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Essential norm of generalized integral type operator from $Q_K(p,q)$ to Zygmund Spaces

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Abstract. Let φ be an analytic self-map on $\mathbb{D}$, $n \in \mathbb{N}$ and $g \in H(\mathbb{D})$. We consider the essential norm of the generalized integral-type operator $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_\mu$ that is defined as follows

$$\left(C^n_{\varphi,g} f\right)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)\, d\xi,$$

for all $f \in Q_K(p,q)$. We give an estimate for the essential norm of the above operator.

1. Introduction

Let $\mathbb{D}$ be unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. The Zygmund space $\mathcal{Z}$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) |f''(z)|. \tag{1}$$

With this norm it is a Banach space (see [3] and [6]). For a multidimensional generalization of the space see, for example, [21]; for the Zygmund-type space on the upper half-plane see [22].

Suppose that μ is a normal function on the interval $[0,1)$. Then $f \in H(\mathbb{D})$ is in the Zygmund-space $\mathcal{Z}_\mu$ (see, e.g., [8]), if

$$\sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)| < \infty. \tag{2}$$

Similar to $\mathcal{Z}$, $\mathcal{Z}_\mu$ is a Banach space with the following norm

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)|, \tag{3}$$

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for all $f \in \mathcal{Z}_\mu$. Note that if we set $\mu(z) = 1 - z^2$, then we obtain $\mathcal{Z}_\mu = \mathcal{Z}$. There has been a considerable interest in studying concrete operators from or to Zygmund type spaces (see, for example, [2–8, 10–12, 16, 17, 21, 22, 24, 32–35], and the related references therein).

Let $p > 0$, $q > -2$ and $K : [0, \infty) \longrightarrow [0, \infty)$ be a nondecreasing continuous function. The space $Q_K(p, q)$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{Q_K(p,q)} = |f(0)| + \sup_{\xi\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \left(1 - |z|^2\right)^q K(g(z, \xi))\, dA(z) < \infty, \tag{4}$$

where dA is the normalized Lebesgue area measure in $\mathbb{D}$, $g(z, \xi) = \log \frac{1}{|\varphi_\xi(z)|}$, and $\varphi_\xi(z) = \frac{\xi - z}{1 - \bar{\xi} z}$. For $p \geq 1$, $Q_K(p, q)$ with the norm $\|f\|_{Q_K(p,q)}$ becomes a Banach space, see [13–15, 28, 29], for more details regarding $Q_K(p, q)$ spaces. Following [28], we assume that the following condition holds

$$\int_0^1 \left(1 - r^2\right)^q \, K(-\log\, r) r\, dr < \infty. \tag{5}$$

If $f \in Q_K(p, q)$ then $f \in \mathcal{B}^{\frac{q+2}{p}}$ and

$$\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} \leq C\|f\|_{Q_K(p,q)}, \tag{6}$$

where $\mathcal{B}^\alpha$, $\alpha > 0$, denotes the Bloch type space (or α-Bloch space), see [28]. We need for the following fact about the functions in $\mathcal{B}^\alpha$ (see [30]):

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| \approx |f'(0)| + \cdots + |f^{(n)}(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^{\alpha+n} |f^{(n+1)}(z)|, \tag{7}$$

where $n \in \mathbb{N}$.

For an analytic self-mapping φ on $\mathbb{D}$, the composition operator C_φ is defined as follows:

$$C_\varphi(f)(z) = f(\varphi(z)),$$

for all $f \in H(\mathbb{D})$. The above operator is generalized by Li and Stević in [4] as follows:

$$\left(C^g_\varphi f\right)(z) = \int_0^z f'(\varphi(\xi)) g(\xi)\, d\xi, \quad f \in H(\mathbb{D}), z \in \mathbb{D},$$

where $g \in H(\mathbb{D})$. This version of composition operator is widely considered by many researchers, for example see [5, 9, 18–20]. The operator can be extended in several ways. For the corresponding integral-type operator on the unit ball in $\mathbb{C}^n$, see e.g., [23, 26, 31]. The following operator is a generalization of C^g_φ on the unit disk

$$\left(C^n_{\varphi,g} f\right)(z) = \int_0^z f^{(n)}(\varphi(\xi)) g(\xi)\, d\xi, \quad f \in H(\mathbb{D}), z \in \mathbb{D},$$

where $n \in \mathbb{N}$. This integral type operator has been investigated by many authors, see, e.g., [1, 2, 14, 25, 33] and the references therein. The boundedness and compactness of the above operator from α-Bloch spaces into Q_K spaces were studied by Stević and Sharma in [25]. The same problems have been studied for the operator $C^n_{\varphi,g}$ from $Q_K(p, q)$ and $Q_{K,0}(p, q)$ to α-Bloch spaces and little α-Bloch spaces in [14]. For the properties of this operator between H^∞ and Zygmund-type spaces, see [33], and between mixed-norm space and Zygmund-type space (little Zygmund-type space), see [2] and between Bloch-type spaces and weighted Dirichlet-type spaces, see [1].

The boundedness and compactness of the operator $C^n_{\varphi,g}$ from $Q_K(p, q)$ and $Q_{K,0}(p, q)$ into Zygmund type spaces were investigated in [15] and it was proved that $C^n_{\varphi,g} : Q_K(p, q) \to \mathcal{Z}_\mu$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)\, |g'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} = \lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}} = 0.$$

In this paper, we consider this integral type operator and investigate the essential norm of this operator from $Q_K(p,q)$ to $\mathcal{Z}_\mu$. As a result we can obtain the above characterization for the compactness. For any operator T between two Banach spaces X and Y, the essential norm of T is denoted by $\|T\|_{e,X\to Y}$ and is defined as follows

$$\|T\|_{e,X\to Y} = \inf\{\|T-S\| : S \text{ is a compact operator from } X \text{ to } Y\}.$$

The operator is compact if and only if $\|T\|_{e,X\to Y} = 0$.

2. Main results

Throughout this paper, we assume that the following condition holds

$$\int_0^1 K(-\log r)(1-r)^{\min\{-1,q\}}\left(\log\frac{1}{1-r}\right)^{\chi_{-1}(q)} rdr < \infty, \tag{8}$$

where $\chi_O(x)$ is the characteristic function of the set O and we denote the essential norm of $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_\mu$ by $\|C^n_{\varphi,g}\|_{e,Q_K(p,q)\to\mathcal{Z}_\mu}$.

The following lemma is proved in a standard way (see, e.g., [27]).

Lemma 2.1. *Let $g \in H(\mathbb{D})$, $n \in \mathbb{N}$ and φ be an analytic self-map of $\mathbb{D}$. Then $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_\mu$ is compact if and only if $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_\mu$ is bounded and for any bounded sequence $\{f_i\}_{i=1}^\infty$ in $Q_K(p,q)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, as $i \to \infty$, we have $\|C^n_{\varphi,g}f_i\|_{\mathcal{Z}_\mu} \to 0$ as $i \to \infty$.*

Lemma 2.2. *Let $p > 0$, $q > -2$, $K : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function, φ be an analytic self-map of $\mathbb{D}$ and g be an analytic function on $\mathbb{D}$. If $\|\varphi\| < 1$ and $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_\mu$ is bounded, then $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_\mu$ is compact.*

Proof. From [15, Theorem 1], $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_\mu$ is bounded if and only if

$$M_1 = \sup_{z\in\mathbb{D}} \frac{\mu(|z|)\,|g'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} < \infty \tag{9}$$

and

$$M_2 = \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}} < \infty. \tag{10}$$

Suppose that $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $Q_K(p,q)$ such that converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. This implies that for any $n \in \mathbb{N}$, $\{f_k^{(n)}\}_{k=1}^\infty$ converges uniformly to 0 on compact subsets of $\mathbb{D}$ as $k \to \infty$. We now set $\rho = \|\varphi\|_\infty$, where $\rho \in (0,1)$.

Since $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_\mu$ is bounded then there exists a positive constant C such that for

$$\|C^n_{\varphi,g}f\|_{\mathcal{Z}_\mu} \le C\|f\|_{Q_K(p,q)} \tag{11}$$

for all $f \in Q_K(p,q)$. Since any polynomial belongs to $Q_K(p,q)$, by taking the function $f_1(z) = \frac{z^n}{n!}$ in the above inequality, we obtain

$$R_1 = \sup_{z\in\mathbb{D}} \mu(|z|)|g'(z)| < \infty, \tag{12}$$

Similarly by using $f_2(z) = \frac{z^{n+1}}{(n+1)!}$, we get

$$\sup_{z\in\mathbb{D}} \mu(|z|)|\varphi'(z)g(z) + \varphi(z)g'(z)| < \infty. \tag{13}$$

According to (12), (13) and boundedness of φ, we obtain that

$$R_2 = \sup_{z\in\mathbb{D}} \mu(|z|)|\varphi'(z)||g(z)| < \infty. \tag{14}$$

Using the fact that $\{f_k^{(n)}\}_{k=1}^{\infty}$, $\{f_k^{(n+1)}\}_{k=1}^{\infty}$ converge uniformly to 0 on compact subsets of $\mathbb{D}$ and inequalities (12) and (14), we have

$$\begin{aligned}
\|C_{g,\varphi}^n f_k\|_{\mathcal{Z}_\mu} =& |(C_{g,\varphi}^n(f_k))'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|(C_{g,\varphi}^n(f_k))''(z)| \\
\le& |f_k^{(n)}(\varphi(0))||g(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|\varphi'(z)||f_k^{(n+1)}(\varphi(z))||g(z)| \\
&+ \sup_{z\in\mathbb{D}} \mu(|z|)|f_k^{(n)}(\varphi(z))||g'(z)| \\
\le& |f_k^{(n)}(\varphi(0))||g(0)| + \sup_{z\in\mathbb{D}, |\varphi(z)|\le\rho} \mu(|z|)|\varphi'(z)||f_k^{(n+1)}(\varphi(z))||g(z)| \\
&+ \sup_{z\in\mathbb{D}, |\varphi(z)|\le\rho} \mu(|z|)|f_k^{(n)}(\varphi(z))||g'(z)| \to 0, \ k \to \infty.
\end{aligned} \tag{15}$$

Thus, $C_{g,\varphi}^n$ is compact by Lemma 2.1. □

Theorem 2.3. *Let $p > 0$, $q > -2$, $K : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function, φ be an analytic self-map of $\mathbb{D}$ and g be an analytic function on $\mathbb{D}$. If $C_{\varphi,g}^n : Q_K(p,q) \to \mathcal{Z}_\mu$ is bounded, then*

$$\|C_{\varphi,g}^n\|_{e, Q_K(p,q)\to\mathcal{Z}_\mu} \approx \max \left\{ \begin{array}{l} \limsup_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}, \\ \limsup_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}} \end{array} \right\} \tag{16}$$

Proof. If $\|\varphi\|_\infty < 1$, then by Lemma 2.2 the proof holds, since we can regard that the quantities on the right-hand side in (16) are automatically equal to zero. So, let $\left\|\varphi\right\|_\infty = 1$. Suppose that $w \in \mathbb{D}$. Define the functions F_w as follows

$$F_w(z) = C_F \frac{1-|w|^2}{(1-z\overline{w})^{\frac{q+2}{p}}} - D_F \frac{(1-|w|^2)^2}{(1-z\overline{w})^{\frac{q+2}{p}+1}} \tag{17}$$

where $C_F = \frac{q+2}{p} + n + 1$ and $D_F = \frac{q+2}{p}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}
F_w^{(n)}(z) =& C_F \prod_{j=0}^{n-1} \left(\frac{q+2}{p} + j\right) \overline{w}^n (1-|w|^2)(1-\overline{w}z)^{-(\frac{q+2}{p}+n)} \\
&- D_F \prod_{j=0}^{n-1} \left(\frac{q+2}{p} + j + 1\right) \overline{w}^n (1-|w|^2)^2 (1-z\overline{w})^{-(\frac{q+2}{p}+n+1)}.
\end{aligned}$$

Choose $\{z_k\} \subseteq \mathbb{D}$ such that $\lim_{k\to\infty} \left|\varphi(z_k)\right| = 1$. Define f_k, for all $k \in \mathbb{N}$ as follows:

$$f_k(z) = F_{\varphi(z_k)}(z) = C_F \frac{1-|\varphi(z_k)^2|}{(1-\overline{\varphi(z_k)}z)^{\frac{q+2}{p}}} - D_F \frac{(1-|\varphi(z_k)|^2)^2}{(1-\overline{\varphi(z_k)}z)^{\frac{q+2}{p}+1}}.$$

Then $f_k \in Q_K(p,q)$ and there exists $0 < C < \infty$ such that $\sup_{k\in\mathbb{N}} \left\|f_k\right\|_{Q_K(p,g)} \le C$. Moreover, f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Also

$$f_k^{(n)}(\varphi(z_k)) = r_{n-1} \frac{|\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\frac{2+q-p}{p}+n}} \tag{18}$$

and

$$f_k^{(n+1)}\left(\varphi\left(z_k\right)\right)=0,$$

where $r_{n-1}=\prod_{j=0}^{n-1}\left(\frac{q+2}{p}+j\right)$. For any compact operator $T:\mathcal{Q}_K(p,g)\to\mathcal{Z}_\mu$, by Lemma 2.10 [27], we have $\lim_{k\to\infty}\|Tf_k\|_{\mathcal{Z}_\mu}=0$. Then

$$\begin{aligned}
C\|C_{\varphi,g}^n-T\| &\geq \limsup_{k\to\infty}\|(C_{\varphi,g}^n-T)f_k\|_{\mathcal{Z}_\mu} \\
&\geq \limsup_{k\to\infty}\left(\|C_{\varphi,g}^n f_k\|_{\mathcal{Z}_\mu}-\|Tf_k\|_{\mathcal{Z}_\mu}\right) \\
&= \limsup_{k\to\infty}\|C_{\varphi,g}^n f_k\|_{\mathcal{Z}_\mu} \\
&\geq \limsup_{k\to\infty}\mu(|z_k|)|(C_{\varphi,g}^n(f_k))''\left(z_k\right)| \\
&= \limsup_{k\to\infty}\mu\left(|z_k|\right)\left|\varphi'\left(z_k\right)f_k^{(n+1)}\left(\varphi\left(z_k\right)\right)g\left(z_k\right)+f_k^{(n)}\left(\varphi\left(z_k\right)\right)g'\left(z_k\right)\right| \\
&= \limsup_{k\to\infty}\mu\left(|z_k|\right)\left|f_k^{(n)}\left(\varphi\left(z_k\right)\right)g'\left(z_k\right)\right| \\
&= r_{n-1}\limsup_{k\to\infty}\frac{\mu\left(|z_k|\right)|\varphi\left(z_k\right)|^n\left|g'\left(z_k\right)\right|}{\left(1-\left|\varphi\left(z_k\right)\right|^2\right)^{\frac{2+q-p}{p}+n}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|C_{\varphi,g}^n\|_{e,\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu} &\geq\|C_{\varphi,g}^n-T\| \\
&\geq\frac{r_{n-1}}{C}\limsup_{k\to\infty}\frac{\mu\left(|z_k|\right)\left|g'\left(z_k\right)\right|}{\left(1-\left|\varphi\left(z_k\right)\right|^2\right)^{\frac{2+q-p}{p}+n}} \\
&=\frac{r_{n-1}}{C}\limsup_{|\varphi(z)|\to1}\frac{\mu\left(|z|\right)\left|g'\left(z\right)\right|}{\left(1-\left|\varphi\left(z\right)\right|^2\right)^{\frac{2+q-p}{p}+n}}.
\end{aligned}$$

Now define the function $h_w(z)$, for all $w\in\mathbb{D}$, as follows:

$$h_w(z)=\left(\frac{q+2}{p}+n+1\right)\frac{(1-|w|^2)^2}{(1-z\overline{w})^{\frac{q+2}{p}+1}}-\left(\frac{q+2}{p}+1\right)\frac{(1-|w|^2)^3}{(1-z\overline{w})^{\frac{q+2}{p}+2}} \tag{19}$$

and set $h_k(z)=h_{\varphi(z_k)}(z)$. Then there exists $0<C<\infty$ such that $\sup\|h_k\|_{\mathcal{Q}_K(p,q)}\leq C$ and $\{h_k\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Moreover

$$h_k^{(n)}\left(\varphi\left(z_k\right)\right)=0$$

and

$$h_k^{(n+1)}(\varphi(z_k))=-r_{n+1}\frac{|\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{\frac{q+2}{p}+n}}$$

where $r_{n+1}=\prod_{j=1}^{n+1}\left(\frac{q+2}{p}+j\right)$. Suppose that $S:\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu$ is a compact operator. Thus we have

$$\lim_{k\to\infty}\|Sh_k\|_{\mathcal{Z}_\mu}=0.$$

Then

$$\begin{aligned} C\|C^n_{\varphi,g}-S\| &\geq \limsup_{k\to\infty}\left(\|C^n_{\varphi,g}h_k\|_{Z_\mu}-\|Sh_k\|_{\mathcal{Z}_\mu}\right)\\ &=\limsup_{k\to\infty}\|C^n_{\varphi,g}h_k\|_{\mathcal{Z}_\mu}\\ &\geq \limsup_{k\to\infty}\mu(|z_k|)\left|\left(C^n_{\varphi,g}(h_k)\right)''(z_k)\right|\\ &\geq \limsup_{k\to\infty}\left|\mu(|z_k|)\varphi'(z_k)h_k^{(n+1)}(\varphi(z_k))g(z_k)+\mu(|z_k|)h_k^{(n)}(\varphi(z_k))g'(z_k)\right|\\ &=\limsup_{k\to\infty}\left|\mu(|z_k|)\varphi'(z_k)h_k^{(n+1)}(\varphi(z_k))g(z_k)\right|\\ &=r_{n+1}\limsup_{k\to\infty}\frac{\mu(|z_k|)|\varphi(z_k)|^{n+1}|\varphi'(z_k)||g(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{q+2-p}{p}+n+1}}. \end{aligned} \tag{20}$$

Hence, we have

$$\|C^n_{\varphi,g}\|_{e,\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu}\geq\frac{r_{n+1}}{C}\limsup_{|\varphi(z)|\to1}\frac{\mu(|z|)|\varphi'(z)||g(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2-p}{p}+n+1}}.$$

Thus,

$$\|C^n_{\varphi,g}\|_{e,\mathcal{Q}_K(p,q)}\geq\left\{\begin{array}{c}\limsup_{|\varphi(z)|\to1}\frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}\\ \limsup_{|\varphi(z)|\to1}\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}}.\end{array}\right\}$$

We now obtain the upper bound for $\|C^n_{\varphi,g}\|_{e,\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu}$. The boundedness of $C^n_{\varphi,g}$ implies that there exists $0<C<\infty$ such that

$$\|C^n_{\varphi,g}f\|_{\mathcal{Z}_\mu}\leq C\|f\|_{\mathcal{Q}_K(p,g)}, \tag{21}$$

for all $f\in\mathcal{Q}_K(p,q)$. Let $\{r_j\}\subset(0,1)$ be a sequence such that $r_j\to1$ as $j\to\infty$. For any j, we have

$$\sup_{z\in\mathbb{D}}\frac{\mu(|z|)\left|r_j\varphi'(z)\right|\left|g(z)\right|}{\left(1-\left|r_j\varphi(z)\right|^2\right)^{\frac{2+q-p}{p}+n+1}}\leq\sup_{z\in\mathbb{D}}\frac{\mu(|z|)\left|r_j\varphi'(z)\right|\left|g(z)\right|}{\left(1-\left|r_j\right|^2\right)^{\frac{2+q-p}{p}+n+1}}<\infty \tag{22}$$

and

$$\sup_{z\in\mathbb{D}}\frac{\mu(|z|)|g'(z)|}{\left(1-\left|r_j\varphi(z)\right|^2\right)^{\frac{2+q-p}{p}+n}}\leq\sup_{z\in\mathbb{D}}\frac{\mu(|z|)\left|g'(z)\right|}{\left(1-\left|r_j\right|^2\right)^{\frac{2+q-p}{p}+n}}<\infty. \tag{23}$$

Thus by [12, Theorem 1], $C^n_{r_j\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu$ is bounded. Since $\left\|r_j\varphi\right\|_\infty<1$, Lemma 2.2 implies that $C^n_{r_j\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu$ is compact. Therefore,

$$\|C^n_{\varphi,g}\|_{e,\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu}\leq\limsup_{j\to\infty}\|C^n_{\varphi,g}-C^n_{r_j\varphi,g}\|.$$

For any $f \in Q_K(p,q)$ with $\|f\|_{Q_K(p,q)} \leq 1$ we have

$$
\begin{aligned}
&\|(C^n_{\varphi,g} - C^n_{r_j\varphi,g})f\|_{\mathcal{Z}_\mu} = |((C^n_{\varphi,g} - C^n_{r_j\varphi,g})f)'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|((C^n_{\varphi,g} - C^n_{r_j\varphi,g})f)''(z)| \\
&\leq |((C^n_{\varphi,g} - C^n_{r_j\varphi,g})f)'(0)| \\
&\quad + \sup_{z\in\mathbb{D}} \mu(|z|)\left|\varphi'(z)\right|\left|f^{(n+1)}(\varphi(z)) - r_j f^{(n+1)}\left(r_j\varphi(z)\right)\right|\left|g(z)\right| \\
&\quad + \sup_{z\in\mathbb{D}} \mu(|z|)\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(r_j\varphi(z)\right)\right|\left|g'(z)\right| \\
&\leq \left|f^{(n)}(\varphi(0)) - f^{(n)}\left(r_j\varphi(0)\right)\right|\left|g(0)\right| \\
&\quad + \sup_{|\varphi(z)|\leq r_N} \mu(|z|)\left|\varphi'(z)\right|\left|f^{(n+1)}(\varphi(z)) - r_j f^{(n+1)}\left(r_j\varphi(z)\right)\right|\left|g(z)\right| \\
&\quad + \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left|\varphi'(z)\right|\left|f^{(n+1)}(\varphi(z)) - r_j f^{(n+1)}\left(r_j\varphi(z)\right)\right|\left|g(z)\right| \\
&\quad + \sup_{|\varphi(z)|\leq r_N} \mu(|z|)\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(r_j\varphi(z)\right)\right|\left|g'(z)\right| \\
&\quad + \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(r_j\varphi(z)\right)\right|\left|g'(z)\right|,
\end{aligned} \tag{24}
$$

where $N \in \mathbb{N}$ is large enough such that $r_j \geq \frac{1}{2}$ for all $j \geq N$. We set

$$
\begin{aligned}
F_1 &= \left|f^{(n)}(\varphi(0)) - f^{(n)}\left(r_j\varphi(0)\right)\right|\left|g(0)\right| \\
F_2 &= \sup_{|\varphi(z)|\leq r_N} \mu(|z|)\left|\varphi'(z)\right|\left|f^{(n+1)}(\varphi(z)) - r_j f^{(n+1)}\left(r_j\varphi(z)\right)\right|\left|g(z)\right| \\
F_3 &= \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left|\varphi'(z)\right|\left|f^{(n+1)}(\varphi(z)) - r_j f^{(n+1)}\left(r_j\varphi(z)\right)\right|\left|g(z)\right| \\
F_4 &= \sup_{|\varphi(z)|\leq r_N} \mu(|z|)\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(r_j\varphi(z)\right)\right|\left|g'(z)\right| \\
F_5 &= \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(r_j\varphi(z)\right)\right|\left|g'(z)\right|.
\end{aligned}
$$

As $j \to \infty$, it is clear that

$$F_1 = \left|f^{(n)}(\varphi(0)) - f^{(n)}\left(r_j\varphi(0)\right)\right|\left|g(0)\right| \to 0.$$

For F_2, according to (14) and noting that $r_j f^{(n+1)}_{r_j} \to f^{(n+1)}$, $f_{r_j}(z) = f(r_j z)$, uniformly on compact subsets of $\mathbb{D}$, we get

$$
\begin{aligned}
\limsup_{j\to\infty} F_2 &= \limsup_{j\to\infty} \sup_{|\varphi(z)|\leq r_N} \mu(|z|)\left|\varphi'(z)\right|\left|f^{(n+1)}(\varphi(z)) - r_j f^{(n+1)}\left(r_j\varphi(z)\right)\right|\left|g(z)\right| \\
&\leq R_2 \limsup_{j\to\infty} \sup_{|\varphi(z)|\leq r_N} \left|f^{(n+1)}(\varphi(z)) - r_j f^{(n+1)}\left(r_j\varphi(z)\right)\right| = 0.
\end{aligned}
$$

Similarly for F_4 we have

$$
\begin{aligned}
\limsup_{j\to\infty} F_4 &= \limsup_{j\to\infty} \sup_{|\varphi(z)|\leq r_N} \mu(|z|)\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(r_j\varphi(z)\right)\right|\left|g'(z)\right| \\
&\leq R_1 \limsup_{j\to\infty} \sup_{|\varphi(z)|\leq r_N} \left|f^{(n)}(\varphi(z)) - f^{(n)}\left(r_j\varphi(z)\right)\right| = 0.
\end{aligned}
$$

Using (6) and (7) for F_3 we obtain

$$\begin{aligned}
F_3 &= \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left|\varphi'(z)\right|\left|f^{(n+1)}(\varphi(z)) - r_j f^{(n+1)}\left(r_j\varphi(z)\right)\right|\left|g(z)\right| \\
&\leq \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left|\varphi'(z)\right|\left(\left|f^{(n+1)}(\varphi(z))\right| + \left|r_j f^{(n+1)}\left(r_j\varphi(z)\right)\right|\right)\left|g(z)\right| \\
&\leq \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left|\varphi'(z)\right|\left|f^{(n+1)}(\varphi(z))\right|\left|g(z)\right| \\
&\quad + \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left|\varphi'(z)\right|\left|r_j f^{(n+1)}\left(r_j\varphi(z)\right)\right|\left|g(z)\right| \\
&\leq C_2 \sup_{r_N<|\varphi(z)|<1} \frac{\mu(|z|)\left|\varphi'(z)\right|\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n+1}}\|f\|_{Q_K(p,q)} \\
&\quad + C_2 \sup_{r_N<|\varphi(z)|<1} \frac{\mu(|z|)\,|\varphi'(z)\,|\left|g(z)\right|}{\left(1-\left|r_j\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n+1}}\|f\|_{Q_K(p,q)} \\
&\leq \sup_{r_N<|\varphi(z)|<1} \frac{C_2\mu(|z|)\left|\varphi'(z)\right|\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n+1}} + \sup_{r_N<|\varphi(z)|<1} \frac{C_2\mu(|z|)\left|\varphi'(z)\right|\left|g(z)\right|}{\left(1-\left|r_j\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n+1}}.
\end{aligned}$$

Whenever $j \to \infty$, we have

$$\limsup_{j\to\infty} F_3 \leq 2C_2 \limsup_{|\varphi(z)|\to 1} \frac{\mu(|z|)\left|\varphi'(z)\right|\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n+1}}.$$

Moreover,

$$\begin{aligned}
F_5 &= \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left|f^{(n)}(\varphi(z)) - f^{(n)}\left(r_j\varphi(z)\right)\right|\left|g'(z)\right| \\
&\leq \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left(\left|f^{(n)}(\varphi(z))\right| + \left|f^{(n)}\left(r_j\varphi(z)\right)\right|\right)\left|g'(z)\right| \\
&\leq \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left|f^{(n)}(\varphi(z))\right|\left|g'(z)\right| \\
&\quad + \sup_{r_N<|\varphi(z)|<1} \mu(|z|)\left|f^{(n)}\left(r_j\varphi(z)\right)\right|\left|g'(z)\right| \\
&\leq C_3 \sup_{r_N<|\varphi(z)|<1} \frac{\mu(|z|)\left|g'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n}}\|f\|_{Q_K(p,q)} \\
&\quad + C_3 \sup_{r_N<|\varphi(z)|<1} \frac{\mu(|z|)\left|g'(z)\right|}{\left(1-\left|r_j\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n}}\|f\|_{Q_K(p,q)} \\
&\leq \sup_{r_N<|\varphi(z)|<1} \frac{C_3\mu(|z|)\left|g'(z)\right|}{\left(1-|\varphi(z)\,|^2\right)^{\frac{q+2-p}{p}+n}} + \sup_{r_N<|\varphi(z)|<1} \frac{C_3\mu(|z|)\left|g'(z)\right|}{\left(1-\left|r_j\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n}}
\end{aligned}$$

Whenever $j \to \infty$, we get

$$\limsup_{j\to\infty} F_5 \leq \limsup_{|\varphi(z)|\to 1} \frac{2C_3\mu(|z|)\left|g'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n}}. \tag{25}$$

Thus, by the obtained inequalities for F_3 and F_5, we have

$$\begin{aligned}
&\|C^n_{\varphi,g}\|_{e,Q_K(p,q)\to\mathcal{Z}_\mu} \leq \limsup_{j\to\infty} \|C^n_{\varphi,g} - C^n_{r_j\varphi,g}\| \\
&\leq \limsup_{|\varphi(z)|\to 1} \frac{2C_2\mu(|z|)\left|\varphi'(z)\right|\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n+1}} + \limsup_{|\varphi(z)|\to 1} \frac{2C_3\mu(|z|)\left|g'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n}} \\
&\leq \max\left\{\limsup_{|\varphi(z)|\to 1} \frac{2C_2\mu\,|(z)|\left|\varphi'(z)\right|\left|g(z)\right|}{(1-|\varphi(z)|^2)^{\frac{q+2-p}{p}+n+1}}, \limsup_{|\varphi(z)|\to 1} \frac{2C_3\mu(|z|)\left|g'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{\frac{q+2-p}{p}+n}}\right\}.
\end{aligned}$$

Therefore, we have found an upper bound for $\left\|C^n_{\varphi,g}\right\|_{e,Q_K(p,q)\to\mathcal{Z}_\mu}$. This completes the proof. □

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