} ds} \leq C, \end{aligned} \tag{3.27}$$

where C does not depend on t. Therefore, we complete the proof by applying Lemma 2.5. □

Theorem 3.11. *Assume that F satistied the condition (H) and $h \in \mathcal{H}$. Then $\mathscr{S}_h^F[0,T]$ is a nonempty and compact subset of $\mathscr{C}([0,T];\mathcal{H})$.*

Proof. The argument is similar to the proof of theorem Theorem 3.10. We consider the multimap $\mathcal{M} : \mathscr{C}([0,T];\mathcal{H}) \to \mathscr{P}(\mathscr{C}([0,T];\mathcal{H}))$ defined by

$$\mathcal{M}(u) := \left\{ v \in \mathscr{C}([0,T];\mathcal{H}) : v(t,.) = \sum_{n=1}^{\infty} e^{-\mu_n t}\langle h, \phi_n\rangle \phi_n(.) + \Psi(f)(t,.), f \in \mathcal{S}_F(u) \right\}.$$

Choose C_1 satisfying

$$\left\| \sum_{n=1}^{\infty} e^{-\mu_n t}\langle h, \phi_n\rangle \phi_n \right\|_{\mathcal{H}} + \|\Psi(f)(t)\|_{\mathcal{H}} \leq C_1 \left(\|h\|_{\mathcal{H}} + \int_0^t \|f(s)\|_{\mathcal{H}} ds \right). \tag{3.28}$$

Using Lemma 3.7 and Lemma 3.9, we derive that $\mathcal{M}$ is u.s.c and ν_{L_0}-condensing, where L_0 in (3.21). We define the weighted space

$$C_{L_0}([0,T];\mathcal{H}) = \left\{ v \in \mathscr{C}([0,T];\mathcal{H}) : \exists K > 0, \|v(t,.)\|_{\mathcal{H}} \leq Ke^{L_0 t} \ \forall t \in [0,T] \right\},$$

endowed with norm

$$\|v\|_{C_{L_0}([0,T];\mathcal{H})} = \sup_{t\in[0,T]} e^{-L_0 t}\|v(t,\cdot)\|_{\mathcal{H}} \quad \forall v \in \mathscr{C}_{L_0}([0,T];\mathcal{H}).$$

Choose $r > C_1\|h\|_{\mathcal{H}} + (r+1)/4$. Let $u \in \overline{B}(r)$, $f \in \mathcal{S}_F(u)$, $v \in \mathcal{M}(u)$. Using the condition (Hc), we have

$$\begin{aligned} e^{-L_0 t}\|v(t,.)\|_{\mathcal{H}} &\leq e^{-L_0 t} \left\| \sum_{n=1}^{\infty} e^{-A_n t}\langle h, \phi_n\rangle \phi_n \right\|_{\mathcal{H}} + e^{-L_0 t} \left\| \Phi(f)(t,.) \right\|_{\mathcal{H}} \\ &\leq C_1 \left(e^{-L_0 t}\|h\|_{\mathcal{H}} + \int_0^t e^{-L_0(t-s)} e^{-L_0 s} s^{\gamma_1} \left(1 + \|u(s,.)\|_{\mathcal{H}}\right) ds \right) \\ &\leq C_1 \left(e^{-L_0 t}\|h\|_{\mathcal{H}} + \int_0^t s^{\gamma_1}(e^{-L_0 s} + r) e^{-L_0(t-s)} ds \right) \\ &\leq C_1 \left(e^{-L_0 t}\|h\|_{\mathcal{H}} + (1+r) \int_0^t s^{\gamma_1} e^{-L_0(t-s)} ds \right) < r. \end{aligned}$$

This implies $v \in \overline{B}(r)$. It follows that $\mathscr{C}_h^F[0,T] \neq \emptyset$ by applying Lemma 2.4. To prove that $\mathscr{C}_h^F[0,T]$ is compact. This is argued similarly to the last part in the proof of the previous theorem. □

3.4. Continuous dependence on parameters

In this subsection, we consider the dependence of the solution of the following parameterized problems (1.6) and (1.7) on the parameter μ in a metric space (E, d). For convenience, we recall the problem

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{K}\mathcal{A}^{\sigma_1}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{A}^{\sigma_2}u(t,x) \in F(t,u,\mu), & (t,x) \in [0,T) \times \Omega \\ u(T,x) = h(x), & x \in \Omega,\end{cases} \tag{3.29}$$

where $\frac{\partial}{\partial t}$, $\mathcal{A}$ are described as in Section 1 and $0 < \sigma_2 \le \sigma_1$.

We establish the continuous dependence on parameters with the following assumptions (H_μ) on non-linearity.

Let $F : [0,T] \times \mathcal{H} \times E \to Kv(\mathcal{H})$ be a multimapping satisfying the following conditions:

H_μ(a) : The multimapping $F(.,u,\mu)$ has a strongly measurable selection for every $(u,\mu) \in \mathcal{H} \times E$;

H_μ(b) : The multimapping $F(t,.,.) : \mathcal{H} \times E \to Kv(\mathcal{H})$ is u.s.c for a.e. $t \in [0,T]$;

H_μ(c) : There exists a function $\alpha \in L^1((0,T);\mathbb{R})$ such that

$$|||F(t,u,\mu)||| := \sup_{v\in F(t,u,\mu)} \|v\|_{\mathcal{H}} \le \alpha(t)(1 + \|u\|_{\mathcal{H}}) \text{ for a.e. } t \in (0,T), \text{ for all } u \in \mathcal{H}, \mu \in E;$$

H_μ(d) : There is $\mathcal{B} \in L^1((0,T);\mathbb{R})$ satisfying

$$\chi(F(t,G,E)) \le \mathcal{B}(t)\chi(G) \text{ for a.e. } t \in (0,T) \text{ and for all } G \in b(\mathcal{H}),$$

here χ is MNC in $\mathcal{H}$ defined

$$\chi(G) = \inf\{\varepsilon > 0 : G \text{ has a finite } \varepsilon\text{-net}\}. \tag{3.30}$$

For $(u,\mu) \in \mathscr{C}([0,T];\mathcal{H}) \times E$, we denote

$$\mathcal{S}_{F,\mu}(u) = \left\{f \in L^1((0,T);\mathcal{H}) | f(t,.) \in F(t,u,\mu), \text{ for a.e. } t \in (0,T)\right\}.$$

For every $\mu \in E$, similarly as Theorem 3.10, we also denote multioperator $\mathcal{M}_\mu : \mathscr{C}([0,T];\mathcal{H}) \to \mathscr{P}(\mathscr{C}([0,T];\mathcal{H}))$ defined

$$\mathcal{M}_\mu(u) := \left\{v \in \mathscr{C}([0,T];\mathcal{H}) : v(t) = \sum_{n=1}^{\infty} e^{\mu_n(T-t)}\langle h, \phi_n\rangle\phi_n - \Phi(f)(t), f \in \mathcal{S}_{F,\mu}(u)\right\}.$$

Denote by $\mathcal{G}_h^{F,\mu}$ the set of all local mild solutions of Problem (3.29), i.e. , $u \in \mathcal{G}_h^{F,\mu}$ if there exists $\tau \in [0,T)$ and $u \in \mathscr{C}([0,T];\mathcal{H})$ such that for all $\overline{\tau} \in (\tau,T]$ and $v_{\overline{\tau}} = u|_{[\overline{\tau},T]}$, it holds

$$v_{\overline{\tau}} \in \left\{w \in \mathscr{C}([\overline{\tau},T];\mathcal{H}) : w(t) = \sum_{n=1}^{\infty} e^{\mu_n(T-t)}\langle h, \phi_n\rangle\phi_n - \Phi(f)(t), f \in \mathcal{S}_{F,\mu}(u)\right\},$$

and $\mathcal{G}_h^{F,\mu}[0,T] := \left\{v \in \mathcal{G}_h^{F,\mu} : v \in \mathcal{M}_\mu(v)\right\}$.

Theorem 3.12. *Assume that the assumption (H_μ) holds, the set $\mathcal{G}_h^{F,\mu_0}[0,T]$ is bounded for some parameter $\mu_0 \in E$ and*

$$\mathcal{G}_h^{F,\mu_0}[\overline{\tau},T] = \mathcal{G}_h^{F,\mu_0}[0,T]|_{[\overline{\tau},T]} \quad \text{for all } \overline{\tau} \in [0,T). \tag{3.31}$$

Then, for every given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$\mathcal{G}_h^{F,\mu}[0,T] \subset \mathcal{N}_\epsilon\left(\mathcal{G}_h^{F,\mu_0}[0,T]\right) \quad \text{for all } \lambda \in \mathcal{B}_{\delta_\epsilon}(\mu_0).$$

Proof. Assume that $r > 0$ such that $|||\mathcal{G}_h^{F,\mu}[0,T]||| < r$. Firstly, we shall show the following statement by contraction argument: There is $\delta > 0$ such that if $\mu \in \mathcal{N}_\delta(\mu_0) \subset E$, then

$$|||\mathcal{G}_h^{F,\mu}(t)||| \leq 3r \quad \text{for all } t \in [0,T]. \tag{3.32}$$

Indeed, we assume by contradiction that (3.32) fails. Then, we can take sequences $\{\mu_n\} \subset E$, $\{t_n\} \subset [0,T]$, $\{u_n\} \subset \mathscr{C}([0,T];\mathcal{H})$, $\mu_n \to \mu_0$ such that $w_n \in \mathcal{M}^{\mu_n}(w_n)$ and

$$\operatorname{dist}\left(w_n(t_n), \mathcal{G}_h^{F,\mu_0}(t_n)\right) \geq 2r, \quad \operatorname{dist}\left(w_n(t), \mathcal{G}_h^{F,\mu_0}(t)\right) < 2r \tag{3.33}$$

for all $t \in (t_n, T]$.

Denote $t_* = \overline{\lim}\{t_n\}$. We shall prove that $t_* \in [0,T)$. Indeed, assume $t_* = T$. Let us choose a sub-sequence of $\{t_n\}$ tends to T, which we also denote by $\{t_n\}$ for convenience. Since $\mathcal{G}_h^{F,\mu_0}$ is bounded and from (3.31), it follows that $\mathcal{G}_h^{F,\mu_0}$ is compact, and so the distance between h and $\mathcal{G}_h^{F,\mu_0}(t_n)$ going to zero. It is clear that

$$\begin{aligned} 2r &\leq \operatorname{dist}\left(w_n(t_n), \mathcal{G}_h^{F,\mu_0}(t_n)\right) \\ &\leq \|w_n(t_n) - h\|_{\mathcal{H}} + \operatorname{dist}\left(h, \mathcal{G}_h^{F,\mu_0}(t_n)\right) \\ &\leq \left\| \sum_{j=1}^{\infty} e^{\mu_j (T-t_n)} \langle h, \phi_j \rangle \phi_j - h \right\|_{\mathcal{H}} + \|\Phi(f_n)(t_n)\|_{\mathcal{H}} + \operatorname{dist}\left(h, \mathcal{G}_h^{F,\mu_0}(t_n)\right), \end{aligned} \tag{3.34}$$

where $f_n \in \mathcal{S}_{F,\mu_0}(w_n)$ for all $n = 1, 2, \ldots$. Letting $n \to \infty$ in (3.34), we derive the contradiction $2r \leq 0$. Summarily, we deduce $t_* < T$.

By the definition of t_*, there exists number γ with $0 \leq t_* < \gamma < T$ such that all solution w_n are defined on $[0, \gamma - t_*]$. We next prove that for every w_n, there exists $\tau_n \in [0, \gamma - t_*] \subsetneq [0,T]$ satisfying

$$\operatorname{dist}\left(w_n(\tau_n), \mathcal{G}_h^{F,\mu_0}(\tau_n)\right) \geq \epsilon. \tag{3.35}$$

For any $t_\dagger \in (t_n, T]$, we can assume that $\|w_n(t_\dagger) - w_\dagger(t_\dagger)\|_{\mathcal{H}} < \epsilon$ for some $w_\dagger \in \mathcal{G}_h^{F,\mu_0}$ by the compactness of $\mathcal{G}_h^{F,\mu_0}$. Then,

$$\begin{aligned} &\|w_n(t_\dagger + t) - w_\dagger(t_\dagger + t)\|_{\mathcal{H}} \\ &\qquad \leq \|w_n(t_\dagger + t) - w_n(t_\dagger)\|_{\mathcal{H}} + \|w_\dagger(t_\dagger + t) - w_\dagger(t_\dagger)\|_{\mathcal{H}} + \|w_n(t_\dagger) - w_\dagger(t_\dagger)\|_{\mathcal{H}}. \end{aligned}$$

With the similar arguments as the first part of Lemma 3.3, one can choose t small enough such that both $\|w_n(t_\dagger+t) - w_n(t_\dagger)\|_{\mathcal{H}}$ and $\|w_\dagger(t_\dagger+t) - w_\dagger(t_\dagger)\|_{\mathcal{H}}$ are less than $\epsilon/4$. Hence, the norms $\|w_n(t_\dagger+t) - w_\dagger(t_\dagger+t)\|_{\mathcal{H}} \leq 3\epsilon/2$, which contradicts (3.33). Namely, (3.35) is proved.

By the same arguments in the proof of Lemma 3.8, we see that the multimap $\mathcal{M}_* : E \times \mathscr{C}([0, \gamma - t_*];\mathcal{H}) \to Kv(C([0, \gamma - t_*];\mathcal{H}))$, $\mathcal{M}_*(\mu, u) = \mathcal{M}_\mu(u)$, is ν_L-condensing for some $L > 0$. This ensures relative compactness of the sequence $\{w_n|_{[0,\gamma - t_*]}\}$. Let us take $w_* = \lim w_n|_{[0,\gamma - t_*]}$, which belongs to $\mathcal{M}_*(\lambda_0, w_*)$ on $[0, \gamma - t_*]$. Thus, letting $n \to \infty$ in (3.35), we obtain

$$\operatorname{dist}\left(w_*(t_*), \mathcal{G}_h^{F,\mu_0}(t_*)\right) \geq \epsilon.$$

Consequently, the solution u_* cannot be extended to the interval $[0,T]$, which contradicts (3.31). Finally, the proof is completed by applying Lemma 2.6. □

Denote by $\mathcal{R}_h^{F,\mu}$ the family of all local mild solutions of Problem (1.7), i.e. , $u \in \mathcal{R}_h^{F,\mu}$ iff there exists $\tau \in (0,T]$ and $u \in \mathscr{C}([0,T];\mathcal{H})$ such that for all $\overline{\tau} \in [0,\tau]$ and $v_{\overline{\tau}} = u|_{[0,\overline{\tau}]}$, it holds

$$v_{\overline{\tau}} \in \left\{ w \in \mathscr{C}([0,\overline{\tau}];\mathcal{H}) : w(t) = \sum_{n=1}^{\infty} e^{-\mu_n t} \langle h, \phi_n \rangle \phi_n + \Psi(f)(t), f \in \mathcal{S}_{F,\mu}(u) \right\},$$

and $\mathcal{R}_h^{F,\mu}[0,T] := \left\{v \in \mathcal{R}_h^{F,\mu} : v \in \mathcal{M}^{\mu}(v)\right\}$, here

$$\mathcal{M}^{\mu}(u) := \left\{ v \in \mathscr{C}([0,T];\mathcal{H}) : v(t) = \sum_{n=1}^{\infty} e^{-\mu_n t}\langle h, \phi_n\rangle\phi_n + \Psi(f)(t), f \in \mathcal{S}_{F,\mu}(u) \right\}.$$

Theorem 3.13. *Assume that the condition* (H_{μ}) *holds, the set* $\mathcal{R}_h^{F,\mu_0}[0,T]$ *is bounded for some* $\mu_0 \in E$, *and*

$$\mathcal{R}_h^{F,\mu_0}[0,\overline{\tau}] = \mathcal{R}_h^{F,\mu_0}[0,T]|_{[0,\overline{\tau}]} \quad \text{for all } \overline{\tau} \in (0,T]. \tag{3.36}$$

Then, for every given $\epsilon > 0$, *there is* $\delta_\epsilon > 0$ *satisfying*

$$\mathcal{R}_h^{F,\mu}[0,T] \subset \mathcal{N}_\epsilon\left(\mathcal{R}_h^{F,\mu_0}[0,T]\right) \quad \text{for all } \lambda \in \mathcal{B}_{\delta_\epsilon}(\mu_0).$$

Proof. Assume that $r > 0$ with $|||\mathcal{R}_h^{F,\mu}[0,T]||| < r$. Firstly, we shall prove the following statement by contraction argument: There exists $\delta > 0$ such that if $\mu \in \mathcal{N}_\delta(\mu_0) \subset E$, then

$$|||\mathcal{R}_h^{F,\mu}(t)||| \le 3r \quad \text{for all } t \in [0,T]. \tag{3.37}$$

Indeed, we assume by contradiction that (3.37) fails. Then, we can take sequences $\{\mu_n\} \subset E$, $\{t_n\} \subset [0,T]$, $\{u_n\} \subset \mathscr{C}([0,T];\mathcal{H})$, $\mu_n \to \mu_0$ such that $w_n \in \mathcal{M}_{\mu_n}(w_n)$ and

$$\operatorname{dist}\left(w_n(t_n), \mathcal{R}_h^{F,\mu_0}(t_n)\right) \ge 2r, \quad \operatorname{dist}\left(w_n(t), \mathcal{R}_h^{F,\mu_0}(t)\right) < 2r \tag{3.38}$$

for all $t \in [0, t_n)$.

Denote $t_* = \underline{\lim}\{t_n\}$. We shall prove that $t_* \in (0,T]$. Indeed, assume that $t_* = 0$. Let us choose a subsequence of $\{t_n\}$ going to 0, which we also denote by $\{t_n\}$ for convenience. Since $\mathcal{R}_h^{F,\mu_0}$ is bounded and from (3.36), it follows that $\mathcal{R}_h^{F,\mu_0}$ is compact, so the distance between h and $\mathcal{R}_h^{F,\mu_0}(t_n)$ tends to zero. We observe that

$$\begin{aligned} 2r &\le \operatorname{dist}\left(w_n(t_n), \mathcal{R}_h^{F,\mu_0}(t_n)\right) \\ &\le \|w_n(t_n) - h\|_{\mathcal{H}} + \operatorname{dist}\left(h, \mathcal{R}_h^{F,\mu_0}(t_n)\right) \\ &\le \left\| \sum_{j=1}^{\infty} e^{\mu_j(T-t_n)}\langle h, \phi_j\rangle\phi_j - h \right\|_{\mathcal{H}} + \|\Phi(f_n)(t_n)\|_{\mathcal{H}} + \operatorname{dist}\left(h, \mathcal{R}_h^{F,\mu_0}(t_n)\right), \end{aligned} \tag{3.39}$$

here $f_n \in \mathcal{S}_{F,\mu_0}(w_n)$ for all $n = 1, 2, ...$ Letting $n \to \infty$ in (3.39), we derive the contradiction $2r \le 0$. Summarily, we deduce $t_* > 0$.

By the definition of t_*, there exists number γ with $0 < \gamma < t_* \le T$ such that all solution w_n are defined on $[0, t_* - \gamma]$. We next prove that for every w_n, there exists $\tau_n \in [0, t_* - \gamma] \subsetneq [0,T]$ satisfying

$$\operatorname{dist}\left(w_n(\tau_n), \mathcal{R}_h^{F,\mu_0}(\tau_n)\right) \ge \epsilon. \tag{3.40}$$

For every n, let any $t_\dagger \in [0, t_n)$. By the compactness of $\mathcal{R}_h^{F,\mu_0}$, we can assume that $\|w_n(t_\dagger) - w_\dagger(t_\dagger)\|_{\mathcal{H}} < \epsilon$ for some $w_\dagger \in \mathcal{R}_h^{F,\mu_0}$. Then

$$\begin{aligned} &\|w_n(t_\dagger + t) - w_\dagger(t_\dagger + t)\|_{\mathcal{H}} \\ &\qquad \le \|w_n(t_\dagger + t) - w_n(t_\dagger)\|_{\mathcal{H}} + \|w_\dagger(t_\dagger + t) - w_\dagger(t_\dagger)\|_{\mathcal{H}} + \|w_n(t_\dagger) - w_\dagger(t_\dagger)\|_{\mathcal{H}}. \end{aligned}$$

With the same arguments as the first part of Lemma 3.3, one can choose t small enough such that both $\|w_n(t_\dagger+t)-w_n(t_\dagger)\|_{\mathcal{H}}$ and $\|w_\dagger(t_\dagger+t)-w_\dagger(t_\dagger)\|_{\mathcal{H}}$ are less than $\epsilon/4$. Hence, the norms $\|w_n(t_\dagger+t)-w_\dagger(t_\dagger+t)\|_{\mathcal{H}} \le 3\epsilon/2$, which contradicts (3.38). Namely, (3.40) is proved.

By similar arguments as obtaining Lemma 3.8, we note that the multimap $\mathcal{M}_* : E \times \mathscr{C}([0, t_* - \gamma]; \mathcal{H}) \to Kv(C([0, t_* - \gamma]; \mathcal{H}))$, $\mathcal{M}_*(\mu, u) = \mathcal{M}^\mu(u)$, is ν_L-condensing for some $L > 0$. This ensures relative compactness of the sequence $\{w_n|_{[0,t_*-\gamma]}\}$. Let us take $w_* = \lim w_n|_{[0,\gamma-t_*]}$, which belongs to $\mathcal{M}_*(\lambda_0, w_*)$ on $[0, t_* - \gamma]$. Thus, by passing to the limit in (3.40), we obtain

$$\mathrm{dist}\Big(w_*(t_*), \mathcal{R}_h^{F,\mu_0}(t_*)\Big) \geq \epsilon.$$

Consequently, the solution u_* cannot be extended to the interval $[0, T]$, which contradicts (3.36). Finally, we complete the proof by applying Lemma 2.6. □

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