Filomat 37:17 (2023), 5813–5828
https://doi.org/10.2298/FIL2317813H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Ritz numerical method and hybrid functions (block-pulse functions and Legendre polynomials) for a class of two-dimensional time-delay optimal control problems

S. M. Hosseini[a], F. Soltanian[a], K. Mamehrashi[a,b]

[a] *Department of Mathematics, Payame Noor University, Tehran, Iran*
[b] *Mathematics Unit, School of Science and Engineering, University of Kurdistan-Hewler, Erbil, Kurdistan Region, Iraq*

Abstract. In this paper, we provided a numerical method to solve a class of two dimensional time-delay optimal control problems (2DTDOCPs) with quadratic cost functional using Ritz method and orthogonal Legendre Block-Pulse functions. First, the state and control vectors are approximated as a series of hybrid functions(block-pulse functions and Legendre polynomials) with unknown coefficients. Then, we derive an equation with unknown coefficients by substituting these approximations in the cost functional. A system of algebraic equations is obtained by applying the optimal conditions for this equation. Solving this system and substituting the coefficients into approximating the guessed functions, the state and control functions are obtained. By increasing the number of blocks, as well as the basic functions, we get more accurate solutions. The convergence of proposed method is discussed, and finally, we will present some examples to show the validity and applicability of proposed method, and evaluate its accuracy and efficiency. Moreover, our results are compared to previous results to show the superiority of this technique.

1. Introduction

Delays often occur in chemical, industrial, biological, electronic, and transportation systems as well as many other areas, such as population growth models, economic growth, and neural networks [1, 2]. Analysis, identification, and optimal control of delay systems were considered by many researchers which are very important from a practical point of view. In applied mathematics, solving delayed differential equations is important. The process of determining the response of delay systems is often very complex and in a wide range of these systems, determining the analytical response is difficult or impossible. For this reason, delay systems are among the most important categories of control systems and providing a suitable and efficient numerical method to solve them is of considerable importance.

In nature, many quantities are the functions of two independent variables, and two-dimensional systems and signals are used to model phenomena with two independent variables. Moreover, the importance and necessity of examining two-dimensional state delay optimal control systems is very important. Continuous-temporal dynamic equations describe many control systems. Considering lack of analytical solutions to

2020 *Mathematics Subject Classification*. Primary 49M05 ; Secondary 65N35, 65M70.
Keywords. Two dimensional time-delay optimal control problem; Ritz method; Block pulse-Legendre; Numerical method.
Received: 27 December 2021; Accepted: 25 January 2023
Communicated by Marko Petković
Email addresses: mohsen.hssn419@gmail.com (S. M. Hosseini), f_soltanian@pnu.ac.ir (F. Soltanian), k_mamehrashi@pnu.ac.ir (K. Mamehrashi)

two-dimensional state delay, optimal control problems of continuous-temporal systems, numerical and semi-analytical methods are suggested to solve a number of such problems. Also, by using Pontryagin's maximum principle, the time-delay optimal control problem reduces to a system of coupled two-point boundary value problem involving both delay and advance terms whose exact solution can not be easily calculated except in very special cases [8]. Numerical computational methods which are currently used in science and engineering are very diverse, and a specific solution can be provided for each specific problem or special condition. Since it is very difficult and in some cases impossible to obtain the solution to the analysis of these types of problems, semi-analytical and numerical methods are used for solving them. In the following, some studies in delay systems are presented. In [3], Banks proposed the optimal control of nonlinear delay systems from a theoretical point of view and proposed an approximate method to solve them. In [4], Banks, Burns and Cliff, analyzed and identified delay systems. The authors of [5] proposed a method to solve multiple delay systems using integral and delay matrices corresponding to Taylor polynomials. Also, Lee presented another method to solve the optimal control of delay functions with non-equal constraints on state variables [6]. Another work, optimal control of a nonlinear time-delay system in batch fermentation process has been studied in [27]. In [28], Dadebo and Luus, investigated the use of iterative dynamic programming employing systematic region contraction and accessible grid points for the optimal control of time-delay systems. Also in [29], a symplectic local pseudospectral method for solving nonlinear state-delayed optimal control problems with inequality constraints is proposed by Wang et al. Peng et al. proposed an iterative Symplectic Pseudospectral Method to solve Nonlinear State-delayed Optimal Control Problems in [30].

In recent years, orthogonal functions have received special attention from the researchers for analyzing the optimal control systems [7]. As an example, a method for analyzing and solving time-delay optimal control problems using orthogonal hybrid functions was proposed in [8] by Marzban and Razzaghi. Hence, Kern, Maurer and Gllmann solved a series of one-dimensional time-delay optimal control problems [9]. Moreover, in [10], the authors studied and solved a range of nonlinear time-delay optimal control problems using fixed piecewise functions.

Two-dimensional systems along with their spatial models were first introduced as part of image processing by Roesser. [11]. Mamehrashi and Yousefi used a method to solve a class of two-dimensional control problems using the Ritz-Galerkin method [12]. In another work, Tsai, Li, and Shieh [13] transformed the Roesser type continuous-temporal state optimal control problem with a quadratic cost functional into a discrete two-dimensional control problem. Furthermore, Nemati and Yousefi used Ritz method to solve a class of two-dimensional control problems [14], and the problems of two-dimensional optimal control of deficit were studied by Rabiei, Ordokhani and Babolian [15].

In recent years, orthogonal Block-Pulse functions were used to solve many state optimal control problems, examples of which can be seen in the articles of researchers and scholars [16–19, 31, 32]. Rakhshan and Effati presented a class of deficit time-delay optimal control problems using an Euler-Lagrange approach [20]. In [21], Nouri, Nazari and Torkzadeh, transformed the problem of time-delay fractional differential equations into a system of nonlinear algebraic equations using hybrid functions. Moreover, Rafiei, Kaffash and Karbassi, solved a number of one-dimensional time-delay control problems using hybrid functions [22].

We present an alternative numerical method to solve a class of two-dimensional time-delay optimal control problems. To the best of our knowledge, it is the first numerical method for 2DTDOCPs. The proposed method is a direct method based on approximating state and control variables via hybrid Block-Pulse functions and Legendre polynomials using the Ritz method [12, 14]. Considering the flexibility of Ritz method in the face of initial and boundary conditions, we used this approach in the proposed method. To calculate the dual integral in a standard function, we have used Gaussian method. By substituting the approximated functions in the constraints of the problem and using the suggested method, the optimal control problem is reduced to an unconstrained optimization problem which can be easily solved.

This article is organized as follows: Section 2 introduces the general problem of two-dimensional time-delay optimal control. Section 3 introduces the Ritz method and the hybrid Legendre Block-Pulse functions. In sections 4 and 5, we describe the proposed numerical method and discuss the convergence of the method, respectively. In section 6, the efficiency and accuracy of the proposed method are examined by providing two examples. Furthermore, a conclusion is given in section 7.

2. Problem statement

Consider the following controllable and observable two-dimensional time delay system [23],

$$z'(x,t) = Az(x,t) + A_d z(x-\tau_1, t-\tau_2) + Bu(x,t) \tag{1}$$

where,

$$z'(x,t) = \begin{bmatrix} \frac{\partial z^h(x,t)}{\partial x} \\ \frac{\partial z^v(x,t)}{\partial t} \end{bmatrix}, z(x,t) = \begin{bmatrix} z^h(x,t) \\ z^v(x,t) \end{bmatrix}, z(x-\tau_1, t-\tau_2) = \begin{bmatrix} z^h(x-\tau_1, t) \\ z^v(x, t-\tau_2) \end{bmatrix}$$

$z^h(x,t)$ and $z^v(x,t)$ are the horizontal and vertical components of the space, A, B and A_d are constant matrices with appropriate dimensions, respectively. Also τ_1, τ_2 are constant delays of horizontal and vertical components. The boundary conditions are as follows,

$$\begin{aligned} z^h(a,t) &= \begin{cases} f_a(t) & -\tau_1 \le a \le 0, 0 \le t \le t_f \\ 0 & -\tau_1 \le a \le 0, t \ge t_f. \end{cases} \\ z^v(x,b) &= \begin{cases} g_b(x) & -\tau_2 \le b \le 0, 0 \le x \le x_f \\ 0 & -\tau_2 \le b \le 0, x \ge x_f. \end{cases} \end{aligned} \tag{2}$$

which, x_f and t_f are fixed positive values, $f_a(t)$ and $g_b(x)$ are given vectors. The purpose of this paper is to determine the control vector $u(x,t)$ and the corresponding state vector $z(x,t)$ such that the following cost functional is minimized according to constraints (1) and (2),

$$J = \frac{1}{2}\xi^T(x_f, t_f) S \xi(x_f, t_f) + \frac{1}{2}\int_0^{t_f}\int_0^{x_f} [\xi^T(x,t) Q \xi(x,t) + u^T(x,t) R u(x,t)] dx dt, \tag{3}$$

where S and Q are semi-definite matrices and R is the positive definite matrix.

3. Function approximation

3.1. Ritz method

The Ritz method is a simple and efficient way to approximate the solution of an optimization problem. In this method, the solution of the functional minimization problem,

$$min \quad L[y(x)] = \int_a^b f(x, y, y') dx$$

with boundary conditions,

$$y(a) = a_0 \, , \; y(b) = b_0$$

is considered as follows,

$$y_n(x) \approx \sum_{i=1}^{n} c_i \varphi_i(x) + \varphi_0(x). \tag{4}$$

We must select the basic functions $\varphi_i(x)$ that satisfy the following conditions,

$$\varphi_0(a) = a_0, \quad \varphi_0(b) = b_0$$
$$\varphi_i(a) = \varphi_i(b) = 0, \quad i = 1, 2, ..., n \tag{5}$$

By substituting $y_n(x)$ into the problem and solving it, the unknown coefficients and the solution of $y_n(x)$ are obtained.
Suppose that $\varphi_i(x) = k(x)p_i(x)$, so Eq. (4) can be written as

$$y_n(x) \approx \sum_{i=1}^{n} k(x)c_ip_i(x) + \varphi_0(x) \tag{6}$$

where, $k(x)$ satisfies the homogeneous conditions and $p_i(x)$ are Legendre polynomials.
If we use the Ritz method to approximate the function $z(x, t)$, Eq. (6) is written as

$$z_{mn}(x, t) = \sum_{i=0}^{m} \sum_{j=0}^{n} k(x, t)c_{ij}p_i(x)p_j(t) + w(x, t), \tag{7}$$

where $k(x, t)$ and $w(x, t)$ satisfy the homogeneous and boundary conditions, respectively [24].

3.2. Two-dimensional Block-Pulse functions

A set of two dimensional Block-Pulse functions (2DBPFs) $\Phi_{i_1,i_2}(x, t)$ for $x \in [0, T_1], t \in [0, T_2]$ is defined as follows

$$\Phi_{i_1,i_2}(x, t) = \begin{cases} 1 & x \in \left[\frac{(i_1 - 1)T_1}{m_1}, \frac{i_1T_1}{m_1}\right), t \in \left[\frac{(i_2 - 1)T_2}{m_2}, \frac{i_2T_2}{m_2}\right) \\ 0 & \text{otherwise} \end{cases}$$

2DBPFs are disjoined with each other,

$$\Phi_{i_1,i_2}(x, t)\Phi_{j_1,j_2}(x, t) = \begin{cases} \Phi_{i_1,i_2}(x, t) & i_1 = j_1 \text{ and } i_2 = j_2 \\ 0 & \text{otherwise} \end{cases}$$

and are orthogonal,

$$\int_0^{T_1} \int_0^{T_2} \Phi_{i_1,i_2}(x, t)\Phi_{j_1,j_2}(x, t)dxdt = \begin{cases} h_1h_2 & i_1 = j_1 \text{ and } i_2 = j_2 \\ 0 & \text{otherwise} \end{cases}$$

in the region $x \in [0, T_1)$ and $t \in [0, T_2)$, where

$$i_1, j_1 = 1, 2, ..., m_1, i_2, j_2 = 1, 2, ..., m_2, \quad h_1 = \frac{T_1}{m_1}, h_2 = \frac{T_2}{m_2}.$$

Also 2DBPFs are complete when both m_1 and m_2 approach infinity [26].
Since each 2DBPF takes only one value in its subregion, the 2DBPFs can be expressed as

$$\Phi_{i_1,i_2}(x, t) = \varphi_{i_1}(x)\Psi_{i_2}(t),$$

where $\varphi_{i_1}(x)$ and $\Psi_{i_2}(t)$ are one-dimensional Block-Pulse functions related to the variables x and t, respectively.

3.3. Hybrid functions of Block-Pulse and Legendre polynomials

The two-dimensional hybrid functions (block-pulse functions and Legendre polynomials) is defined as follows

$$\Psi_{i_1 j_1 i_2 j_2}(x,t) = \begin{cases} L_{j_1}(\frac{2N_1 x}{x_f} - 2i_1 + 1) L_{j_2}(\frac{2N_2 t}{t_f} - 2i_2 + 1) : (x,t) \in [\frac{i_1-1}{N_1}x_f, \frac{i_1}{N_1}x_f] \times [\frac{i_2-1}{N_2}t_f, \frac{i_2}{N_2}t_f] \\ 0 \qquad\qquad \text{otherwise} \end{cases} \tag{8}$$

where, $i_1 = 1, 2, ..., N_1$, $i_2 = 1, 2, ..., N_2$ and $j_1, j_2 = 0, 1, ...M - 1$ are the order of Block-Pulse functions and Legendre polynomials, respectively and $L_{j_1}(x), L_{j_2}(t)$ are the well known Legendre polynomials [8].

4. Proposed state-control parameterization method

In this section, we present a numerical method based on approximating the state and control variables to solve a class of 2DTDOCPs.
Consider the following problem:

$$\text{Min } J = \frac{1}{2} z^T(x_f, t_f) S z(x_f, t_f) + \frac{1}{2}\int_0^{t_f}\int_0^{x_f} [z^T(x,t) Q z(x,t) + u^T(x,t) R u(x,t)] dxdt \tag{9}$$

subject to the time-delay system dynamics,

$$z'(x,t) = Az(x,t) + A_d z(x - \tau_1, t - \tau_2) + Bu(x,t) \tag{10}$$

with the following boundary conditions

$$z^h(a,t) = \begin{cases} f_a(t) & -\tau_1 \le a \le 0, 0 \le t \le t_f \\ 0 & -\tau_1 \le a \le 0, t \ge t_f. \end{cases} \tag{11}$$

$$z^v(x,b) = \begin{cases} g_b(x) & -\tau_2 \le b \le 0, 0 \le x \le x_f \\ 0 & -\tau_2 \le b \le 0, x \ge x_f. \end{cases} \tag{12}$$

Suppose that $Q \subset PC^2([0, x_f] \times [0, t_f])$ is a set of all continuous piecewise functions that satisfy the boundary condition (11)-(12). The cost functional J is a function of $z(x,t)$ and $u(x,t)$, so problem (9)-(12) can be considered as a problem of minimizing the value of J on Q. Suppose $Q_{N_1 N_2 (M-1)(M-1)} \subset Q$ is a set of the Legendre Block-Pulse hybird functions consisting of $N_1 N_2$ polynomials and the degree of each polynomial is at most $(M - 1)(M - 1)$. The state variable is approximated using a finite number of the hybrid functions (block-pulse functions and Legendre polynomials) as

$$z(x,t) = \sum_{i_1=1}^{N_1}\sum_{j_1=0}^{M-1}\sum_{i_2=1}^{N_2}\sum_{j_2=0}^{M-1} f_{i_1,j_1,i_2,j_2} \Psi_{i_1,j_1,i_2 j_2}(x,t) \tag{13}$$

Let

$$\alpha = [f_{1,0,1,0}, f_{1,0,1,1}, ..., f_{1,0,1,M-1}, ..., f_{N_1,M-1,N_2,M-1}] \tag{14}$$

Now by selecting N_1,N_2 as

$$N_1 = \begin{cases} \dfrac{x_f}{\tau_1} & \dfrac{x_f}{\tau_1} \text{ is integer} \\ \left[\dfrac{x_f}{\tau_1}\right] + 1 & \text{otherwise} \end{cases} \tag{15}$$

$$N_2 = \begin{cases} \dfrac{t_f}{\tau_2} & \dfrac{t_f}{\tau_2} \text{ is integer} \\ \left[\dfrac{t_f}{\tau_2}\right] + 1 & \text{otherwise} \end{cases} \tag{16}$$

the interval $[0, x_f] \times [0, t_f]$ is converted to the following $N_1 \times N_2$ sub intervals

$$[0, \frac{1}{N_1}x_f] \times [0, \frac{1}{N_2}t_f], [0, \frac{1}{N_1}x_f] \times [\frac{1}{N_2}t_f, \frac{2}{N_2}t_f] \cdots [\frac{N_1-1}{N_1}x_f, x_f] \times [\frac{N_2-1}{N_2}t_f, t_f]$$

As a result, the state variable (13) is written as

$$\hat{z}(x,t) = \begin{cases} \hat{z}_{11}(x,t) = \sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} f_{1,j_1,1,j_2}\psi_{1,j_1,1,j_2} & : (x,t) \in [0, \frac{1}{N_1}x_f] \times [0, \frac{1}{N_2}t_f] \\ \vdots & \\ \hat{z}_{N_1N_2}(x,t) = \sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} f_{N_1,j_1,N_2,j_2}\psi_{N_1,j_1,N_2,j_2} & : (x,t) \in [\frac{N_1-1}{N_1}x_f, x_f] \times [\frac{N_2-1}{N_2}t_f, t_f] \end{cases} \tag{17}$$

To approximate $z(x-\tau_1, t-\tau_2)$ without interfering the whole discussion, we explain the steps for $z^v(x, t-\tau_2)$. Eq.(12) yields

$$z^v(x,t) = g(x,t) \quad 0 \le x \le x_f, -\tau_2 \le t \le 0.$$

If $s = t + \frac{1}{N_2}t_f$, then $t = s - \frac{1}{N_2}t_f$ and considering the definition of N_2, we can write $t = s - \tau_2$, therefore, Eq. (12) becomes as

$$z^v_{11}(x, s-\tau_2) = g(x, s-\tau_2) \quad 0 \le s \le \tau \tag{18}$$

Also, $z^v(x, s-\tau_2)$ for

$$(x,t) \in [0, \frac{1}{N_1}x_f] \times [\frac{1}{N_2}t_f, \frac{2}{N_2}t_f]$$

can be defined as

$$\hat{z_{11}}(x,t) = \sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} f_{1,j_1,1,j_2}\psi_{1,j_1,1,j_2}(x,t).$$

Considering $s = t + \frac{1}{N_2}t_f$, we can write

$$z^v_{12}(x, s-\tau_2) = \sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} f_{1,j_1,1,j_2}\psi_{1,j_1,1,j_2}(x, s-\tau_2) \quad , (x,s) \in [0, \frac{1}{N_1}x_f] \times [\frac{1}{N_2}t_f, \frac{2}{N_2}t_f].$$

Now, according to the definition of $\Psi_{i_1j_1i_2j_2}(x,t)$, we have

$$\psi_{1,j_1,1,j_2}(x, s-\tau_2) = \psi_{1,j_1,1,j_2}(x,s)$$

therefore,

$$z^v_{12}(x,s-\tau_2)=\sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} f_{1,j_1,1,j_2}\psi_{1,j_1,2,j_2}(x,s) \quad : (x,s)\in[0,\frac{1}{N_1}x_f]\times[\frac{1}{N_2}t_f,\frac{2}{N_2}t_f].$$

Similarly, $z^v(x,s-\tau_2)$ can be specified for other intervals and as a result, it is written as

$$z^v(x,s-\tau_2)=\begin{cases} g(x,t) & (x,t)\in[0,\frac{1}{N_1}x_f]\times[0,\frac{1}{N_2}t_f]\\ \sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} f_{1,j_1,1,j_2}\psi_{1,j_1,1,j_2}(x,t) & (x,t)\in[0,\frac{1}{N_1}x_f]\times[\frac{1}{N_2}t_f,\frac{2}{N_2}t_f]\\ \vdots \end{cases} \tag{19}$$

The same procedure is applied to approximate the other component of the state function $z^h(x-\tau_1,t)$. Now, by substituting the values of $z(x,t)$ and $z(x-\tau_1,t-\tau_2)$ in (10), the control variable $u(x,t)$ is approximated. Then, substituting the approximations of $z(x,t)$ and $u(x,t)$ into the cost functional (9) an unconstrained optimization problem is obtained as

$$\text{Min } J(\alpha),\quad \alpha=[f_{1,0,1,0},f_{1,0,1,1},\dots f_{N_1,M-1,N_2,M-1}].$$

To minimize this, we solve the following system of equations using Newton's iterative method

$$\frac{\partial J[\alpha]}{\partial f_{i_1j_1i_2j_2}}=0,\quad i_1=1,2,...,N_1,\quad i_2=1,2,...,N_2,\quad j_1,j_2=1,2,...,M-1. \tag{20}$$

By solving the above algebraic system for α, the unknown coefficients $f_{i_1j_1i_2j_2}$ are achieved. Then, by determining $f_{i_1j_1i_2j_2}$, we can find the approximate value of $z^h(x,t), z^v(x,t)$ and $u(x,t)$ from (13) and (10), respectively.

5. Convergence analysis

In this section, we recall two theorems and present a lemma which ensure the convergence analysis of suggested method. Here, the approximation convergence of a function is derived with respect to the Legendre Block-Pulse bases.

Consider the restriction of the cost functional J to $Q_{N_1N_2(M-1)(M-1)}\subset Q$

$$J[\hat{z}(x,t)]=J[\sum_{i_1=1}^{N_1}\sum_{j_1=0}^{M-1}\sum_{i_2=1}^{N_2}\sum_{j_2=0}^{M-1} f_{i_1,j_1,i_2,j_2}\Psi_{i_1,j_1,i_2j_2}(x,t)] \tag{21}$$

as a function of $N_1N_2(M-1)(M-1)$ variables. The coefficients $f_{i_1j_1i_2j_2}$ are chosen in such a way as to minimize (21). Let $\beta_{N_1N_2(M-1)(M-1)}$ indicate the minimum value of J restricted to $Q_{N_1N_2(M-1)(M-1)}$.

The following theorem is a remarkable result of the Weierstrass famous theorem for two-dimensional space.

Theorem 5.1. *For any $\hat{z}(x,t)\in Q\subset PC^2([0,x_f]\times[0,t_f])$, there exists a sequence of polynomials $\{\Psi_{i_1j_1i_2j_2}(x,t)\}_{i_1,j_1,i_2,j_2=0}^{\infty}\in Q$ that converges uniformly to $\hat{z}(x,t)$.*

Proof. See [33]. □

The convergence of the proposed method is provided by the following Lemma. We use the facts that have been mentioned in Theorem (5.1).

Lemma 5.2. *If*

$$\beta_{N_1N_2(M-1)(M-1)} = \inf_{Q_{N_1N_2(M-1)(M-1)}} J, \qquad for \quad N_1, N_2, M = 1, 2, 3, ..,$$

where $Q_{N_1N_2(M-1)(M-1)}$ is a subset of Q including the Legendre Block-Pulse hybird functions involving N_1N_2 polynomials of degree at most $(M-1)(M-1)$, then

$$\lim_{N_1,N_2,M\to\infty} \beta_{N_1N_2(M-1)(M-1)} = \inf_{Q} J.$$

.

Proof. Let

$$\beta_{N_1N_2(M-1)(M-1)} = \min_{\alpha_{N_1N_2(M-1)(M-1)}} J(\alpha_{N_1N_2(M-1)(M-1)}),$$

then,

$$\beta_{N_1N_2(M-1)(M-1)} = J(\alpha^*{}_{N_1N_2(M-1)(M-1)}),$$

where

$$\alpha^*{}_{N_1N_2(M-1)(M-1)} \in \text{Argmin}\{J(\alpha_{N_1N_2(M-1)(M-1)}) : \alpha_{N_1N_2(M-1)(M-1)} \in R^{2N_1N_2M}\}.$$

Now, let

$$(z^*{}_{N_1N_2(M-1)(M-1)}(x,t), u^*{}_{N_1N_2(M-1)}(x,t)) \in$$
$$\text{Argmin}\{J(z(x,t), u(x,t)) : (z(x,t), u(x,t)) \in Q_{N_1N_2(M-1)(M-1)}\},$$

then

$$J(z^*{}_{N_1N_2(M-1)(M-1)}(x,t), u^*{}_{N_1N_2(M-1)(M-1)}(x,t)) = \min_{(z(x,t),u(x,t))\in Q_{N_1N_2(M-1)(M-1)}} J(z(x,t), u(x,t)),$$

where $Q_{N_1N_2(M-1)(M-1)}$ is a class of combinations of the continuous hybrid functions(block-pulse functions and Legendre polynomials) involving N_1N_2 polynomials of degree at most $(M-1)(M-1)$, so

$$\beta_{N_1N_2(M-1)(M-1)} = J(z^*{}_{N_1N_2(M-1)(M-1)}(x,t), u^*{}_{N_1N_2(M-1)(M-1)}(x,t)).$$

Furthermore, according to $Q_{N_1N_2(M-1)(M-1)} \subset Q_{N_1N_2MM}$, we have

$$\min_{(z(x,t),u(x,t))\in Q_{N_1N_2MM}} J(z(x,t), u(x,t)) \le \min_{(z(x,t),u(x,t))\in Q_{N_1N_2(M-1)(M-1)}} J(z(x,t), u(x,t)).$$

Thus, $\beta_{N_1N_2MM} \le \beta_{N_1N_2(M-1)(M-1)}$ is achieved which means $\beta_{N_1N_2(M-1)(M-1)}$ is a non-increasing sequence. Also, this sequence is lower bounded, so its infimum is the limit. Due to the continuity J and by taking the limit when $N_1, N_2, M \to \infty$, we can write,

$$\lim_{N_1,N_2,M\to\infty} \beta_{N_1N_2(M-1)(M-1)} = \min_{(z(x,t),u(x,t))\in Q} J(z(x,t), u(x,t)).$$

which completes the proof. □

6. Numerical results

In this section, three optimal control examples are considered to illustrate the theoretical result. The cost functional for different values of τ is computed. If τ =0, the 2DTDOCP reduces to two-dimensional quadratic optimal control problem was studied by other researchers. The numerical results were computed using Maple 2018 programming.

Example 6.1. *Consider the following 2DTDOCP*

$$Min\ J=\int_0^3\int_0^3\Big\{\big[z^h(x,t)\big]^2+[z^v(x,t)]^2+[u(x,t)]^2\Big\}dxdt, \tag{22}$$

s.t

$$\begin{aligned}
&\tfrac{\partial z^h(x,t)}{\partial x}=-3z^h(x,t)+3.2z^v(x,t)+0.3u(x,t)\\
&\tfrac{\partial z^v(x,t)}{\partial t}=z^h(x,t)-z^v(x,t-\tau)\\
&z^v(x,0)=e^{-3x}\cos(2\pi x)\\
&z^h(0,t)=-e^{-2t}\\
&z^v(x,\theta)=1,-\tau\le\theta\le 0.
\end{aligned} \tag{23}$$

According to the suggested method in section 4, based on the Ritz method and using two-dimensional Block-Pulse functions, the state variable $z^v(x,t)$ is approximated as

$$z^v(x,t)=\sum_{i_1=1}^{N_1}\sum_{j_1=0}^{M-1}\sum_{i_2=1}^{N_2}\sum_{j_2=0}^{M-1}txf_{i_1,j_1,i_2,j_2}\Psi_{i_1,j_1,i_2,j_2}(x,t)+e^{-3x-2t}\cos(2\pi x),$$

where N_1,N_2 are the numbers of blocks and M is the number of Legendre polynomials. For simplicity, we assume that $N_1=N_2$. In each subinterval

$$(x,t)\in[\frac{i_1-1}{N},\frac{i_1}{N}]\times[\frac{i_2-1}{N},\frac{i_21}{N}],$$

$z^v(x,t)$ can be written for $i_1,i_2=1,2,...,N$ and $j_1,j_2=0,1,...,M-1$ as

$$z^v\,[i_1,i_2]=\sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1}L_{j_1}(\frac{2Nx}{x_f}-2i_1+1)L_{j2}(\frac{2Nt}{t_f}-2i_2+1)f_{i_1j_1i_2j_2} \tag{24}$$

The approximation of $z^v(x,t-\tau)$ is also obtained using (19).
Now from (28)*, we have*

$$\begin{aligned}
z^h(x,t)&=\frac{\partial z^v(x,t)}{\partial t}+z^v(x,t-\tau)\\
u(x,t)&=\frac{10}{3}[\frac{\partial z^h(x,t)}{\partial x}+3z^h(x,t)-3.2z^v(x,t)]
\end{aligned} \tag{25}$$

According to (25)*, $z^h[i_1,i_2]$ and $u[i_1,i_2]$ can be easily obtained in any corresponding subinterval. By substituting the approximated values $z^v(x,t),z^h(x,t)$ and $u(x,t)$ in* (27)*, the 2DTDOPCP reduces to an optimization problem which can be easily solved using existing optimization methods. By solving obtained optimization problem, the unknown coefficients f_{i_1,j_1,i_2,j_2} are determined. Using f_{i_1,j_1,i_2,j_2}, we can determine the approximate value of $z^h(x,t),z^v(x,t)$ and $u(x,t)$. The obtained vertical, horizontal and control variables from proposed method for this example are shown in Fig. 1,2 and 3, respectively. The cost functional values for $\tau=1,\tau=\frac{1}{2},\tau=\frac{1}{3}$ and different values of M are shown in table 1. As seen from the results reported in this table, by increasing M we can get better solutions for the cost functional J. If $\tau=0$, the example 6.1 reduces to two-dimensional optimal control problem without time-delay that has been studied by other researchers. The comparison of the estimated values of J by different methods is shown in Table 2. By comparing the results for similar values of M and N, it is clear that the results of the proposed method are better.*

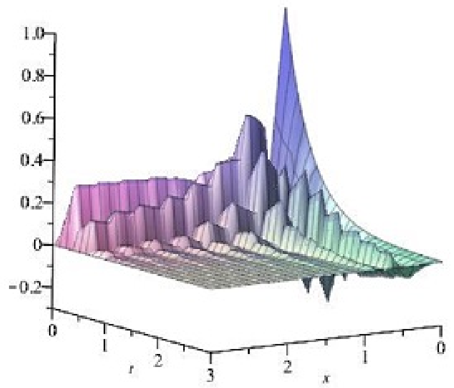

Figure 1: Approximate solution of state vertical component $z^v(x,t)$ with $\tau = 1$ for Example 6.1

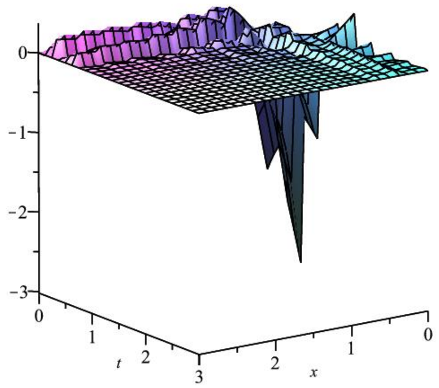

Figure 2: Approximate solution of state horizontal component $z^h(x,t)$ with $\tau = 1$ for Example 6.1

Table 1: Estimated values of J for various values of N, M and τ

	$N = 3, \tau = 1$	$N = 6, \tau = \frac{1}{2}$	$N = 9, \tau = \frac{1}{3}$
$M = 2$	9.9091	4.7208	2.1969
$M = 3$	6.6079	1.7672	1.2483
$M = 7$	2.5895	1.1256	0.7543

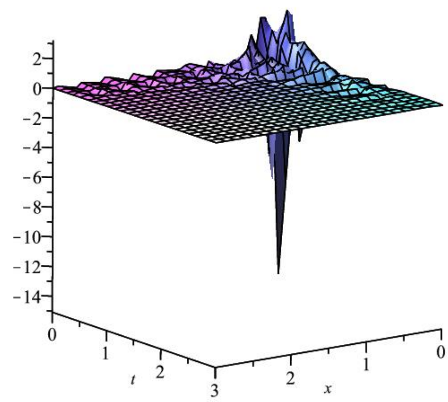

Figure 3: Approximate solution of control variable $u(x, t)$ with $\tau = 1$ for Example 6.1

Table 2: Comparison of estimated value of J in case $\tau = o$ for different methods

Methods	J
Method of [13]	
$X = 0.1, T = 0.1$	0.7348
$X = 0.05, T = 0.05$	0.5510
$X = 0.03, T = 0.03$	0.4760
Method of [12]	
$M = 7, N = 8$	0.6202
$M = 8, N = 3$	0.2792
$M = 8, N = 8$	0.2026
Method of [25]	
$M = 7, N = 6$	0.3094
$M = 9, N = 8$	0.0951
$M = 10, N = 8$	0.0608
Present Method	
$M = 7, N = 8$	0.0075
$M = 8, N = 3$	0.0659
$M = 8, N = 8$	0.0072
$M = 7, N = 6$	0.0693
$M = 9, N = 8$	0.0069
$M = 10, N = 8$	0.0051

Example 6.2. *Consider the following problem*

$$Min\ J = \frac{1}{2}\int_0^5\int_0^5 10^7[z^v(x,t) - sin(x+t)]^2 + [u(x,t)]^2 dxdt,$$

$$\frac{\partial z^h(x,t)}{\partial x} = -3z^h(x,t) + 3.2z^v(x,t) + 0.3u(x,t)$$

$$\frac{\partial z^v(x,t)}{\partial t} = z^h(x,t) - z^v(x,t-\tau) \tag{26}$$

$$z^v(x,0) = e^{-3x}\cos(2\pi x)$$

$$z^h(0,t) = -e^{-2t}$$

$$z^v(x,\theta) = 1 : -\tau \le \theta \le 0.$$

Following the mentioned methodology to solve 2DTDOCPs in section 4, we approximate the function $z^v(x,t)$ in terms of Legendre Block-Pulse basis. We obtain $z^h(x,t)$ and $u(x,t)$ by applying the method similar to Example 6.1.
The achieved vertical, horizontal and control variables by solving the mentioned example are shown in Fig. 4, 5 and 6, respectively. The cost functional values for $\tau = 3, \tau = 2, \tau = 1$ and different values of M are shown in Table 3. If $\tau = 0$, similar to example 6.1, comparing the estimated values of J by different methods is shown in Table 4. As seen from the reported results in Table 4, our solution is better compared with the methods presented in [12] and [25]. By comparing the results for similar values of M and N, it is clear that the results of the proposed method are better.

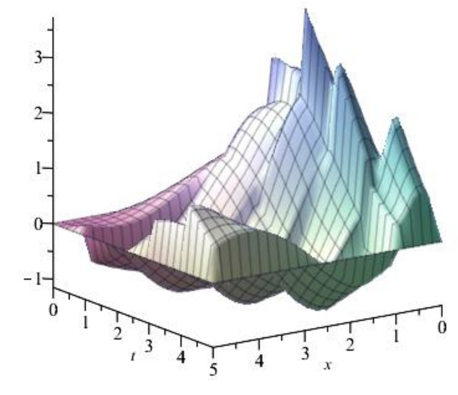

Figure 4: Approximate solution of state vertical component $z^v(x,t)$ with $\tau = 1$ for Example 6.2

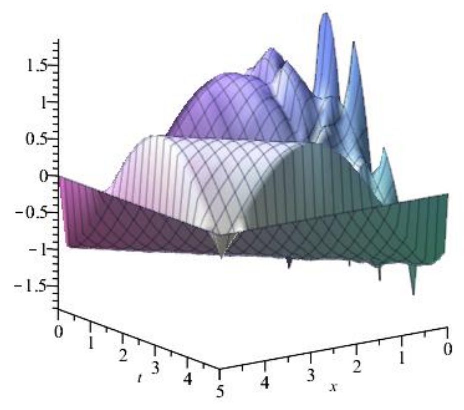

Figure 5: Approximate solution of state horizontal component $z^h(x,t)$ with $\tau = 1$ for Example 6.2

Example 6.3. *Consider the following 2DTDOCP*

$$Min\ J = \int_0^3 \int_0^3 \left\{ \left[z^h(x,t) \right]^2 + [z^v(x,t)]^2 + [u(x,t)]^2 \right\} dxdt, \tag{27}$$

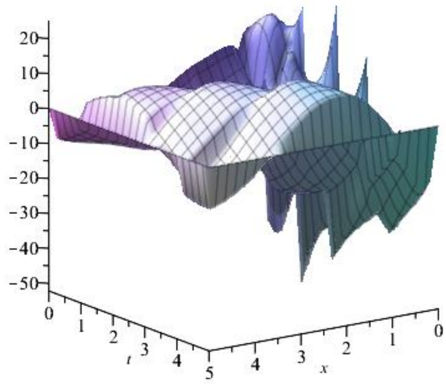

Figure 6: Approximate solution of control variable $u(x, t)$ with $\tau = 1$ for Example 6.2

Table 3: Estimated values of J for various values of N, M andτ

	$N = 2, \tau = 3$	$N = 3, \tau = 2$	$N = 5, \tau = 1$
$M = 3$	11.86268×10^6	8.18034×10^6	4.42061×10^6
$M = 5$	7.54674×10^6	5.42092×10^6	2.23495×10^6
$M = 7$	5.16402×10^6	3.70614×10^6	0.75272×10^6

Table 4: Comparison of estimated value of J in case $\tau = o$ for different methods

Methods	J
Method of [12]	
$M = 4, N = 7$	2.56765×10^6
$M = 6, N = 6$	2.03080×10^6
$M = 7, N = 5$	2.15360×10^6
Method of [25]	
$M = 3, N = 4$	4.090680×10^6
$M = 5, N = 6$	2.398277×10^6
$M = 8, N = 9$	1.199138×10^6
Present Method	
$M = 4, N = 7$	0.000586×10^6
$M = 6, N = 6$	0.410512×10^6
$M = 7, N = 5$	0.610010×10^6
$M = 3, N = 4$	0.909921×10^6
$M = 5, N = 6$	0.423217×10^6
$M = 8, N = 9$	0.000062×10^6

$s.t$

$$\begin{aligned}
&\frac{\partial z^h(x,t)}{\partial x} = -3z^h(x-\tau,t) + 3.2z^v(x,t) + 0.3u(x,t) \\
&\frac{\partial z^v(x,t)}{\partial t} = z^h(x,t) - z^v(x,t-\tau) \\
&z^v(x,0) = e^{-3x}\cos(2\pi x) \\
&z^h(0,t) = -e^{-2t} \\
&z^v(x,\theta) = 1, -\tau \le \theta \le 0. \\
&z^h(\beta,t) = 1, -\tau \le \beta \le 0.
\end{aligned} \tag{28}$$

Following the mentioned methodology for solving the 2DTDOCPs in section 4, we approximate the function $z^v(x,t)$ in terms of the Legendre Block-Pulse basis. We obtain $z^h(x,t)$ and $u(x,t)$ by applying the method similar to Example 6.1.

The achieved vertical, horizontal and control variables by solving the mentioned example are shown in Fig. 7, 8 and 9, respectively. The cost functional values for $\tau = \frac{1}{2}, \tau = \frac{1}{3}, \tau = \frac{1}{4}$ and different values of M are shown in Table 5. If $\tau = 0$, the problem such as example 6.1 reduces to the two-dimensional optimal control problem without time-delay that has been studied by other researchers. The comparison of the estimated values of J by different methods is shown in Table 2.

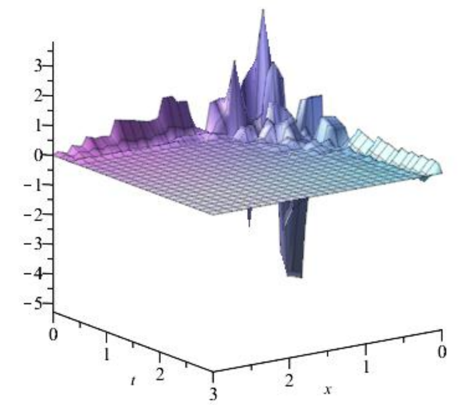

Figure 7: Approximate solution of state vertical component $z^v(x,t)$ with $\tau = \frac{1}{3}$ for Example 6.3

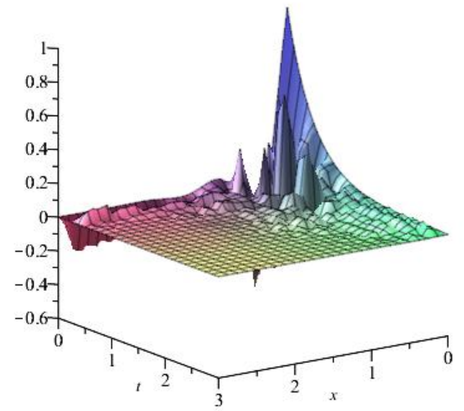

Figure 8: Approximate solution of state horizontal component $z^h(x,t)$ with $\tau = \frac{1}{3}$ for Example 6.3

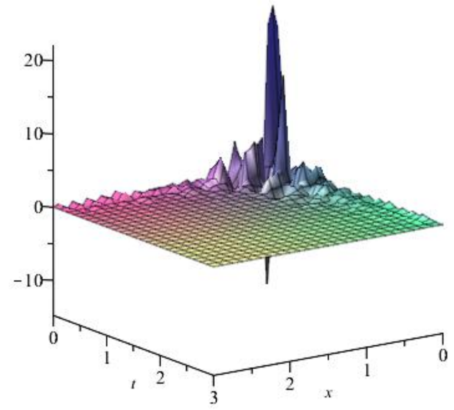

Figure 9: Approximate solution of control variable $u(x, t)$ with $\tau = \frac{1}{3}$ for Example 6.3

Table 5: Estimated values of J for various values of N, M andτ

	$N = 6, \tau = \frac{1}{2}$	$N = 9, \tau = \frac{1}{3}$	$N = 12, \tau = \frac{1}{4}$
$M = 2$	26.0203	22.9754	13.6724
$M = 3$	11.2012	10.1130	4.1256
$M = 5$	5.4121	4.1283	1.7811

7. Conclusion

This paper presents an efficient method to solve a class of 2DTDOCPs. The state and control variables are approximated using Ritz method and the hybrid functions (block-pulse functions and Legendre polynomials). Using the proposed method, without using derivative and multiplicative functional matrices, the optimal control problem reduces to an optimization problem which can be easily solved. We described the proposed numerical method and discuss the convergence of the method. The obtained results showed that our method gives a good approximate of the solution and these results are more accurate than other results that have been obtained from existing methods while $\tau = 0$. Moreover, we obtained satisfactory results only in a small number of polynomial orders compared to other published methods in literatures. To solve 2DTDOCPs, we introduced a building block which can be extended. The problems with inequality constraints and Multiple Time-Varying Delays are under consideration as our future research work.

References

[1] H. Chehardoli, Robust optimal control and identification of adaptive cruise control systems in the presence of time delay and parameter uncertainties, J. Vib. Control 26(2020) 1590–1601.
[2] F. A. Rihan, S. Lakshmanan, H. Maurer, Optimal control of tumour-immune model with time-delay and immuno-chemotherapy, Appl. Math. Comput 353(2019) 147–165.
[3] H. T. Banks, Approximation of nonlinear functional differential equation control systems, J. Optimiz. Theory App 29(1979) 383–408.
[4] H. T. Banks, J. A. Burns, E. M. Cliff, Parameter estimation and identification for systems with delays, SIAM J. Control Optim 19(1981) 791–828.
[5] M. Razzaghi, M. Razzaghi, Taylor series analysis of time-varying multi-delay systems, Int. J. Control 50(1989) 183–192.
[6] L. Y. Lee, Numerical solution of time-delayed optimal control problems with terminal inequality constraints, Optim. Control Appl. Methods 14(1993) 203–210.
[7] M. R. Ali, A. R. Hadhoud, Hybrid Orthonormal Bernstein and Block-Pulse functions wavelet scheme for solving the 2D Bratu problem, Results Phys 12(2019) 525–530.

[8] H. R. Marzban, M. Razzaghi, Optimal control of linear delay systems via hybrid of block-pulse and Legendre polynomials, J. Franklin Inst 341(2004) 279–293.
[9] L. Gllmann, D. Kern, H. Maurer, Optimal control problems with delays in state and control variables subject to mixed control state constraints, Optim. Control Appl. Methods 30(2009) 341–365.
[10] C. Yu, Q. Lin, R. Loxton, K. L. Teo, G. Wang, A hybrid time-scaling transformation for time-delay optimal control problems, J. Optim. Theory Appl 169(2016) 876–901.
[11] R. Roesser, A discrete state-space model for linear image processing, IEEE Trans. Automat. Contr 20(1975) 1–10.
[12] K. Mamehrashi, S. A. Yousefi, Numerical solution of a class of two dimensional quadratic optimal control problems by using Ritz method, Optim. Control Appl. Methods 37(2016) 765–781.
[13] J. H. Tsai, J. S. Li, L. S. Shieh, Discretized quadratic optimal control for continuous-time two-dimensional systems, IEEE Trans. Circuits Syst 49(2002) 116–125.
[14] A. Nemati, S. A. Yousefi, A numerical scheme for solving two-dimensional fractional optimal control problems by the Ritz method combined with fractional operational matrix, IMA J. Math. Control Inf 34(2017) 1079–1097.
[15] K. Rabiei, Y. Ordokhani, E. Babolian, Numerical Solution of 1D and 2D fractional optimal control of system via Bernoulli polynomials, Int. J. Appl. Comput. Math 4(2018) 1–17.
[16] M. Razzaghi, Y. Ordokhani, N. Haddadi, Direct method for variational problems by using hybrid of block-pulse and Bernoulli polynomials, Romanian Journal Of Mathematics and Computer Science 2(2012) 1–17.
[17] S. Hosseinpour, A. Nazemi, A collocation method via block-pulse functions for solving delay fractional optimal control problems, IMA Journal of Mathematical Control and Information 34(2017) 1215–123.
[18] S. Mashayekhi, M. Razzaghi, An approximate method for solving fractional optimal control problems by hybrid functions, Journal of Vibration and Control 24(2018) 1621–1631.
[19] A. Rivaz, F, Yousefi, H. Salehinejad, Using block pulse functions for solving two-dimensional fuzzy Fredholm integral equations of the second kind, Int. J. Appl. Math 25(2012) 571–582.
[20] S. A. Rakhshan, S. Effati, Fractional optimal control problems with time-varying delay: A new delay fractional Euler-Lagrange equations, J. Franklin Inst 357(2020) 5954–5988.
[21] K. Nouri, M. Nazari, L. Torkzadeh, Numerical approximation of the system of fractional differential equations with delay and its applications, Eur. Phys. J. Plus 135(2020) 1–14.
[22] Z. Rafiei, B. Kafash, S. Karbassi, State-control parameterization method based on using hybrid functions of block-pulse and Legendre polynomials for optimal control of linear time delay systems, Appl. Math. Model 45(2017) 1008–1019.
[23] Z. Duan, I. Ghous, B. Wang, J. Shen, Necessary and sufficient stability criterion and stabilization for positive 2D continuous-time systems with multiple delays, Asian J. Control 21(2019) 1355–1366.
[24] K. Mamehrashi, F. Soltanian, A numerical technique for solving a class of 2D variational problems using Legendre spectral method, Computational Methods for Differential Equations 6(2018) 471–482.
[25] A. Nemati, Numerical solution of 2D fractional optimal control problems by the spectral method along with Bernstein operational matrix, Int. J. Control 91(2018) 2632–2645.
[26] Z. H. Jiang, W. Schaufelberger, Block pulse functions and their applications in control systems, Lecture notes in control and information sciences 179(1992) 1–12.
[27] Y. Yu, Optimal control of a nonlinear time-delay system in batch optimal process, Math probl Eng 5(2014) 1–7.
[28] S. Dadebo, R. Luus, Optimal control of time-delay systems by dynamic programming, Optimal Control App Methods 13(1992) 29–41.
[29] X. Wang, H. Peng, S. Zhang, B. Chen, W. Zhong, A symplectic local pseudospectral method for solving nonlinear state-delayed optimal control problems with inequality constraints, Int J Robust Nonlinear Control 28(2018) 2097–2120.
[30] H. Peng, X. Wang, S. Zhang, B. Chen, An iterative Symplectic Pseudospectral Method to solve Nonlinear State-delayed Optimal Control Problems, Communications in Nonlinear Science and Numerical Simulation 48(2017) 95–114.
[31] P. Rahimkhani, Y. Ordokhani, Numerical solution a class of 2D fractional optimal control problems by using 2D Muntz-Legendre wavelets, Optim Control Appl Meth 39(2018) 1916–1934.
[32] S. Sabermahani, Y. Ordokhani, S. Yousefi, Fractional-order Lagrange polynomials: An application for solving delay fractional optimal control problems, Transactions of the Institute of Measurement and Control (2019) 1–13.
[33] T. J. Rivlin, An introduction to the approximation of the functions, John Wiley and Sons, Inc 1978.
[34] L. Elsgolts, Differential Equations and the Calculus of Variations, Translated from the Russian by G. Yankovsky, Mir Publisher: Moscow, 1977.
[35] L. P. Lebedev, M. J. Cloud, The Calculus of Variations and Functional Analysis with Optimal Control and Applications in Mechanic, World Scientific Publishing Co. Pte. Ltd: 5 Toh Tuck Link, Singapore, 2003.