



## Some properties of degenerate Hermite Appell polynomials in three variables

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**Abstract.** This study is about degenerate Hermite Appell polynomials in three variables or  $\Delta_h$ -Hermite Appell polynomials which include both discrete and degenerate cases. After we recall the definition of these polynomials and special cases, we investigate some properties of them such as recurrence relation, lowering operators (LO), raising operators (RO), difference equation (DE), integro-difference equation (IDE) and partial difference equation (PDE). We also obtain the explicit expression in terms of the Stirling numbers of the first kind. Moreover, we introduce  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials,  $3D$ - $\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials,  $3D$ - $\Delta_h$ -Hermite degenerate Apostol-Euler polynomials and  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Boole polynomials as special cases of  $\Delta_h$ -Hermite Appell polynomials. Furthermore, we derive the explicit representation, determinantal form, recurrence relation, LO, RO and DE for these special cases. Finally, we introduce new approximating operators based on  $h$ -Hermite polynomials in three variables and examine the weighted Korovkin theorem. The error of approximation is also calculated in terms of the modulus of continuity and Peetre's  $K$ -functional.

### 1. Introduction

Appell polynomials have attracted intensive interest in recent years because of their diverse application areas. Among the families of polynomials, Bernoulli, Euler, Genocchi and Hermite polynomials [1] are the best known and they have many applications in numerical analysis, asymptotic approximation and special function theory, and thanks to these applications, they have a wide range of uses in engineering and applied sciences. It is because of these application areas that many extensions of Appell polynomials such as  $\Delta_h$ -Appell polynomials in [2], twice iterated Appell polynomials [3], Hermite-based Appell polynomials in [4], Laguerre-based Appell polynomials [5, 6], truncated exponential-based Appell polynomials in [7], twice iterated  $\Delta_h$ -Appell polynomials [8] and  $\Delta_h$ -Gould-Hopper Appell polynomials ( $\Delta_h$ -GHAP) [9] have been the subject of intensive research especially in the last decade. In the literature, extensions of several structures are very important, if this extension is also a unification of the existing structures, since this

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unification focuses researchers on the investigation of advanced properties rather than focusing on the investigation of modified families having similar properties to the existing area.

Hermite-based special numbers and polynomials are based on the work done by Milne-Thomson, in 1933 [10]. Taking into account Milne-Thomson polynomials, Dere et al. have done research in from 2011 to 2013 [11–13]. Later, in 2019, Milne-Thomson type polynomials and numbers were defined by Simsek, ending the generalization of almost all Hermite-based numbers [14]. Moreover, in 2020, Kilar and Simsek built the special numbers and polynomials based on multivariate Hermite polynomials [15]. For some of the work and done in this field, see papers [16–21].

It is the main purpose of this paper to examine  $3D-\Delta_h$ -Hermite Appell polynomials, which unify (therefore include) existing Appell polynomials such as usual Appell polynomials,  $\Delta_h$ -Appell polynomials, bivariate Appell polynomials, Hermite-based Appell polynomials and  $\Delta_h$ -GHAP, etc. ([2, 4, 9, 22–30]). Degenerate Hermite Appell polynomials in three variables first introduced and studied in Baran’s Ph.D thesis in September 2021 [31] and later, by Riyaset et al. in November 2022 [32]. The degenerate Hermite Appell polynomials in three variables  $\mathcal{A}_j(x, y, z; h) := \mathcal{A}_j^h$  introduced in [31, 32] via the generating relation

$$\vartheta(t) (1 + ht^3)^{\frac{z}{h}} (1 + ht^2)^{\frac{y}{h}} (1 + ht)^{\frac{x}{h}} = \sum_{j=0}^{\infty} \mathcal{A}_j(x, y, z; h) \frac{t^j}{j!}, \tag{1}$$

where  $\mathcal{A}_j(0, 0, 0; h) = \lambda_{j,h}$  ( $j = 0, 1, 2, \dots$ ) are the degenerate numbers given by the series

$$\vartheta(t) = \sum_{i=0}^{\infty} \lambda_{i,h} \frac{t^i}{i!}, \quad \lambda_{0,h} \neq 0. \tag{2}$$

Throughout the paper, we assume that the conditions  $|ht| < 1, |ht^2| < 1, |ht^3| < 1$  are satisfied and if there is a need for additional condition on the variable  $t$  which might come from  $\vartheta(t)$  will be mentioned in addition.  $\Delta_h$  is the finite difference operator (see for instance [33]) given by

$$\Delta_h[k](x) = k(x + h) - k(x), \quad h \in \mathbb{R}^+.$$

We have from [33] that, the rate of  $m$  ( $m \in \mathbb{N}$ ) the operator finite difference can be derived from the above definition to give

$$\Delta^m[f](x) = \Delta(\Delta^{m-1}[f](x)) = \sum_{r=0}^m (-1)^{m-r} \binom{m}{r} f(x + rh) \tag{3}$$

and  $\Delta^0 : I, \Delta^1 : \Delta, I$  is the identity operator.  $3D-\Delta_h$ -Hermite Appell polynomials provide the following difference operator properties [31]:

$${}_x\Delta_h(\mathcal{A}_j^h) = jh\mathcal{A}_{j-1}^h, \tag{4}$$

$${}_y\Delta_h(\mathcal{A}_j^h) = j(j-1)h\mathcal{A}_{j-2}^h \tag{5}$$

and

$${}_z\Delta_h(\mathcal{A}_j^h) = j(j-1)(j-2)h\mathcal{A}_{j-3}^h. \tag{6}$$

It can be seen from equalities (4)-(6) that  $3D-\Delta_h$ -Hermite Appell polynomials satisfy the following difference equations [31]:

$$(h {}_y\Delta_h - {}_x\Delta_h^2) \mathcal{A}_j^h = 0 \tag{7}$$

and

$$(h^2 {}_z\Delta_h - {}_x\Delta_h^3) \mathcal{A}_j^h = 0. \tag{8}$$

It should be noted that  $3D-\Delta_h$ -Hermite Appell type polynomials include the following polynomial families:

- Taking  $z = 0$  in (1), we have the  $\Delta_h$ -GHAP [9]

$$\vartheta(t) (1 + ht^2)^{\frac{y}{h}} (1 + ht)^{\frac{x}{h}} = \sum_{j=0}^{\infty} A_j(x, y; h) \frac{t^j}{j!}, \tag{9}$$

the numbers  $A_j(0, 0; h) := \vartheta_{j,h}$  ( $j = 0, 1, 2, \dots$ ) which are degenerate numbers in (2).

Özarslan and Yaşar [9] obtained explicit representation, determinantal form, recurrence relation, shift operators, DE and PDE for  $\Delta_h$ -GHAP.

- Taking the limit  $h \rightarrow 0$  in (1), we get the Hermite-based Appell polynomials [4, 22]

$$\vartheta(t) \exp(xt + yt^2 + zt^3) = \sum_{j=0}^{\infty} {}_A\mathcal{H}_j(x, y, z) \frac{t^j}{j!}.$$

In [22], for Hermite-based Appell polynomials, they found DE, IDE and PDE by using the factorization method. Also they derived the Hermite-based Bernoulli polynomials, the Hermite-based Euler polynomials and the Hermite-based Genocchi polynomials.

In the case  $\vartheta(t) = 1$ , we recover the 3D-Hermite polynomials [34],

$$\exp(xt + yt^2 + zt^3) = \sum_{j=0}^{\infty} \mathcal{H}_j(x, y, z) \frac{t^j}{j!}.$$

- Taking  $y = z = 0$ , we get  $\Delta_h$ -Appell polynomials [23]. In [2], the power series expansion of  $\vartheta(t) (1 + ht)^{\frac{x}{h}}$  is used for defining  $\Delta_h$ -Appell polynomial were defined

$$\sum_{j=0}^{\infty} \mathcal{A}_j(x; h) \frac{t^j}{j!} = \vartheta(t) (1 + ht)^{\frac{x}{h}} = \mathcal{A}_0(x; h) + \frac{t}{1!} \mathcal{A}_1(x; h) + \dots + \frac{t^j}{j!} \mathcal{A}_j(x; h) + \dots, \tag{10}$$

where  $\vartheta(t)$  is the power series of  $t$  given as

$$\vartheta(t) = \vartheta_{0,h} + \frac{t}{1!} \vartheta_{1,h} + \frac{t^2}{2!} \vartheta_{2,h} + \dots + \frac{t^j}{j!} \vartheta_{j,h} + \dots, \vartheta_{0,h} \neq 0.$$

In [2], Costabile and Longo defined  $\Delta_h$ -Appell sequences by

$$\Delta_h(\mathcal{A}_j(x)) = jh\mathcal{A}_{j-1}(x).$$

- Taking  $h \rightarrow 1$  in (1), we get  $\Delta$ -Appell polynomials in three variables as

$$\vartheta(t) (1 + t^3)^z (1 + t^2)^y (1 + t)^x = \sum_{j=0}^{\infty} A_j(x, y, z) \frac{t^j}{j!},$$

where in the case  $\vartheta(t) = \exp(-at)$  and then taking  $y = z = 0$ , we obtain Charlier polynomials  $C_j^{(a)}(x)$  in [37]

$$\exp(-at) (1 + t)^x = \sum_{n=0}^{\infty} C_j^{(a)}(x) \frac{t^j}{j!}, \quad a \neq 1.$$

- Setting  $z = 0$  and then letting  $h \rightarrow 0$ , we obtain bivariate Appell polynomials [24]

$$\vartheta(t) e^{xt+yt^2} = \sum_{j=0}^{\infty} R_j^{(2)}(x, y) \frac{t^j}{j!}$$

- Taking  $y = z = 0$  and then letting  $h \rightarrow 0$ , we get the usual Appell polynomials which were investigated extensively from different aspects. Appell polynomials [35, 36] are defined via the generating relation as

$$\vartheta(t) e^{xt} = \sum_{j=0}^{\infty} A_j(x) \frac{t^j}{j!}, \quad (11)$$

where  $\vartheta(t)$  is a formal power series as follows

$$\vartheta(t) = \sum_{i=0}^{\infty} \lambda_i \frac{t^i}{i!}, \quad \lambda_0 \neq 0. \quad (12)$$

An equivalent definition of the Appell polynomials is the following

$$D_x(A_j(x)) = jA_{j-1}(x), \quad j = 1, 2, 3, \dots,$$

where  $D_x := \frac{d}{dx}$  is the derivative operator.

- Taking  $\vartheta(t) = 1$ , we have  $h$ -Hermite polynomials in three variables in [31] defined by

$$(1 + ht)^{\frac{x}{h}} (1 + ht^2)^{\frac{y}{h}} (1 + ht^3)^{\frac{z}{h}} = \sum_{j=0}^{\infty} G_j^h(x, y, z) \frac{t^j}{j!}. \quad (13)$$

Note that these polynomials include three variable Hermite polynomials as  $h \rightarrow 0$ ,  $\Delta_h$ -GHAP for  $z = 0$ , bivariate Hermite polynomials for  $z = 0$  and  $h \rightarrow 0$ .

By making special choices of  $\vartheta(t)$ , we construct some new families of polynomials in three variables. The special cases that we consider in the present investigation are  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials,  $3D$ - $\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials,  $3D$ - $\Delta_h$ -Hermite degenerate Apostol-Euler polynomials and  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Boole polynomials

We organize the paper as follows:

In section 2, after we recall  $3D$ - $h$ -Hermite polynomials and their explicit representation, we investigate some properties of  $3D$ - $\Delta_h$ -Hermite Appell polynomials recurrence relation, LO, RO, DE, IDE and PDE. In section 3, we obtain an explicit representation in terms of the Stirling numbers of the first kind and some addition formulas. In section 4, we consider some special examples of  $3D$ - $\Delta_h$ -Hermite Appell polynomials. More precisely, we introduce  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials,  $3D$ - $\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials,  $3D$ - $\Delta_h$ -Hermite degenerate Apostol-Euler polynomials and  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Boole polynomials and obtain their explicit representation, determinantal form, recurrence relation, lowering and raising operators, difference equations. In section 5, we construct new operators based on  $h$ -Hermite polynomials in three variables and examine the weighted Korovkin theorem for them. We also obtain the error of approximation with the help of Peetre's  $K$ -functional and modulus of continuity.

## 2. $3D$ - $h$ -Hermite Appell polynomials and their properties

The properties given in this section were studied in a part of Baran's doctoral thesis [31]. For the completion of the paper, we recall the results with their proofs. We should also mention that Theorem 2.6 is new result.

We consider the  $3D$ - $h$ -Hermite polynomials  $G_n^h(x, y, z) := G_n^h$  defined in (13) since they are the usual members of the  $3D$ - $h$ -Hermite Appell polynomials (in the case  $\vartheta(t) = 1$ ), satisfy the following difference operator properties:

$${}_x\Delta_h(G_j^h) = hjG_{j-1}^h, \quad (14)$$

$${}_y\Delta_h(G_j^h) = hj(j-1)G_{j-2}^h \quad (15)$$

and

$${}_z\Delta_h(G_j^h) = hj(j-1)(j-2)G_{j-3}^h. \tag{16}$$

Therefore 3D-h-Hermite polynomials satisfy the following difference equations

$$(h_y\Delta_h - x\Delta_h^2)G_j^h = 0 \tag{17}$$

and

$$(h^2{}_z\Delta_h - x\Delta_h^3)G_j^h = 0. \tag{18}$$

The polynomial  $G_j^h$  have the following explicit representation

$$G_j^h = \sum_{l=0}^{\lfloor \frac{j}{3} \rfloor} \sum_{s=0}^{\lfloor \frac{j-3l}{2} \rfloor} \binom{j-3l}{2s} \binom{j}{3l} (x)_{j-2s-3l}^h (y)_s^h (z)_l^h \frac{(2s)!}{s!} \frac{(3l)!}{l!} \tag{19}$$

where

$$(\eta)_j^h = \left(-\frac{\eta}{h}\right)_j (-h)^j,$$

with

$$(\eta)_j = \eta(\eta+1)(\eta+2)\cdots(\eta+j-1) = \frac{\Gamma(\eta+j)}{\Gamma(\eta)}, \quad j \geq 1, \quad (\eta)_0 := 1.$$

The polynomial sequence  $\{\mathcal{A}_j(x, y, z; h)\}_{j \in \mathbb{N}}$  has the following explicit representation

$$\mathcal{A}_j^h = \sum_{s=0}^j \binom{j}{s} \sum_{l=0}^{\lfloor \frac{s}{3} \rfloor} \sum_{m=0}^{\lfloor \frac{s-3l}{2} \rfloor} \lambda_{j-s,h} \binom{s-3l}{2m} \binom{s}{3l} (x)_{s-2m-3l}^h (y)_m^h (z)_l^h \frac{(2m)!}{m!} \frac{(3l)!}{l!} \tag{20}$$

where

$$\vartheta(t) = \sum_{j=0}^{\infty} \lambda_{j,h} \frac{t^j}{j!}.$$

For  $j = 0, 1, 2, \dots$ , the determinantal form of the 3D- $\Delta_h$ -Hermite Appell polynomials is given by

$$\mathcal{A}_j^h = \frac{(-1)^j}{(\gamma_{0,h})^{j+1}} \begin{vmatrix} G_0^h & G_1^h & G_2^h & \cdots & G_{j-1}^h & G_j^h \\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{j-1,h} & \gamma_{j,h} \\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{j-1}{1}\gamma_{j-2,h} & \binom{j}{1}\gamma_{j-1,h} \\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{j-1}{2}\gamma_{j-3,h} & \binom{j}{2}\gamma_{j-2,h} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{j}{j-1}\gamma_{1,h} \end{vmatrix} \tag{21}$$

where the coefficients of the Maclaurin series of  $\frac{1}{\vartheta(t)}$  are the numbers  $\gamma_{s,h}$  ( $s = 0, 1, 2, \dots$ ) and

$$\frac{1}{\vartheta(t)} = \sum_{s=0}^{\infty} \gamma_{s,h} \frac{t^s}{s!}, \quad (22)$$

$$G_j^h = \sum_{s=0}^j \binom{j}{s} \gamma_{s,h} \mathcal{A}_{j-s}^h. \quad (23)$$

In the following theorem, we obtain the recurrence equation of the 3D- $\Delta_h$ -Hermite Appell polynomials.

**Theorem 2.1.** 3D- $h$ -Hermite Appell polynomials have the following recurrence relation

$$\begin{aligned} \mathcal{A}_{j+1}^h &= (x + \beta_{0,h}) \mathcal{A}_j^h + \sum_{s=0}^{j-1} \binom{j}{s} \beta_{j-s,h} \mathcal{A}_s^h + xj! \sum_{s=1}^j \frac{(-h)^s}{(j-s)!} \mathcal{A}_{j-s}^h \\ &\quad + 2jy \mathcal{A}_{j-1}^h + 2yj! \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-h)^s}{(j-2s-1)!} \mathcal{A}_{j-2s-1}^h + 3zj(j-1) \mathcal{A}_{j-2}^h \\ &\quad + 3zj! \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-h)^s}{(j-3s-2)!} \mathcal{A}_{j-3s-2}^h, \quad j \geq 2 \end{aligned} \quad (24)$$

with

$$\mathcal{A}_{-s}^h = 0, \quad (s = 1, 2, \dots),$$

where

$$\frac{\vartheta'(t)}{\vartheta(t)} = \sum_{s=0}^{\infty} \beta_{s,h} \frac{t^s}{s!}. \quad (25)$$

*Proof.* Differentiating (1) wrt  $t$ , we get

$$\begin{aligned} \sum_{j=0}^{\infty} \mathcal{A}_{j+1}^h \frac{t^j}{j!} &= \frac{\vartheta'(t)}{\vartheta(t)} \vartheta(t) (1+ht)^{\frac{x}{h}} (1+ht^2)^{\frac{y}{h}} (1+ht^3)^{\frac{z}{h}} + \frac{1}{1+ht} x \vartheta(t) (1+ht)^{\frac{x}{h}} (1+ht^2)^{\frac{y}{h}} (1+ht^3)^{\frac{z}{h}} \\ &\quad + 2y \frac{t}{1+ht^2} \vartheta(t) (1+ht)^{\frac{x}{h}} (1+ht^2)^{\frac{y}{h}} (1+ht^3)^{\frac{z}{h}} + 3z \frac{t^2}{1+ht^3} \vartheta(t) (1+ht)^{\frac{x}{h}} (1+ht^2)^{\frac{y}{h}} (1+ht^3)^{\frac{z}{h}}. \end{aligned} \quad (26)$$

Considering the expansions

$$\frac{1}{1+ht} = \sum_{s=0}^{\infty} (-ht)^s, \quad \frac{1}{1+ht^2} = \sum_{s=0}^{\infty} (-1)^s h^s t^{2s}, \quad \frac{1}{1+ht^3} = \sum_{s=0}^{\infty} (-1)^s h^s t^{3s},$$

in (26), we obtain

$$\begin{aligned} \sum_{j=0}^{\infty} \mathcal{A}_{j+1}^h \frac{t^j}{j!} &= \sum_{s=0}^{\infty} \beta_{s,h} \frac{t^s}{s!} \sum_{j=0}^{\infty} \mathcal{A}_j^h \frac{t^j}{j!} + x \sum_{s=0}^{\infty} (-ht)^s \sum_{j=0}^{\infty} \mathcal{A}_j^h \frac{t^j}{j!} \\ &\quad + 2y \sum_{s=0}^{\infty} (-1)^s h^s t^{2s} \sum_{j=0}^{\infty} \mathcal{A}_j^h \frac{t^{j+1}}{j!} + 3z \sum_{s=0}^{\infty} (-1)^s h^s t^{3s} \sum_{j=0}^{\infty} \mathcal{A}_j^h \frac{t^{j+2}}{j!}. \end{aligned}$$

Then using Cauchy product rule in the above equation, we get

$$\begin{aligned} \mathcal{A}_1^h + \mathcal{A}_2^h t + \sum_{j=2}^{\infty} A_{j+1}^h \frac{t^j}{j!} &= (x + \beta_{0,h}) A_0^h + \sum_{j=2}^{\infty} \sum_{s=0}^j \binom{j}{s} \beta_{s,h} \mathcal{A}_{j-s}^h \frac{t^j}{j!} (x + \beta_{0,h}) \mathcal{A}_1^h t + (\beta_{1,h} - xh + 2y) \mathcal{A}_1^h t \\ &+ x \sum_{j=2}^{\infty} \sum_{s=0}^j (-h)^s \frac{j!}{(j-s)!} \mathcal{A}_{j-s}^h \frac{t^j}{j!} + 2y \sum_{j=2}^{\infty} \sum_{s=0}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s \frac{j!}{(j-2s-1)!} \mathcal{A}_{j-2s-1}^h \frac{t^j}{j!} \\ &+ 3z \sum_{j=2}^{\infty} \sum_{s=0}^{\lfloor \frac{j-2}{3} \rfloor} (-h)^s \frac{j!}{(j-3s-2)!} \mathcal{A}_{j-3s-2}^h \frac{t^j}{j!}, \quad j \geq 2. \end{aligned} \tag{27}$$

Besides, by (20), we have the equalities  $A_0^h = \lambda_{0,h}$ ,  $\mathcal{A}_1^h = \lambda_{1,h} + \lambda_{0,h}x$  and  $\mathcal{A}_1^h = \lambda_{2,h} + 2\lambda_{1,h} + \lambda_{0,h}x(x-h) + 2y\lambda_{0,h}$ . Then, using the equations (2) and (27), we have

$$\mathcal{A}_1^h + \mathcal{A}_2^h t = (x + \beta_{0,h}) A_0^h + (x + \beta_{0,h}) \mathcal{A}_1^h t + (\beta_{1,h} - xh + 2y) \mathcal{A}_1^h t. \tag{28}$$

Using (28) in (27), if polynomial equality is used in terms of  $\frac{t^j}{j!}$ , we get the recurrence relation for the 3D- $\Delta_h$ -Hermite Appell polynomials.  $\square$

**Remark 2.2.** Since the last four terms of the right hand side of (24) can be written as

$$\begin{aligned} &2jy\mathcal{A}_{j-1}^h + 2yj! \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s \frac{\mathcal{A}_{j-2s-1}^h}{(j-2s-1)!} + 3zj(j-1)\mathcal{A}_{j-2}^h + 3zj! \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} (-h)^s \frac{\mathcal{A}_{j-3s-2}^h}{(j-3s-2)!} \\ &= 2yj! \sum_{s=0}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s \frac{\mathcal{A}_{j-2s-1}^h}{(j-2s-1)!} + 3zj! \sum_{s=0}^{\lfloor \frac{j-2}{3} \rfloor} (-h)^s \frac{\mathcal{A}_{j-3s-2}^h}{(j-3s-2)!}, \end{aligned}$$

it should be remarked that (24) is valid for  $j \geq 2$ . Similar consideration will be taken into account in the rest of the paper.

**Theorem 2.3.** For the 3D- $\Delta_h$ -Hermite Appell polynomials, we have the LO as

$${}_xL_j^- = \frac{1}{jh} {}_x\Delta_h,$$

and the RO as

$$\begin{aligned} {}_xL_j^+ &= x + \beta_{0,h} + \sum_{s=0}^{j-1} \frac{\beta_{j-s,h}}{(j-s)!h^{j-s}} {}_x\Delta_h^{j-s} + x \sum_{s=1}^j (-1)^s {}_x\Delta_h^s \\ &+ 2y \frac{{}_x\Delta_h}{h} + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^s}{h^{s+1}} {}_x\Delta_h^{2s+1} + 3z \frac{{}_x\Delta_h^2}{h^2} + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-1)^s}{h^{2s+2}} {}_x\Delta_h^{3s+2}, \quad j \geq 2. \end{aligned} \tag{29}$$

The difference equation satisfied by the 3D- $\Delta_h$ -Hermite Appell polynomials is given by

$$\begin{aligned} &\left[ \left( \frac{x}{h} + 1 + \frac{\beta_{0,h}}{h} \right) {}_x\Delta_h + \sum_{s=1}^{j-1} \frac{\beta_{j-s,h}}{(j-s)!h^{j-s+1}} {}_x\Delta_h^{j-s+1} + \left( \frac{x}{h} + 1 \right) \sum_{s=1}^j (-1)^s {}_x\Delta_h^{s+1} \right. \\ &\left. + \sum_{s=1}^j (-1)^s {}_x\Delta_h^s + 2y \frac{{}_x\Delta_h^2}{h^2} + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^s}{h^{s+2}} {}_x\Delta_h^{2s+2} + 3z \frac{{}_x\Delta_h^3}{h^3} + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-1)^s}{h^{2s+3}} {}_x\Delta_h^{3s+3} - j \right] \mathcal{A}_j^h = 0, \quad j \geq 2. \end{aligned} \tag{30}$$

*Proof.* The LO can be seen easily from (4). To give the RO, we write  $\mathcal{A}_s^h, \mathcal{A}_{j-s}^h, \mathcal{A}_{j-1}^h, \mathcal{A}_{j-2s-1}^h, \mathcal{A}_{j-2}^h$  and  $\mathcal{A}_{j-3s-2}^h$  in terms of the lowering operator as follows

$$\begin{aligned} \mathcal{A}_s^h &= [xL_{s+1}^- xL_{s+2}^- \cdots xL_j^-] \mathcal{A}_j^h \\ &= \left[ \frac{1}{(s+1)h} x\Delta_h \frac{1}{(s+2)h} x\Delta_h \cdots \frac{1}{jh} x\Delta_h \right] \mathcal{A}_j^h \\ &= \left[ \frac{s!}{j!h^{j-s}} x\Delta_h^{j-s} \right] \mathcal{A}_j^h, \\ \mathcal{A}_{j-s}^h &= \frac{(j-s)!}{j!h^s} x\Delta_h^s (\mathcal{A}_j^h), \\ \mathcal{A}_{j-1}^h &= \frac{1}{jh} x\Delta_h (\mathcal{A}_j^h), \\ \mathcal{A}_{j-2s-1}^h &= \frac{(j-2s-1)!}{j!h^{2s+1}} x\Delta_h^{2s+1} (\mathcal{A}_j^h), \\ \mathcal{A}_{j-2}(x, y, z; h) &= \frac{1}{j(j-1)h^2} x\Delta_h^2 (\mathcal{A}_j^h), \\ \mathcal{A}_{j-3s-2}^h &= \frac{(j-3s-2)!}{j!h^{3s+2}} x\Delta_h^{3s+2} (\mathcal{A}_j^h). \end{aligned}$$

Using these relations in (24), we get

$$\begin{aligned} \mathcal{A}_{j+1}^h &= (x + \beta_{0,h}) \mathcal{A}_j^h + \sum_{s=0}^{j-1} \frac{\beta_{j-s,h}}{(j-s)!h^{j-s}} x\Delta_h^{j-s} (\mathcal{A}_j^h) + x \sum_{s=1}^j (-1)^s x\Delta_h^s (\mathcal{A}_j^h) + 2y \frac{x\Delta_h (\mathcal{A}_j^h)}{h} \\ &\quad + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^k}{h^{s+1}} x\Delta_h^{2s+1} (\mathcal{A}_j^h) + 3z \frac{x\Delta_h^2 (\mathcal{A}_j^h)}{h^2} + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-1)^s}{h^{2s+2}} x\Delta_h^{3s+2} (\mathcal{A}_j^h), \quad n \geq 2. \end{aligned} \tag{31}$$

Hence, we have the equation (29).

To obtain the DE, we apply the factorization method, which is given simply by

$$xL_{j+1}^- (xL_j^+ \mathcal{A}_j^h) = \mathcal{A}_j^h.$$

Then with the help of the product rule

$$x\Delta_h (u(x) v(x, y, z)) = u(x+h) x\Delta_h v(x, y, z) + v(x, y, z) x\Delta_h u(x),$$

we can write that difference equation (30). So the resultant equation gives the claim.  $\square$

**Theorem 2.4.** *3D- $\Delta_h$ -Hermite Appell polynomials satisfy the following integro-LO, integro-RO, and integro-DE*

$$\begin{aligned} xL_j^- &= \frac{1}{j} x\Delta_h^{-1} y\Delta_h, \\ xL_j^+ &= x + \beta_{0,h} + \sum_{s=0}^{j-1} \frac{\beta_{j-s,h}}{(j-s)!} x\Delta_h^{-(j-s)} y\Delta_h^{j-s} + x \sum_{s=1}^n (-h)^s x\Delta_h^{-s} y\Delta_h^s \\ &\quad + 2y x\Delta_h^{-1} y\Delta_h + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s x\Delta_h^{-(2s+1)} y\Delta_h^{2s+1} + 3z x\Delta_h^{-2} y\Delta_h^2 \\ &\quad + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} (-h)^s x\Delta_h^{-(3s+2)} y\Delta_h^{3s+2}, \quad j \geq 2, \end{aligned}$$

$$\left[ \begin{aligned} & \left( (x + \beta_{0,h}) \frac{y\Delta_h}{h} + \frac{1}{h} \sum_{s=1}^{j-1} \frac{\beta_{j-s,h}}{(j-s)!} x\Delta_h^{-(j-s)} y\Delta_h^{j-s+1} + x \sum_{s=1}^j (-1)^s h^{s-1} x\Delta_h^{-s} y\Delta_h^{s+1} + 2 \left( \frac{y}{h} + 1 \right) x\Delta_h^{-1} y\Delta_h^2 \right. \\ & + 2 x\Delta_h^{-1} y\Delta_h + 2 \left( \frac{y}{h} + 1 \right) \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s x\Delta_h^{-(2s+1)} y\Delta_h^{2s+2} + 2 \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s x\Delta_h^{-(2s+1)} y\Delta_h^{2s+1} \\ & \left. + \frac{3z}{h} x\Delta_h^{-2} y\Delta_h^3 + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} (-1)^s h^{s-1} x\Delta_h^{-(3s+2)} y\Delta_h^{3s+3} - (j+1) \frac{x\Delta_h}{h} \right] \mathcal{A}_j^h = 0, \quad j \geq 2, \end{aligned}$$

respectively.

*Proof.* Applying the inverse difference operator on both sides of (4), we get

$$\begin{aligned} x\Delta_h^{-1} (\mathcal{A}_{j-1}^h) &= \frac{1}{jh} \mathcal{A}_j^h, \\ x\Delta_h^{-s} (\mathcal{A}_{j-1}^h) &= \frac{1}{j(j+1) \cdots (j+s-1) h^s} \mathcal{A}_{j+s-1}^h, \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{A}_s^h &= \frac{s!}{j!} x\Delta_h^{-(j-s)} y\Delta_h^{j-s} (\mathcal{A}_j^h), \\ \mathcal{A}_{j-s}^h &= \frac{(j-s)!}{j!} x\Delta_h^{-s} y\Delta_h^s (\mathcal{A}_j^h), \\ \mathcal{A}_{j-1}^h &= \frac{1}{j} x\Delta_h^{-1} y\Delta_h (\mathcal{A}_j^h), \\ \mathcal{A}_{j-2s-1}^h &= \frac{(j-2s-1)!}{j!} x\Delta_h^{-(2s+1)} y\Delta_h^{2s+1} (\mathcal{A}_j^h), \\ \mathcal{A}_{j-2}^h &= \frac{1}{j(j-1)} x\Delta_h^{-2} y\Delta_h^2 (\mathcal{A}_j^h), \\ \mathcal{A}_{j-3s-2}^h &= \frac{(j-3s-2)!}{j!} x\Delta_h^{-(3s+2)} y\Delta_h^{3s+2} (\mathcal{A}_j^h), \end{aligned}$$

by writing the above equation in the recurrence relation (24), we get the integro-raising operator.

To obtain the integro-DE, we use the factorization method

$$xL_{j+1}^- (xL_j^+ (\mathcal{A}_j^h)) = \mathcal{A}_j^h,$$

and the product rule given in the proof of Theorem 2.3.  $\square$

**Theorem 2.5.** *3D- $\Delta_h$ -Hermite Appell polynomials satisfy the PDE*

$$\begin{aligned} & \left[ \left( \frac{x}{hj} + \frac{\beta_{0,h}}{hj} \right) x\Delta_h^{j-1} y\Delta_h + \frac{1}{hj} \sum_{s=1}^j \frac{\beta_{j-s,h}}{(j-s)!} x\Delta_h^{s-1} y\Delta_h^{j-s+1} \right] \mathcal{A}_j^h \\ & + \frac{1}{h^{j-1}} \sum_{i=0}^{j-1} (-1)^{j-1-i} \binom{j-1}{i} (x+ih) \sum_{s=1}^j (-1)^s h^{s-1} x\Delta_h^{-s} y\Delta_h^{s+1} A_j(x+ih, y, z; h) \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{h^{j-1}} \left[ 2 \left( \frac{y}{h} + 1 \right) x \Delta_h^{j-2} y \Delta_h^2 + 2 x \Delta_h^{n-2} y \Delta_h \right. \\
 & + 2 \left( \frac{y}{h} + 1 \right) \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s x \Delta_h^{-2s+j-2} y \Delta_h^{2s+2} \left. \right] \mathcal{A}_j(x, y, z; h) \\
 & + \frac{1}{h^{j-1}} \left[ 2 \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s x \Delta_h^{-2s+j-2} y \Delta_h^{2s+1} + \frac{3z}{h} x \Delta_h^{j-3} y \Delta_h^3 \right. \\
 & \left. + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} (-1)^k h^{s-1} x \Delta_h^{-3s+j-3} y \Delta_h^{3s+3} - (j+1) \frac{x \Delta_h^j}{h} \right] \mathcal{A}_j^h = 0, \quad j \geq 2.
 \end{aligned}$$

*Proof.* We use the IDE and taking  $(j - 1)$  multiples of the operator wrt  $x$  in (3) and use the  $h$ -summation by parts formula [9] given by

$$x \Delta_h^{j-1} [uv](x) = \sum_{i=0}^{j-1} (-1)^{j-1-i} \binom{j-1}{i} u(x+ih) v(x+ih),$$

and lastly, dividing both sides by  $h^{j-1}$ , we get the PDE for  $3D$ - $\Delta_h$ -Hermite Appell polynomials.  $\square$

**Theorem 2.6.**  $3D$ - $\Delta_h$ -Hermite Appell polynomials satisfy the following integro-LO, Integro-RO and IDE in terms of  ${}_y \Delta_h^{-1} z \Delta_h$ :

$${}_y L_j^- = \frac{1}{j} {}_y \Delta_h^{-1} z \Delta_h, \tag{32}$$

$$\begin{aligned}
 {}_y L_j^+ &= x + \beta_{0,h} + \sum_{s=0}^{j-1} \frac{\beta_{j-s,h}}{(j-s)!} {}_y \Delta_h^{-(j-s)} z \Delta_h^{j-s} + x \sum_{s=1}^j (-h)^s {}_y \Delta_h^{-s} z \Delta_h^s \\
 & + 2y {}_y \Delta_h^{-1} z \Delta_h + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s {}_y \Delta_h^{-(2s+1)} z \Delta_h^{2s+1} + 3z {}_y \Delta_h^{-2} z \Delta_h^2 \\
 & + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} (-h)^s {}_y \Delta_h^{-(3s+2)} z \Delta_h^{3s+2}, \quad j \geq 2,
 \end{aligned} \tag{33}$$

$$\begin{aligned}
 & \left[ (x + \beta_{0,h}) \frac{z \Delta_h}{h} + \frac{1}{h} \sum_{s=1}^{j-1} \frac{\beta_{j-s,h}}{(j-s)!} {}_y \Delta_h^{-(j-s)} z \Delta_h^{j-s+1} + x \sum_{s=1}^j (-1)^s h^{s-1} {}_y \Delta_h^{-s} z \Delta_h^{s+1} + \frac{2y}{h} {}_y \Delta_h^{-1} z \Delta_h^{-1} \right. \\
 & + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-1)^s h^{s-1} {}_y \Delta_h^{-(2s+1)} z \Delta_h^{2s+2} + 3 \left( \frac{z}{h} + 1 \right) {}_y \Delta_h^{-1} z \Delta_h^2 + 3 {}_y \Delta_h^{-1} z \Delta_h^2 \\
 & \left. + 3 \left( \frac{z}{h} + 1 \right) \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} (-1)^s (h)^{s-1} {}_y \Delta_h^{-(3s+2)} z \Delta_h^{3s+3} + 3 \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} (-1)^s h^s {}_y \Delta_h^{-(3s+2)} z \Delta_h^{3s+2} - (j+1) \frac{y \Delta_h}{h} \right] \mathcal{A}_j^h = 0, \quad j \geq 2,
 \end{aligned} \tag{34}$$

respectively.

*Proof.* Replacing  $(j - 1)$  instead of  $j$  for the integro-lowering operator

$${}_y\Delta_h (\mathcal{A}_j(x, y, z; h)) = j(j - 1)h\mathcal{A}_{j-2}(x, y, z; h),$$

we get

$$\begin{aligned} {}_y\Delta_h (\mathcal{A}_{j-1}(x, y, z; h)) &= (j - 1)(j - 2)h\mathcal{A}_{j-3}(x, y, z; h), \\ \mathcal{A}_{j-3}(x, y, z; h) &= \frac{1}{(j - 1)(j - 2)h} {}_y\Delta_h (\mathcal{A}_{j-1}(x, y, z; h)) \end{aligned}$$

and

$$\begin{aligned} {}_z\Delta_h (\mathcal{A}_j(x, y, z; h)) &= j(j - 1)(j - 2)h\mathcal{A}_{j-3}(x, y, z; h) \\ &= j(j - 1)(j - 2)h \frac{1}{(j - 1)(j - 2)h} {}_y\Delta_h (\mathcal{A}_{j-1}(x, y, z; h)), \\ {}_z\Delta_h (\mathcal{A}_j(x, y, z; h)) &= j {}_y\Delta_h (\mathcal{A}_{j-1}(x, y, z; h)), \\ \mathcal{A}_{j-1}(x, y, z; h) &= \frac{1}{j} {}_y\Delta_h^{-1} {}_z\Delta_h (\mathcal{A}_j(x, y, z; h)). \end{aligned}$$

Then we can write the integro-lowering operator (32). Also we give the integro-raising operator

$$\begin{aligned} \mathcal{A}_s(x, y, z; h) &= [{}_yL_{s+1}^- {}_yL_{s+2}^- \cdots {}_yL_j^-] \mathcal{A}_j(x, y, z; h) \\ &= \left[ \frac{1}{s+1} {}_y\Delta_h^{-1} {}_z\Delta_h \frac{1}{s+2} {}_y\Delta_h^{-1} {}_z\Delta_h \cdots \frac{1}{j} {}_y\Delta_h^{-1} {}_z\Delta_h \right] \mathcal{A}_j(x, y, z; h) \\ &= \frac{s!}{j!} {}_y\Delta_h^{-(j-s)} {}_z\Delta_h^{j-s} \mathcal{A}_j(x, y, z; h) \\ \mathcal{A}_s(x, y, z; h) &= \frac{s!}{j!} {}_y\Delta_h^{-(j-s)} {}_z\Delta_h^{j-s} \mathcal{A}_j(x, y, z; h). \end{aligned}$$

We can write the following equations instead of  $\mathcal{A}_{j-s}(x, y, z; h)$ ,  $\mathcal{A}_{j-1}(x, y, z; h)$ ,  $\mathcal{A}_{j-2s-1}(x, y, z; h)$ ,  $\mathcal{A}_{j-2}(x, y, z; h)$  and  $\mathcal{A}_{j-3s-2}(x, y, z; h)$

$$\begin{aligned} \mathcal{A}_{j-s}(x, y, z; h) &= \frac{(j - s)!}{j!} {}_y\Delta_h^{-(j-j+s)} {}_z\Delta_h^{j-j+s} \mathcal{A}_j(x, y, z; h) \\ &= \frac{(j - s)!}{j!} {}_y\Delta_h^{-s} {}_z\Delta_h^s \mathcal{A}_j(x, y, z; h), \\ \mathcal{A}_{j-1}(x, y, z; h) &= \frac{(j - 1)!}{j!} {}_y\Delta_h^{-(j-j+1)} {}_z\Delta_h^{j-j+1} \mathcal{A}_j(x, y, z; h) \\ &= \frac{(j - 1)!}{j!} {}_y\Delta_h^{-1} {}_z\Delta_h \mathcal{A}_j(x, y, z; h), \\ \mathcal{A}_{j-2s-1}(x, y, z; h) &= \frac{(j - 2s - 1)!}{j!} {}_y\Delta_h^{-(2s+1)} {}_z\Delta_h^{2s+1} \mathcal{A}_j(x, y, z; h), \\ \mathcal{A}_{j-2}(x, y, z; h) &= \frac{(j - 2)!}{j!} {}_y\Delta_h^{-2} {}_z\Delta_h^2 \mathcal{A}_j(x, y, z; h), \\ \mathcal{A}_{j-3s-2}(x, y, z; h) &= \frac{(j - 3s - 2)!}{j!} {}_y\Delta_h^{-(3s+2)} {}_z\Delta_h^{3s+2} \mathcal{A}_j(x, y, z; h). \end{aligned}$$

Using the recurrence relation

$$\begin{aligned}
 \mathcal{A}_{j+1}(x, y, z; h) &= (x + \beta_{0,h}) \mathcal{A}_j(x, y, z; h) + \sum_{s=0}^{j-1} \frac{\beta_{j-s,h}}{(j-s)!} {}_y\Delta_h^{-(j-s)} {}_z\Delta_h^{j-s} \mathcal{A}_j(x, y, z; h) \\
 &+ x \sum_{s=1}^j (-h)^s {}_y\Delta_h^{-s} {}_z\Delta_h^s \mathcal{A}_j(x, y, z; h) + 2y {}_y\Delta_h^{-1} {}_z\Delta_h \mathcal{A}_j(x, y, z; h) \\
 &+ 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s {}_y\Delta_h^{-(2s+1)} {}_z\Delta_h^{2s+1} \mathcal{A}_j(x, y, z; h) + 3z {}_y\Delta_h^{-2} {}_z\Delta_h^2 \mathcal{A}_j(x, y, z; h) \\
 &+ 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} (-h)^s {}_y\Delta_h^{-(3s+2)} {}_z\Delta_h^{3s+2} \mathcal{A}_j(x, y, z; h), \quad j \geq 2,
 \end{aligned}
 \tag{35}$$

we can write that the integro-raising operator (33). For the integro-difference equation, in (35) applying to both sides the following equation in [9]

$${}_z\Delta_h(f(z)g(y, z)) = f(z+h) {}_z\Delta_h g(y, z) + g(y, z) {}_z\Delta_h f(z).$$

We can write the integro-difference equation (34).  $\square$

### 3. Some summation formulas for 3D- $\Delta_h$ -Hermite Appell polynomials

Now, we investigate the connections between the Stirling numbers of the first kind and 3D- $\Delta_h$ -Hermite Appell polynomials. We recall that the Stirling numbers  $S(i, k)$  of the first kind are defined the following generating function [38]

$$\{\log(1+z)\}^k = k! \sum_{i=k}^{\infty} S(i, k) \frac{z^i}{i!}, \quad |z| < 1.$$

From the definition of  $S(i, k)$  [38], the Pochhammer symbol can be written as

$$(z)_i = \sum_{k=0}^i (-1)^{i-k} S(i, k) z^k.
 \tag{36}$$

**Theorem 3.1.**  $\Delta_h$ -GHAP can be represented by

$$\mathcal{A}_j(x, y; h) = \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{j-2l}{2} \rfloor} \frac{j!}{(j-2s-2l)!(s+l)!} \mathcal{A}_{j-2s-2l}(x; h) h^s S(s+l, l) y^l.
 \tag{37}$$

*Proof.* Using the definition of  $\Delta_h$ -Appell polynomials (10) and  $\Delta_h$ -GHAP (9), we have

$$\begin{aligned}
 \sum_{j=0}^{\infty} \mathcal{A}_j(x, y; h) \frac{t^j}{j!} &= \vartheta(t) (1+ht^2)^{\frac{y}{h}} (1+ht)^{\frac{x}{h}} \\
 &= \left( \sum_{j=0}^{\infty} \mathcal{A}_j(x; h) \frac{t^j}{j!} \right) \left( \sum_{s=0}^{\infty} \left( -\frac{y}{h} \right)_s (-h)^s \frac{t^{2s}}{s!} \right).
 \end{aligned}$$

Using Cauchy product rule and then if polynomial equality is used in terms of  $\frac{t^j}{j!}$  in the resultant equation, we get

$$\mathcal{A}_j(x, y; h) = \sum_{s=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{(j-2s)!s!} (-h)^s \mathcal{A}_{j-2s}(x; h) \left(-\frac{y}{h}\right)_s.$$

Using in the above equality (36), we get

$$\begin{aligned} \mathcal{A}_j(x, y; h) &= \left( \sum_{s=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{(j-2s)!s!} (-h)^s \mathcal{A}_{j-2s}(x; h) \right) \left( \sum_{l=0}^s (-1)^{s-l} S(s, l) (-h)^{-l} y^l \right) \\ &= \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{s=l}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{(j-2s)!s!} (-h)^{s-l} \mathcal{A}_{j-2s}(x; h) S(s, l) (-1)^{s-l} y^l \\ &= \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{j-2l}{2} \rfloor} \frac{j!}{(j-2s-2l)!(s+l)!} (-h)^s \\ &\quad \times \mathcal{A}_{j-2s-2l}(x; h) (-1)^s S(s+l, l) y^l. \end{aligned}$$

Whence the result.  $\square$

**Theorem 3.2.** *The 3D- $\Delta_h$ -Hermite Appell polynomials have the following representation*

$$\begin{aligned} \mathcal{A}_j^h &= \sum_{v=0}^{\lfloor \frac{j}{3} \rfloor} \sum_{m=0}^{\lfloor \frac{j-3v}{3} \rfloor} \frac{j!}{(j-3m-3v)!(m+v)!} \sum_{l=0}^{\lfloor \frac{j-3m-3v}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{j-3m-3v-2l}{2} \rfloor} \frac{(j-3m-3v)!}{(j-3m-3v-2s-2l)!} \\ &\quad \times \frac{1}{(s+l)!} (h)^{s+m} \mathcal{A}_{j-3m-3v-2s-2l}(x; h) s(m+v, v) S(s+l, l) y^l z^v. \end{aligned} \tag{38}$$

*Proof.* Using the equation (9) in the equation (1), we have

$$\begin{aligned} \sum_{j=0}^{\infty} \mathcal{A}_j^h \frac{t^j}{j!} &= \vartheta(t) (1+ht)^{\frac{x}{h}} (1+ht^2)^{\frac{y}{h}} (1+ht^3)^{\frac{z}{h}} \\ &= \left( \sum_{j=0}^{\infty} \mathcal{A}_j(x, y; h) \frac{t^j}{j!} \right) \left( \sum_{m=0}^{\infty} \left(-\frac{z}{h}\right)_m (-h)^m \frac{t^{3m}}{m!} \right) \\ &= \sum_{j=0}^{\infty} \sum_{m=0}^{\lfloor \frac{j}{3} \rfloor} \frac{j!}{(j-3m)!m!} \mathcal{A}_{j-3m}(x, y; h) (-h)^m \left(-\frac{z}{h}\right)_m \frac{t^j}{m!}. \end{aligned}$$

If polynomial equality is used in terms of  $\frac{t^j}{j!}$  in this equation, we obtain

$$\mathcal{A}_j^h = \sum_{m=0}^{\lfloor \frac{j}{3} \rfloor} \frac{j!}{(j-3m)!m!} \mathcal{A}_{j-3m}(x, y; h) (-h)^m \left(-\frac{z}{h}\right)_m.$$

Using (36) for the term  $\left(-\frac{z}{h}\right)_m$  in the above equality, we have

$$\mathcal{A}_j^h = \sum_{v=0}^{\lfloor \frac{j}{3} \rfloor} \sum_{m=0}^{\lfloor \frac{j-3v}{3} \rfloor} \frac{j!}{(j-3m-3v)!(m+v)!} \mathcal{A}_{j-3m-3v}(x, y; h) (h)^m S(m+v, v) z^v.$$

Using the equation corresponding to  $\mathcal{A}_{j-3m-3v}(x, y; h)$  from (37), we have

$$\mathcal{A}_j^h = \sum_{v=0}^{\lfloor \frac{j}{3} \rfloor} \sum_{m=0}^{\lfloor \frac{j-3v}{3} \rfloor} \frac{j!}{(j-3m-3v)!(m+v)!} \sum_{l=0}^{\lfloor \frac{j-3m-3v}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{j-3m-3v-2l}{2} \rfloor} \frac{(j-3m-3v)!}{(j-3m-3v-2s-2l)!} \times \frac{1}{(s+l)!} (h)^{s+m} \mathcal{A}_{j-3m-3v-2s-2l}(x; h) S(m+v, v) S(s+l, l) y^l z^v,$$

which completes the proof.  $\square$

**Theorem 3.3.** *The 3D- $\Delta_h$ -Hermite Appell polynomials satisfy the following identity*

$$\mathcal{A}_j^h = \sum_{l=0}^{\lfloor \frac{j}{3} \rfloor} \sum_{s=0}^{\lfloor \frac{j-3l}{3} \rfloor} \frac{j!}{(j-3s-3l)!(s+l)!} \mathcal{A}_{j-3s-3l}(x, y; h) h^s S(s+l, l) z^l. \tag{39}$$

*Proof.* Using the definition of 3D- $\Delta_h$ -Hermite-Appell polynomials in (1), we get

$$\sum_{j=0}^{\infty} \mathcal{A}_j^h \frac{t^j}{j!} = \left( \sum_{j=0}^{\infty} \mathcal{A}_j(x, y; h) \frac{t^j}{j!} \right) \left( \sum_{s=0}^{\infty} \left( \frac{-z}{h} \right)_s (-h)^s \frac{t^{3s}}{s!} \right).$$

Then using the Cauchy rule, we have

$$\mathcal{A}_j^h = \sum_{s=0}^{\lfloor \frac{j}{3} \rfloor} \binom{j}{3s} \frac{(3s)!}{s!} \mathcal{A}_{j-3s}(x, y; h) \left( \frac{-z}{h} \right)_s (-h)^s.$$

Using Stirling numbers of the first kind in (36) instead of  $\left( \frac{-z}{h} \right)_s$  and then using Cauchy product rule, we arrive the desired result.  $\square$

**Corollary 3.4.** *Taking  $z = 1$  in (39), we have*

$$\mathcal{A}_j(x, y, 1; h) = \sum_{l=0}^{\lfloor \frac{j}{3} \rfloor} \sum_{s=0}^{\lfloor \frac{j-3l}{3} \rfloor} \frac{j!}{(j-3s-3l)!(s+l)!} \mathcal{A}_{j-3s-3l}(x, y; h) h^s S(s+l, l).$$

**Theorem 3.5.** *The 3D- $\Delta_h$ -Hermite Appell polynomials have the following addition formulas*

$$\mathcal{A}_j(x + \omega, y, z; h) = \sum_{l=0}^j \sum_{s=0}^{j-l} \frac{j!}{(j-s-l)!(s+l)!} \mathcal{A}_{j-s-l}(x, y, z; h) h^s S(s, l) \omega^l, \tag{40}$$

$$\mathcal{A}_j(x, y + \omega, z; h) = \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{j-2l}{2} \rfloor} \frac{j!}{(j-2s-2l)!(s+l)!} \mathcal{A}_{j-2s-2l}(x, y, z; h) h^s S(s, l) \omega^l, \tag{41}$$

$$\mathcal{A}_j(x, y, z + \omega; h) = \sum_{l=0}^{\lfloor \frac{j}{3} \rfloor} \sum_{s=0}^{\lfloor \frac{j-3l}{3} \rfloor} \frac{j!}{(j-3s-3l)!(s+l)!} \mathcal{A}_{j-3s-3l}(x, y, z; h) h^s S(s, l) \omega^l. \tag{42}$$

*Proof.* Taking  $x + \omega$  instead of  $x$  in the equation (40), we have

$$\sum_{j=0}^{\infty} A_j(x + \omega, y, z; h) = \left( \sum_{j=0}^{\infty} A_j^h \frac{t^j}{j!} \right) \left( \sum_{s=0}^{\infty} (\omega)_s^h \frac{t^s}{s!} \right).$$

Using the Cauchy rule and after comparing the coefficients of  $\frac{t^j}{j!}$  on both sides of the resulting equation, we have

$$\mathcal{A}_j(x + \omega, y, z; h) = \sum_{s=0}^j \binom{j}{s} \mathcal{A}_{j-s}(x, y, z; h) (\omega)_s^h.$$

Then, using (36) for  $(\omega)_s^h$ , we have the claim (40). In a similar way, we can obtain the equations (41) and (42).  $\square$

#### 4. Special examples of $3D$ - $\Delta_h$ -Hermite Appell polynomials

We introduce some special cases of  $3D$ - $\Delta_h$ -Hermite Appell polynomials and obtain their properties such as explicit representation, determinantal form, recurrence relation, LO, RO and DE for them. The special cases, we consider are the  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials,  $3D$ - $\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials,  $3D$ - $\Delta_h$ -Hermite degenerate Apostol-Euler polynomials and  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Boole polynomials

##### 4.1. $3D$ - $\Delta_h$ -Hermite $\lambda$ -Charlier polynomials

Here, we examine explicit representation, determinantal form, recurrence relation, LO, RO and DE provided by  $3D$ - $\Delta_h$ -Hermite Charlier polynomials  $C_{j,\lambda}^a(x, y, z; h)$ .

We introduce  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials via the following generating function

$$\lambda^{-a^ht} (1 + ht)^{\frac{x}{h}} (1 + ht^2)^{\frac{y}{h}} (1 + ht^3)^{\frac{z}{h}} = \sum_{j=0}^{\infty} C_{j,\lambda}^a(x, y, z; h) \frac{t^j}{j!}, \quad \lambda > 0.$$

If  $\lambda = e$ ,  $h \rightarrow 1$ ,  $y = 0$  and  $z = 0$  in generating function, we obtain the Charlier polynomials.

**Corollary 4.1.**  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials satisfy the explicit representation as follows

$$C_{j,\lambda}^a(x, y, z; h) = \sum_{s=0}^j \binom{j}{s} \sum_{l=0}^{\lfloor \frac{s}{3} \rfloor} \sum_{m=0}^{\lfloor \frac{s-3l}{2} \rfloor} C_{j-s,\lambda}^{a,h} \binom{s-3l}{2m} \binom{s}{3l} (x)_{s-2m-3l}^h (y)_m^h (z)_l^h \frac{(2m)!}{m!} \frac{(3l)!}{l!}$$

in which the  $\lambda$ -Charlier numbers  $C_{j,\lambda}^a(0, 0, 0; h) = C_{j,\lambda}^{a,h}$  for  $x = y = z = 0$ ,  $\lambda = e$ , [37, 39], are given by the following series

$$\lambda^{-a^ht} = \sum_{j=0}^{\infty} C_{j,\lambda}^{a,h} \frac{t^j}{j!}.$$

The first four  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials are as follows.

Table 1: The first four 3D- $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials.

$j$	$C_{j,\lambda}^a(x, y, z; h)$
0	1
1	$x - a^h \ln \lambda$
2	$x^2 - a^{2h} \ln^2 \lambda - 2a^h x \ln \lambda - hx + 2y$
3	$x^3 - 3hx^2 + 2h^2x + 3xa^{2h} \ln^2 \lambda + 3xa^h \ln \lambda h - 3a^h \ln \lambda x^2 - a^{3h} \ln^3 \lambda - 6ya^h \ln \lambda + 6xy + 6z$

**Corollary 4.2.** 3D- $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials satisfy the following determinantal form

$$C_{j,\lambda}^a(x, y, z; h) = (-1)^j \begin{vmatrix} 1 & G_1^h & G_2^h & \cdots & G_{j-1}^h & G_j^h \\ 1 & a^h \ln \lambda & a^{2h} \ln^2 \lambda & \cdots & a^{h(j-1)} \ln^{j-1} \lambda & a^{hj} \ln^j \lambda \\ 0 & 1 & \binom{2}{1} a^h \ln \lambda & \cdots & \binom{j-1}{1} a^{h(j-2)} \ln^{j-2} \lambda & \binom{j}{1} a^{h(j-1)} \ln^{j-1} \lambda \\ 0 & 0 & 1 & \cdots & \binom{j-1}{2} a^{h(j-3)} \ln^{j-3} \lambda & \binom{j}{2} a^{h(j-2)} \ln^{j-2} \lambda \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{j}{j-1} a^h \ln \lambda \end{vmatrix},$$

where

$$\frac{1}{\lambda^{-a^h t}} = \sum_{s=0}^{\infty} a_{s,h}(\lambda) \frac{t^s}{s!} \quad \text{and} \quad G_j^h = \sum_{s=0}^j \binom{j}{s} a_{s,h}(\lambda) C_{j-s,\lambda}^a(x, y, z; h).$$

*Proof.* Taking  $\vartheta(t) = \lambda^{-a^h t}$ , it gives

$$\frac{1}{\lambda^{-a^h t}} = \sum_{s=0}^{\infty} a_{s,h}(\lambda) \frac{t^s}{s!}$$

from equation (22) and

$$G_j^h = \sum_{s=0}^j \binom{j}{s} a_{s,h}(\lambda) C_{j-s,\lambda}^a(x, y, z; h)$$

from equation (23). Therefore, these equations are provided by substituting them in equation (21).  $\square$

**Corollary 4.3.** The recurrence relation for 3D- $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials is given by

$$\begin{aligned} C_{j+1,\lambda}^a(x, y, z; h) &= (x - a^h \ln \lambda) C_{j,\lambda}^a(x, y, z; h) \\ &+ xj! \sum_{s=1}^j (-h)^s \frac{C_{j-s,\lambda}^a(x, y, z; h)}{(j-s)!} + 2jy C_{j-1,\lambda}^a(x, y, z; h) \\ &+ 2yj! \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s \frac{C_{j-2s-1,\lambda}^a(x, y, z; h)}{(j-2s-1)!} + 3j(j-1)z C_{j-2,\lambda}^a(x, y, z; h) \\ &+ 3zj! \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} (-h)^s \frac{C_{j-3s-2,\lambda}^a(x, y, z; h)}{(j-3s-2)!}, \quad j \geq 2. \end{aligned}$$

*Proof.* It is proved when  $\lambda^{-a^h t}$  is written in case of  $\vartheta(t)$  in Theorem 2.1.  $\square$

**Corollary 4.4.** *The 3D- $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials have the following LO, RO and DE*

$$\begin{aligned}
 {}_x L_j^- &= \frac{1}{j! h} x \Delta_h, \\
 {}_x L_j^+ &= x - a^h \ln \lambda + x \sum_{s=1}^j (-1)^s x \Delta_h^s + 2y \frac{x \Delta_h}{h} + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^s}{h^{s+1}} x \Delta_h^{2s+1} \\
 &\quad + 3z \frac{x \Delta_h^2}{h^2} + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-1)^s}{h^{2s+2}} x \Delta_h^{3s+2}, \quad j \geq 2, \\
 &\left[ \left( \frac{x}{h} + 1 - \frac{a^h \ln \lambda}{h} \right) x \Delta_h + \left( \frac{x}{h} + 1 \right) \sum_{s=1}^j (-1)^s x \Delta_h^{s+1} + \sum_{s=1}^j (-1)^s x \Delta_h^s + 2y \frac{x \Delta_h^2}{h^2} \right. \\
 &\quad \left. + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^s}{h^{s+2}} x \Delta_h^{2s+2} + 3z \frac{x \Delta_h^3}{h^3} + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-1)^s}{h^{2s+3}} x \Delta_h^{3s+3} - j \right] C_{j,\lambda}^a(x, y, z; h) = 0, \quad j \geq 2,
 \end{aligned}$$

respectively.

*Proof.* It is proved when  $\lambda^{-a^h t}$  is written in case of  $\vartheta(t)$  in Theorem 2.3.  $\square$

#### 4.2. 3D- $\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials

This part includes determinantal form, recurrence relation, LO, RO and DE for degenerate 3D- $\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials  $\mathcal{B}_j(x, y, z; \lambda; h)$ .

We introduce the 3D- $\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials via the following generating function

$$\frac{t}{\lambda(1+ht)^{\frac{1}{h}} - 1} (1+ht)^{\frac{x}{h}} (1+ht^2)^{\frac{y}{h}} (1+ht^3)^{\frac{z}{h}} = \sum_{j=0}^{\infty} \mathcal{B}_j(x, y, z; \lambda; h) \frac{t^j}{j!}, \quad |t| < \left| \ln \left( \frac{1}{\lambda} \right) \right|.$$

If  $h \rightarrow 0$  and  $y = 0, z = 0$  in generating function, we get the degenerate Apostol-Bernoulli polynomials [40].

**Corollary 4.5.** *The polynomials  $\mathcal{B}_j(x, y, z; \lambda; h)$  have the explicit representation*

$$\mathcal{B}_j(x, y, z; \lambda; h) = \sum_{s=0}^j \binom{j}{s} \sum_{l=0}^{\lfloor \frac{s}{3} \rfloor} \sum_{m=0}^{\lfloor \frac{s-3l}{2} \rfloor} \mathcal{B}_{j-s,h}(\lambda) \binom{s-3l}{2m} \binom{s}{3l} (x)_{s-2m-3l}^h (y)_m^h (z)_l^h \frac{(2m)!}{m!} \frac{(3l)!}{l!}$$

in which  $\mathcal{B}_j(0, 0, 0; \lambda; h) = \mathcal{B}_{j,h}(\lambda)$  (degenerate Apostol-Bernoulli numbers) [40] as follows

$$\frac{t}{\lambda(1+ht)^{\frac{1}{h}} - 1} = \sum_{j=0}^{\infty} \mathcal{B}_{j,h}(\lambda) \frac{t^j}{j!}.$$

The first four 3D- $\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials are as follows.

Table 2: The first four  $3D-\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials.

$j$	$\mathcal{B}_j(x, y, z; \lambda; h)$
0	0
1	$\frac{1}{\lambda-1}$
2	$-\frac{2}{(\lambda-1)^2} (x + \lambda - x\lambda)$
3	$\frac{3}{(\lambda-1)^3} [(\lambda-1)^2 x^2 + (\lambda-1)(1-\lambda h-2\lambda)x + \lambda^2(1+h) + \lambda(1-h) + 2y(\lambda-1)^2]$

**Corollary 4.6.** *The  $3D-\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials satisfy the following determinantal form*

$$\mathcal{B}_j(x, y, z; \lambda; h) = \frac{(-1)^j}{\lambda^{j+1}} \begin{vmatrix} 0 & G_1^h & G_2^h & \cdots & G_{j-1}^h & G_j^h \\ \lambda & \lambda \frac{(1)_2^h}{2} & \lambda \frac{(1)_3^h}{3} & \cdots & \frac{(1)_j^h}{j} & \lambda \frac{(1)_{j+1}^h}{j+1} \\ 0 & \lambda & \lambda \binom{2}{1} \frac{(1)_2^h}{2} & \cdots & \lambda \binom{j-1}{1} \frac{(1)_{j-1}^h}{j-1} & \lambda \binom{j}{1} \frac{(1)_j^h}{j} \\ 0 & 0 & \lambda & \cdots & \lambda \binom{j-1}{2} \frac{(1)_{j-2}^h}{j-2} & \lambda \binom{j}{2} \frac{(1)_{j-1}^h}{j-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & \lambda \binom{j}{j-1} \frac{(1)_2^h}{2} \end{vmatrix},$$

where

$$\frac{\lambda(1+ht)^{\frac{1}{h}} - 1}{t} = \sum_{s=0}^{\infty} a_{s,h}(\lambda) \frac{t^s}{s!} \quad \text{and} \quad G_j^h = \sum_{s=0}^j \binom{j}{s} a_{s,h}(\lambda) \mathcal{B}_{j-s}(x, y, z; \lambda; h).$$

*Proof.* Taking  $\vartheta(t) = \frac{t}{\lambda(1+ht)^{\frac{1}{h}} - 1}$ , it gives

$$\frac{\lambda(1+ht)^{\frac{1}{h}} - 1}{t} = \sum_{s=0}^{\infty} a_{s,h}(\lambda) \frac{t^s}{s!}$$

from equation (22) and

$$\frac{\lambda(1+ht)^{\frac{1}{h}} - 1}{t} = \sum_{s=0}^{\infty} a_{s,h}(\lambda) \frac{t^s}{s!} \quad \text{and} \quad G_j^h = \sum_{s=0}^j \binom{j}{s} a_{s,h}(\lambda) \mathcal{B}_{j-s}(x, y, z; \lambda; h)$$

from equation (23). Therefore, these equations are proved by substituting them in equation (21).  $\square$

**Corollary 4.7.** *3D- $\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials satisfy the following recurrence relation*

$$\begin{aligned} & \left(1 + \frac{1}{j}\right)(x-1) \mathcal{B}_j(x, y, z; \lambda; h) - \left(1 + \frac{1}{j}\right) \sum_{s=0}^{j-1} \binom{j}{s} (-h)^{j-s} (j-s)! \mathcal{B}_s(x, y, z; \lambda; h) \\ & - \left(1 + \frac{1}{j}\right) \sum_{s=0}^j \sum_{k=0}^{j-s+1} \binom{j}{s} \frac{(j-s)! (-h)^k}{(j-s+1-k)!} \mathcal{B}_{j-s+1-k, h}(\lambda) \mathcal{B}_s(x, y, z; \lambda; h) + \left(1 + \frac{1}{j}\right) xj! \sum_{s=1}^j \frac{(-h)^s \mathcal{B}_{j-s}(x, y, z; \lambda; h)}{(j-s)!} \\ & + \left(1 + \frac{1}{j}\right) 2jy \mathcal{B}_{j-1}(x, y, z; \lambda; h) + \left(1 + \frac{1}{j}\right) 2yj! \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-h)^s \mathcal{B}_{j-2s-1}(x, y, z; \lambda; h)}{(j-2s-1)!} \\ & + \left(1 + \frac{1}{j}\right) 3j(j-1)z \mathcal{B}_{j-2}(x, y, z; \lambda; h) + \left(1 + \frac{1}{j}\right) 3zj! \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-h)^s \mathcal{B}_{j-3s-2}(x, y, z; \lambda; h)}{(j-3s-2)!} \\ & = \mathcal{B}_{j+1}(x, y, z; \lambda; h), \quad j \geq 2, \end{aligned}$$

where

$$\frac{t}{\lambda(1+ht)^{\frac{1}{h}} - 1} = \sum_{j=0}^{\infty} \mathcal{B}_{j, h}(\lambda) \frac{t^j}{j!}.$$

*Proof.* It is proved when  $\frac{t}{\lambda(1+ht)^{\frac{1}{h}} - 1}$  is written in case of  $\vartheta(t)$  in Theorem 2.1.  $\square$

**Corollary 4.8.** *The LO, RO and DE of the 3D- $\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials are*

$$\begin{aligned} {}_x L_j^- &= \frac{1}{jh} {}_x \Delta_h, \\ {}_x L_j^+ &= \left(1 + \frac{1}{j}\right)(x-1) - \left(1 + \frac{1}{j}\right) \sum_{s=0}^{j-1} (-1)^{j-s} {}_x \Delta_h^{j-s} - \left(1 + \frac{1}{j}\right) \sum_{s=0}^j \sum_{k=0}^{j-s+1} (-h)^k \frac{\mathcal{B}_{j-s+1-k, h}(\lambda)}{(j-s+1-k)! h^{j-s}} {}_x \Delta_h^{j-s} \\ & + \left(1 + \frac{1}{j}\right) x \sum_{s=1}^j (-1)^s {}_x \Delta_h^s + \left(1 + \frac{1}{j}\right) 2y \frac{{}_x \Delta_h}{h} + 2 \left(1 + \frac{1}{j}\right) y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^s}{h^{s+1}} {}_x \Delta_h^{2s+1} + \left(1 + \frac{1}{j}\right) 3z \frac{{}_x \Delta_h^2}{h^2} \\ & + \left(1 + \frac{1}{j}\right) 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-1)^s}{h^{2s+2}} {}_x \Delta_h^{3s+2}, \quad j \geq 2, \\ & \left[ \left(1 + \frac{1}{j}\right) \left(\frac{x}{h} + 1 - \frac{1}{h}\right) {}_x \Delta_h - \left(1 + \frac{1}{j}\right) \sum_{s=0}^j \sum_{k=0}^{j-s+1} (-h)^k \frac{\mathcal{B}_{j-s+1-k, h}(\lambda)}{(j-s+1-k)! h^{j-s+1}} {}_x \Delta_h^{j-s+1} \right. \\ & + \left(1 + \frac{1}{j}\right) \left(\frac{x}{h} + 1\right) \sum_{s=1}^j (-1)^s {}_x \Delta_h^{s+1} + \left(1 + \frac{1}{j}\right) \sum_{s=1}^j (-1)^s {}_x \Delta_h^s - \left(1 + \frac{1}{j}\right) \sum_{s=0}^{j-1} (-1)^{j-s} \frac{{}_x \Delta_h^{j-s+1}}{h} + 2 \left(1 + \frac{1}{j}\right) y \frac{{}_x \Delta_h^2}{h^2} \\ & + 2 \left(1 + \frac{1}{j}\right) y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^s}{h^{s+2}} {}_x \Delta_h^{2s+2} + \left(1 + \frac{1}{j}\right) 3z \frac{{}_x \Delta_h^3}{h^3} \\ & \left. + \left(1 + \frac{1}{j}\right) 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-1)^s}{h^{2s+3}} {}_x \Delta_h^{3s+3} - \left(j - \frac{1}{j}\right) \right] \mathcal{B}_j(x, y, z; \lambda; h) = 0, \quad j \geq 2, \end{aligned}$$

respectively.

*Proof.* It is proved when  $\frac{t}{\lambda(1+ht)^{\frac{1}{h}} - 1}$  is written in case of  $\vartheta(t)$  in Theorem 2.3.  $\square$

4.3.  $3D\text{-}\Delta_h\text{-Hermite}$  degenerate Apostol-Euler polynomials

Now, we introduce the determinantal form, recurrence relation, LO, RO and DE for  $3D\text{-}\Delta_h\text{-Hermite}$  degenerate Apostol-Euler polynomials  $E_j(x, y, z; \lambda; h)$ .

We introduce the  $3D\text{-}\Delta_h\text{-Hermite}$  degenerate Apostol-Euler polynomials via the following generating function

$$\frac{2}{\lambda(1+ht)^{\frac{1}{h}} + 1} (1+ht)^{\frac{x}{h}} (1+ht^2)^{\frac{y}{h}} (1+ht^3)^{\frac{z}{h}} = \sum_{j=0}^{\infty} E_j(x, y, z; \lambda; h) \frac{t^j}{j!}, \quad |t| < \left| \ln\left(\frac{-1}{\lambda}\right) \right|.$$

**Corollary 4.9.**  $3D\text{-}\Delta_h\text{-Hermite}$  degenerate Apostol-Euler polynomials have the following explicit representation

$$E_j(x, y, z; \lambda; h) = \sum_{s=0}^j \binom{j}{s} \sum_{l=0}^{\lfloor \frac{s}{3} \rfloor} \sum_{m=0}^{\lfloor \frac{s-3l}{2} \rfloor} \mathcal{E}_{j-s, h}(\lambda) \binom{s-3l}{2m} \binom{s}{3l} (x)_{s-2m-3l}^h (y)_m^h (z)_l^h \frac{(2m)! (3l)!}{m! l!},$$

in which  $E_j(0, 0, 0; \lambda; h) = \mathcal{E}_{j, h}(\lambda)$  (degenerate Apostol-Euler numbers) [40] are given by

$$\frac{2}{\lambda(1+ht)^{\frac{1}{h}} + 1} = \sum_{j=0}^{\infty} \mathcal{E}_{j, h}(\lambda) \frac{t^j}{j!}.$$

The first four  $3D\text{-}\Delta_h\text{-Hermite}$  degenerate Apostol-Euler polynomials are as follows.

Table 3: The first four  $3D\text{-}\Delta_h\text{-Hermite}$  degenerate Apostol-Euler polynomials.

$j$	$E_j(x, y, z; \lambda; h)$
0	$\frac{2}{\lambda+1}$
1	$\frac{2}{(\lambda+1)^2} (x + x\lambda - \lambda)$
2	$\frac{2}{(\lambda+1)^3} [(\lambda+1)^2 x^2 - (\lambda+1)(h+2\lambda+h\lambda)x + 2y(\lambda+1)^2 + \lambda^2(1+h) - \lambda(1-h)]$
3	$\frac{2}{(\lambda+1)^4} [(\lambda+1)^3 x^3 - 3(\lambda+1)^2(\lambda+h\lambda+h)x^2 + (\lambda+1)(2(\lambda+1)^2 h^2) + 6\lambda(\lambda+1)h + 6y(\lambda+1)^2 + 3\lambda(\lambda-1)x + (-2h^2 - 3h - 1 - 6y + 6z)\lambda^3 + (-4h^2 + 4 - 12y + 18z)\lambda^2 + (-2h^2 + 3h - 1 - 6y + 18z)\lambda + 6z]$

**Corollary 4.10.**  $3D\text{-}\Delta_h\text{-Hermite}$  degenerate Apostol-Euler polynomials satisfy the following determinantal form

$$E_j(x, y, z; \lambda; h) = (-1)^j \left( \frac{2}{\lambda+1} \right)^{j+1} \begin{vmatrix} 1 & G_1^h & G_2^h & \cdots & G_{j-1}^h & G_j^h \\ \frac{(\lambda+1)}{2} & \frac{\lambda}{2} & \frac{\lambda}{2} (1)_2^h & \cdots & \frac{\lambda}{2} (1)_{j-1}^h & \frac{\lambda}{2} (1)_n^h \\ 0 & \frac{(\lambda+1)}{2} & \binom{2}{1} \frac{\lambda}{2} & \cdots & \binom{j-1}{1} \frac{\lambda}{2} (1)_{j-2}^h & \binom{j}{1} \frac{\lambda}{2} (1)_{j-1}^h \\ 0 & 0 & \frac{(\lambda+1)}{2} & \cdots & \binom{j-1}{2} \frac{\lambda}{2} (1)_{j-3}^h & \binom{j}{2} \frac{\lambda}{2} (1)_{j-2}^h \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(\lambda+1)}{2} & \binom{j}{j-1} \frac{\lambda}{2} \end{vmatrix},$$

where

$$\frac{(1+ht)^{\frac{1}{h}} + 1}{2} = \sum_{s=0}^{\infty} a_{s, h}(\lambda) \frac{t^s}{s!} \quad \text{and} \quad G_j^h = \sum_{s=0}^j \binom{j}{s} a_{s, h}(\lambda) E_{j-s}(x, y, z; \lambda; h).$$

Proof. Taking  $\vartheta(t) = \frac{2}{\lambda(1+ht)^{\frac{1}{h}} + 1}$ , it gives

$$\frac{\lambda(1+ht)^{\frac{1}{h}} + 1}{2} = \sum_{s=0}^{\infty} a_{s,h}(\lambda) \frac{t^s}{s!}$$

from equation (22) and

$$G_j^h = \sum_{s=0}^j \binom{j}{s} a_{s,h}(\lambda) E_{j-s}(x, y, z; \lambda; h)$$

from equation (23). Therefore, these equations are proved by substituting them in equation (21).  $\square$

**Corollary 4.11.** The following recurrence relation is satisfied for 3D- $\Delta_h$ -Hermite degenerate Apostol-Euler polynomials

$$\begin{aligned} & \left(x - \frac{\lambda}{\lambda + 1}\right) E_j(x, y, z; \lambda; h) - \sum_{s=0}^{j-1} \binom{j}{s} (-h)^{j-s} (j-s)! E_s(x, y, z; \lambda; h) \\ & + \frac{1}{2} \sum_{s=0}^{j-1} \sum_{k=0}^{j-s} \binom{j}{s} \binom{j-s}{k} (-h)^k k! \mathcal{E}_{j-k-s,h}(\lambda) E_s(x, y, z; \lambda; h) \\ & + xj! \sum_{s=1}^j (-h)^s \frac{E_{j-s}(x, y, z; \lambda; h)}{(j-s)!} + 2jyE_{j-1}(x, y, z; \lambda; h) \\ & + 2yj! \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s \frac{E_{j-2s-1}(x, y, z; \lambda; h)}{(j-2s-1)!} + 3j(j-1)zE_{j-2}(x, y, z; \lambda; h) \\ & + 3zj! \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} (-h)^s \frac{E_{j-3s-2}(x, y, z; \lambda; h)}{(j-3s-2)!} = E_{j+1}(x, y, z; \lambda; h), \quad j \geq 2 \end{aligned}$$

where

$$\frac{2}{\lambda(1+ht)^{\frac{1}{h}} + 1} = \sum_{j=0}^{\infty} \mathcal{E}_{j,h}(\lambda) \frac{t^j}{j!}$$

Proof. It is proved when  $\frac{2}{\lambda(1+ht)^{\frac{1}{h}} + 1}$  is written in case of  $\vartheta$  in Theorem 2.1.  $\square$

**Corollary 4.12.** The LO, RO and DE satisfied by  $E_j(x, y, z; \lambda; h)$  are given as

$$\begin{aligned} {}_xL_j^- &= \frac{1}{jh} {}_x\Delta_h, \\ {}_xL_j^+ &= x - \frac{\lambda}{\lambda + 1} + \frac{1}{2} \sum_{s=0}^{j-1} \sum_{k=0}^{j-s} (-1)^k \frac{\mathcal{E}_{j-k-s,h}(\lambda)}{(j-k-s)! h^{j-k-s}} {}_x\Delta_h^{j-s} + x \sum_{s=1}^j (-1)^s {}_x\Delta_h^s + 2y \frac{{}_x\Delta_h}{h} + \sum_{s=0}^{j-1} (-1)^{j-s+1} {}_x\Delta_h^{j-s} \\ & + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^s}{h^{s+1}} {}_x\Delta_h^{2s+1} + 3z \frac{{}_x\Delta_h^2}{h^2} + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-1)^s}{h^{2s+2}} {}_x\Delta_h^{3s+2}, \quad j \geq 2, \end{aligned}$$

$$\left[ \left( \frac{x}{h} + 1 - \frac{\lambda}{h(\lambda + 1)} \right) x \Delta_h + \frac{1}{2h} \sum_{s=0}^{j-1} \sum_{k=0}^{j-s} (-1)^k \frac{\mathcal{E}_{j-k-s,h}(\lambda)}{(j-k-s)! h^{j-k-s}} x \Delta_h^{j-s+1} + \left( \frac{x}{h} + 1 \right) \sum_{s=1}^j (-1)^s x \Delta_h^{s+1} \right. \\ \left. + \sum_{s=1}^j (-1)^s x \Delta_h^s + 2y \frac{x \Delta_h^2}{h^2} + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^s}{h^{s+2}} x \Delta_h^{2s+2} + 3z \frac{x \Delta_h^3}{h^3} + \sum_{s=0}^{j-1} (-1)^{j-s+1} \frac{x \Delta_h^{j-s+1}}{h} \right. \\ \left. + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-1)^s}{h^{2s+3}} x \Delta_h^{3s+3} - j \right] E_j(x, y, z; \lambda; h) = 0, \quad j \geq 2.$$

*Proof.* It is proved when  $\frac{2}{\lambda(1+ht)^{\frac{1}{h}+1}}$  is written in case of  $\vartheta$  in Theorem 2.3.  $\square$

#### 4.4. 3D- $\Delta_h$ -Hermite $\lambda$ -Boole polynomials

In this part we derive explicit representation, determinantal form, recurrence relation, LO, RO and DE for 3D- $\Delta_h$ -Hermite  $\lambda$ -Boole polynomials  $\mathcal{Bl}_j(x, y, z; \mu; \lambda; h)$ .

We introduce 3D- $\Delta_h$ -Hermite  $\lambda$ -Boole polynomials via the following generating function

$$\frac{1}{1 + (\lambda + ht)^{\frac{\mu}{h}}} (1 + ht)^{\frac{x}{h}} (1 + ht^2)^{\frac{y}{h}} (1 + ht^3)^{\frac{z}{h}} = \sum_{j=0}^{\infty} \mathcal{Bl}_j(x, y, z; \mu; \lambda; h) \frac{t^j}{j!}.$$

If  $h = 1, y = 0$  and  $z = 0$  in generating function, we give the  $\lambda$ -Boole polynomials with the generating function as follows

$$\frac{1}{1 + (\lambda + t)^\mu} (1 + t)^x = \sum_{j=0}^{\infty} \mathcal{Bl}_j(x; \mu; \lambda) \frac{t^j}{j!}.$$

**Corollary 4.13.** 3D- $\Delta_h$ -Hermite  $\lambda$ -Boole polynomials have the following explicit representation

$$\mathcal{Bl}_j(x, y, z; \mu; \lambda; h) \sum_{s=0}^j \binom{j}{s} \sum_{l=0}^{\lfloor \frac{s}{3} \rfloor} \sum_{m=0}^{\lfloor \frac{s-3l}{2} \rfloor} \mathcal{Bl}_{j-s,h}(\mu; \lambda) \binom{s-3l}{2m} \binom{m}{3l} (x)_{s-2m-3l}^h (y)_m^h (z)_l^h \frac{(2m)!}{m!} \frac{(3l)!}{l!},$$

where the  $\lambda$ -Boole numbers  $\mathcal{Bl}_j(0, 0, 0; \mu; \lambda; h) = \mathcal{Bl}_{j,h}(\mu; \lambda)$  (for  $\lambda = 1$  in [9]), are given by the

$$\frac{1}{1 + (\lambda + ht)^{\frac{\mu}{h}}} = \sum_{j=0}^{\infty} \mathcal{Bl}_{j,h}(\mu; \lambda) \frac{t^j}{j!}.$$

**Corollary 4.14.** 3D- $\Delta_h$ -Hermite  $\lambda$ -Boole polynomials satisfy the determinantal form as follows

$$\mathcal{Bl}_j(x, y, z; \mu; \lambda; h) = \frac{(-1)^j}{\left(\lambda^{\frac{\mu}{h}} + 1\right)^{j+1}} \begin{vmatrix} 1 & G_1^h & G_2^h & \cdots & G_{j-1}^h & G_j^h \\ \lambda^{\frac{\mu}{h}} + 1 & \lambda^{\frac{1}{h}(\mu-h)} (\mu)_1^h & \lambda^{\frac{1}{h}(\mu-2h)} (\mu)_2^h & \cdots & \lambda^{\frac{1}{h}(\mu-(j-1)h)} (\mu)_{j-1}^h & \lambda^{\frac{1}{h}(\mu-jh)} (\mu)_j^h \\ 0 & \lambda^{\frac{\mu}{h}} + 1 & \binom{2}{1} \lambda^{\frac{1}{h}(\mu-h)} (\mu)_1^h & \cdots & \binom{j-1}{1} \lambda^{\frac{1}{h}(\mu-(j-2)h)} (\mu)_{j-2}^h & \binom{j}{1} \lambda^{\frac{1}{h}(\mu-(j-1)h)} (\mu)_{j-1}^h \\ 0 & 0 & \lambda^{\frac{\mu}{h}} + 1 & \cdots & \binom{j-1}{2} \lambda^{\frac{1}{h}(\mu-(j-3)h)} (\mu)_{j-3}^h & \binom{j}{2} \lambda^{\frac{1}{h}(\mu-(j-2)h)} (\mu)_{j-2}^h \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^{\frac{\mu}{h}} + 1 & \binom{j}{j-1} \lambda^{\frac{1}{h}(\mu-h)} (\mu)_1^h \end{vmatrix},$$

where

$$1 + (\lambda + ht)^{\frac{\mu}{h}} = \sum_{s=0}^{\infty} a_{s,h}(\mu; \lambda) \frac{t^s}{s!} \quad \text{and} \quad G_j^h = \sum_{s=0}^j \binom{j}{s} a_{s,h}(\mu; \lambda) \mathcal{Bl}_{j-s}(x, y, z; \mu; \lambda; h).$$

Proof. Taking  $\vartheta(t) = \frac{1}{\lambda+(1+ht)^{\frac{\mu}{h}}}$ , it gives

$$\lambda + (1 + ht)^{\frac{\mu}{h}} = \sum_{s=0}^{\infty} a_{s,h}(\mu; \lambda) \frac{t^s}{s!}$$

from equation (22) and

$$G_j^h = \sum_{s=0}^j \binom{j}{s} a_{s,h}(\mu; \lambda) \mathcal{B}l_{j-s}(x, y, z; \mu; \lambda; h)$$

from equation (23). Therefore, these equations are proved by substituting them in main determinantal form.  $\square$

**Corollary 4.15.**  $3D-\Delta_h$ -Hermite  $\lambda$ -Boole polynomials satisfied the following recurrence relation as follows

$$\begin{aligned} & \left(x - \frac{\mu}{\lambda}\right) \mathcal{B}l_j(x, y, z; \mu; \lambda; h) - \frac{\mu}{\lambda} \sum_{s=0}^{j-1} \binom{j}{s} \frac{(-h)^{j-s}}{\lambda^{j-s}} (j-s)! \mathcal{B}l_s(x, y, z; \mu; \lambda; h) \\ & + \frac{\mu}{\lambda} \sum_{s=0}^j \sum_{k=0}^{j-s} \binom{j}{s} \binom{j-s}{k} \frac{(-h)^k}{\lambda^k} k! \mathcal{B}l_{j-k-s,h}(\mu; \lambda) \mathcal{B}l_s(x, y, z; \mu; \lambda; h) \\ & + xj! \sum_{s=1}^j (-h)^s \frac{\mathcal{B}l_{j-s}(x, y, z; \mu; \lambda; h)}{(j-s)!} + 2jy \mathcal{B}l_{j-1}(x, y, z; \mu; \lambda; h) \\ & + 2yj! \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} (-h)^s \frac{\mathcal{B}l_{j-2s-1}(x, y, z; \mu; \lambda; h)}{(j-2s-1)!} + 3j(j-1)z \mathcal{B}l_{j-2}(x, y, z; \mu; \lambda; h) \\ & + 3zj! \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} (-h)^s \frac{\mathcal{B}l_{j-3s-2}(x, y, z; \mu; \lambda; h)}{(j-3s-2)!} = \mathcal{B}l_{j+1}(x, y, z; \mu; \lambda; h), \quad j \geq 2 \end{aligned}$$

where

$$\frac{1}{1 + (\lambda + ht)^{\frac{\mu}{h}}} = \sum_{j=0}^{\infty} \mathcal{B}l_{j,h}(\mu; \lambda) \frac{t^j}{j!}, \quad \left| \frac{ht}{\lambda} \right| < 1.$$

Proof. It is proved when  $\frac{1}{\lambda+(1+ht)^{\frac{\mu}{h}}}$  is written in case of  $\vartheta$  in Theorem 2.1.  $\square$

**Corollary 4.16.**  $3D-\Delta_h$ -Hermite  $\lambda$ -Boole polynomials satisfy the following LO, RO and DE as follows

$$\begin{aligned} {}_xL_j^- &= \frac{1}{jh} {}_x\Delta_h, \\ {}_xL_j^+ &= x - \frac{\mu}{\lambda} + \frac{\mu}{\lambda} \sum_{s=0}^j \sum_{k=0}^{j-s} \frac{(-1)^k}{\lambda^k} \frac{\mathcal{B}l_{j-k-s,h}(\mu; \lambda)}{(j-s-k)! h^{j-s-k}} {}_x\Delta_h^{j-s} + x \sum_{s=1}^j (-1)^s {}_x\Delta_h^s - \frac{\mu}{\lambda} \sum_{s=0}^{j-1} \frac{(-1)^{j-s}}{\lambda^{j-s}} {}_x\Delta_h^{j-s} \\ & + 2y \frac{{}_x\Delta_h}{h} + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^s}{h^{s+1}} {}_x\Delta_h^{2s+1} + 3z \frac{{}_x\Delta_h^2}{h^2} + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-1)^s}{h^{2s+2}} {}_x\Delta_h^{3s+2}, \quad j \geq 2, \end{aligned}$$

$$\left[ \left( \frac{x}{h} + 1 - \frac{\mu}{\lambda h} \right) x \Delta_h + \frac{\mu}{h\lambda} \sum_{s=0}^j \sum_{k=0}^{j-s} \frac{(-1)^k}{\lambda^k} \frac{\mathcal{B}l_{j-k-s,h}(\mu; \lambda)}{(j-s-k)! h^{j-s-k}} x \Delta_h^{j-s+1} + \left( \frac{x}{h} + 1 \right) \sum_{s=1}^j (-1)^s x \Delta_h^{s+1} \right. \\ \left. - \frac{\mu}{h\lambda} \sum_{s=0}^{j-1} \frac{(-1)^{j-s}}{\lambda^{j-s}} x \Delta_h^{j-s+1} + \sum_{s=1}^j (-1)^s x \Delta_h^s + 2y \frac{x \Delta_h^2}{h^2} + 2y \sum_{s=1}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^s}{h^{s+2}} x \Delta_h^{2s+2} \right. \\ \left. + 3z \frac{x \Delta_h^3}{h^3} + 3z \sum_{s=1}^{\lfloor \frac{j-2}{3} \rfloor} \frac{(-1)^s}{h^{2s+3}} x \Delta_h^{3s+3} - j \right] \mathcal{B}l_{j,\lambda}(x, y, z; \mu; \lambda; h) = 0, \quad j \geq 2.$$

*Proof.* It is proved when  $\frac{1}{\lambda+(\lambda+ht)\frac{\mu}{h}}$  is written in case of  $\vartheta$  in Theorem 2.3.  $\square$

### 5. Approximating operators on $h$ -Hermite polynomials in three variables

In this section, we examine new operators including  $h$ -Hermite polynomials in three variables and give the weighted Korovkin theorem, modulus of continuity and Peetre’s  $K$ -functional for these operators.

Throughout this section let  $n \in \mathbb{N} := \{1, 2, \dots\}$ ,  $\forall x \in [0, \infty)$  and fixed  $y, z \geq 0, h \in (-1, 0)$ . Consider the following linear positive operators:

$$\mathfrak{L}_n(u; x) = \frac{1}{(1+h)^{\frac{nx}{h}} (1+h)^{\frac{y}{h}} (1+h)^{\frac{z}{h}}} \sum_{k=0}^{\infty} \frac{G_k(nx, y, z; h)}{k!} u \left( \frac{k(h+1)}{n} \right), \tag{43}$$

where  $f$  is sufficiently nice function which guaranties the convergence of the above series.

**Lemma 5.1.** *We have the following properties for these operators:*

$$\mathfrak{L}_n(1; x) = 1, \tag{44}$$

$$\mathfrak{L}_n(t; x) = x + \frac{2y + 3z}{n}, \tag{45}$$

$$\mathfrak{L}_n(t^2; x) = x^2 + \frac{1}{n^2} \left[ 4yh \left( \frac{y}{h} - 1 \right) + 12yz + 9zh \left( \frac{z}{h} - 1 \right) \right. \\ \left. + (1+h)(2y+6z) + (1+h)(2y+3z) \right] \\ + \frac{1}{n} [x + 4xy + 6xz], \tag{46}$$

where for each  $x \in [0, \infty)$ .

*Proof.* For  $\mathfrak{L}_n(1; x)$ , if take  $u = 1$  in (43), we get

$$\mathfrak{L}_n(1; x) = \frac{1}{(1+h)^{\frac{nx}{h}} (1+h)^{\frac{y}{h}} (1+h)^{\frac{z}{h}}} \sum_{k=0}^{\infty} \frac{G_k(nx, y, z; h)}{k!}.$$

On the other hand, since  $-1 < h < 0, |-ht| < 1, |-ht^2| < 1, |-ht^3| < 1$ , it is seen that it converges when replacing  $x$  by  $nx$  and taking  $t = 1$  in (13)

$$\sum_{k=0}^{\infty} \frac{G_k(nx, y, z, h)}{k!} = (1+h)^{\frac{nx}{h}} (1+h)^{\frac{y}{h}} (1+h)^{\frac{z}{h}},$$

and thus we get

$$\mathfrak{L}_n(1; x) = \frac{1}{(1+h)^{\frac{nx}{h}} (1+h)^{\frac{y}{h}} (1+h)^{\frac{z}{h}}} (1+h)^{\frac{nx}{h}} (1+h)^{\frac{y}{h}} (1+h)^{\frac{z}{h}} \\ = 1.$$

For  $\mathfrak{Q}_n(t; x)$ , if take  $u = t$  in (43), we get

$$\begin{aligned}\mathfrak{Q}_n(t; x) &= \frac{1}{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}} \sum_{k=0}^{\infty} \frac{G_k(nx, y, z; h) k(h+1)}{k! n} \\ &= \frac{h+1}{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}} n} \sum_{k=0}^{\infty} \frac{G_{k+1}(nx, y, z; h)}{k!}.\end{aligned}$$

Taking derivative with respect to  $t$  on both sides of the generating function, we have

$$\begin{aligned}\sum_{k=0}^{\infty} G_{k+1}(x, y, z; h) \frac{t^k}{k!} &= x(1+ht)^{\frac{x}{h}-1} (1+ht^2)^{\frac{y}{h}} (1+ht^3)^{\frac{z}{h}} \\ &\quad + 2yt(1+ht)^{\frac{x}{h}} (1+ht^2)^{\frac{y}{h}-1} (1+ht^3)^{\frac{z}{h}} \\ &\quad + 3zt^2(1+ht)^{\frac{x}{h}} (1+ht^2)^{\frac{y}{h}} (1+ht^3)^{\frac{z}{h}-1}.\end{aligned}\tag{47}$$

In (47), we replacing  $x$  by  $nx$  and take  $t = 1$ , we get

$$\sum_{k=0}^{\infty} \frac{G_{k+1}(x, y, z; h)}{k!} = (1+h)^{\frac{nx}{h}} (1+h)^{\frac{y}{h}} (1+h)^{\frac{z}{h}} (1+h)^{-1} [nx + 2y + 3z],$$

and thus we have

$$\mathfrak{Q}_n(t; x) = x + \frac{2y + 3z}{n}.$$

For  $\mathfrak{Q}_n(t^2; x)$ , if take  $u = t^2$  in (43), we get

$$\begin{aligned}\mathfrak{Q}_n(t^2; x) &= \frac{(h+1)^2}{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}} n^2} \sum_{k=0}^{\infty} \frac{G_{k+2}(nx, y, z; h)}{k!} \\ &\quad + \frac{(h+1)^2}{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}} n^2} \sum_{k=0}^{\infty} \frac{G_{k+1}(nx, y, z; h)}{k!}.\end{aligned}$$

On the other hand, in (47), taking derivative with respect to  $t$  then replace  $x$  by  $nx$  and  $t = 1$  to obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{G_{k+2}(nx, y, z; h)}{k!} &= \frac{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}}{(1+h)^2} \left[ n^2x^2 - nxh + 4nxy + 6nxz \right. \\ &\quad \left. + 4yh\left(\frac{y}{h} - 1\right) + 12yz + 9zh\left(\frac{z}{h} - 1\right) \right] \\ &\quad + \frac{(1+h)^{\frac{nx}{h}}(1+h)^{\frac{y}{h}}(1+h)^{\frac{z}{h}}}{(1+h)} (2y + 6z),\end{aligned}$$

and thus we get

$$\begin{aligned}\mathfrak{Q}_n(t^2; x) &= x^2 + \frac{1}{n^2} \left[ 4yh\left(\frac{y}{h} - 1\right) + 12yz + 9zh\left(\frac{z}{h} - 1\right) + (1+h)(2y + 6z) \right. \\ &\quad \left. + (1+h)(2y + 3z) \right] + \frac{1}{n} [x + 4xy + 6xz].\end{aligned}$$

Hence the proof is completed.  $\square$

**Lemma 5.2.** We have the following results

$$\mathfrak{Q}_n(t-x; x) = \frac{2y+3z}{n} \quad (48)$$

and

$$\begin{aligned} \mathfrak{Q}_n((t-x)^2; x) &= \frac{1}{n^2} \left[ 4yh \left( \frac{y}{h} - 1 \right) + 12yz + 9zh \left( \frac{z}{h} - 1 \right) + (1+h)(2y+6z) \right. \\ &\quad \left. + (1+h)(2y+3z) \right] + \frac{x}{n}, \end{aligned} \quad (49)$$

where for each  $x \in [0, \infty)$ .

Let  $\varphi(x) = 1 + x^2$ . Then the function spaces  $B_\varphi[0, \infty)$ ,  $C_\varphi[0, \infty)$  and  $C_\varphi^k[0, \infty)$  are defined as follows:

$$B_\varphi[0, \infty) = \{u : [0, \infty) \rightarrow \mathbb{R}, |u(x)| \leq M_u \varphi(x)\},$$

$$C_\varphi[0, \infty) = \{u \in B_\varphi[0, \infty) : u \text{ is continuous}\},$$

$$C_\varphi^k[0, \infty) = \left\{ u \in C_\varphi[0, \infty) : \lim_{x \rightarrow \infty} \frac{u(x)}{\varphi(x)} = k \right\},$$

where  $\varphi(x) = 1 + x^2$  is a weight function and  $k$  and  $M_u$  are constants and the norm on  $B_\varphi[0, \infty)$  is

$$\|u\|_\varphi = \sup_{x \geq 0} \frac{|u(x)|}{\varphi(x)},$$

see [42].

**Lemma 5.3.**  $\mathfrak{Q}_n : C_\varphi \rightarrow B_\varphi$  is a sequence of linear positive operators.

*Proof.* We need to prove that linear positive operators  $\mathfrak{Q}_n$ , from  $C_\varphi$  to  $B_\varphi$ .

$$\begin{aligned} \|\mathfrak{Q}_n(u; x)\|_{\varphi=1+x^2} &= \sup_{x \in [0, \infty)} \frac{|\mathfrak{Q}_n\left(\left(\frac{u}{\varphi}\right)(t)\varphi(t); x\right)|}{\varphi(x)} \\ &\leq \|u\|_\varphi \sup_{x \in [0, \infty)} \frac{\mathfrak{Q}_n(1; x) + \mathfrak{Q}_n(t^2; x)}{1+x^2} \\ &\leq \|u\|_\varphi \sup_{x \in [0, \infty)} \frac{1+x^2+C}{1+x^2} \\ &\leq M \|u\|_\varphi \end{aligned}$$

where

$$\begin{aligned} C &= \frac{1}{n^2} \left[ 4yh \left( \frac{y}{h} - 1 \right) + 12yz + 9zh \left( \frac{z}{h} - 1 \right) + (1+h)(2y+6z) \right. \\ &\quad \left. + (1+h)(2y+3z) \right] + \frac{1}{n} [x + 4xy + 6xz]. \end{aligned}$$

Therefore the given operators are uniformly bounded.  $\square$

**Theorem 5.4.** Let  $\mathfrak{Q}_n : C_\varphi \rightarrow B_\varphi$  linear positive operators and  $\varphi(x) = 1 + x^2$ . If

$$\lim_{n \rightarrow \infty} \left\| \mathfrak{Q}_n(t^i; x) - x^i \right\|_\varphi = 0, \quad i = 0, 1, 2 \quad (50)$$

then for all  $u \in C_\varphi^k$ , we have

$$\lim_{n \rightarrow \infty} \|\mathfrak{Q}_n(u; x) - u\|_\varphi = 0. \quad (51)$$

*Proof.* From (44)-(46), we obtain

$$\|\mathfrak{Q}_n(1; x) - 1\|_\varphi = 0,$$

$$\|\mathfrak{Q}_n(t; x) - x\|_\varphi \leq \frac{1}{n} \rightarrow 0, \text{ as } n \rightarrow \infty,$$

and

$$\left\| \mathfrak{Q}_n(t^2; x) - x^2 \right\|_\varphi \leq \frac{A + Bn}{n^2} \rightarrow 0, \text{ as } n \rightarrow \infty,$$

where

$$A = 4yh \left( \frac{y}{h} - 1 \right) + 12yz + 9zh \left( \frac{z}{h} - 1 \right) + (1+h)(2y+6z) + (1+h)(2y+3z)$$

and

$$B = x + 4xy + 6xz.$$

The proof completed by applying Korovkin's theorem [41] (see also [42],[43]).  $\square$

Let  $u \in \widetilde{C}[0, \infty)$ . Then for  $\zeta > 0$  the modulus of continuity of  $u$  defined by  $\omega(u; \zeta)$  in [44],

$$\omega(u; \zeta) = \sup_{\substack{x, y \in [0, \infty) \\ |x-y| \leq \zeta}} |u(x) - u(y)|$$

where  $\widetilde{C}[0, \infty)$  denotes the space of uniformly continuous functions on  $[0, \infty)$ . Thereafter it is best known that one can write

$$|u(x) - u(y)| \leq \left( 1 + \frac{|x-y|}{\zeta} \right) \omega(u; \zeta). \quad (52)$$

for any  $\zeta > 0$  and each  $x \in [0, \infty)$ .

**Theorem 5.5.** For  $u \in \widetilde{C}[0, \infty)$ , we have

$$|\mathfrak{Q}_n(u; x) - u(x)| \leq 2\omega(u; \zeta), \quad (53)$$

with  $\zeta = \zeta_n(x) = \sqrt{\mathfrak{Q}_n((t-x)^2; x)}$  where  $\mathfrak{Q}_n((t-x)^2; x)$  is obtained in Lemma 5.1.

*Proof.* Using linearity of the operators  $\mathfrak{Q}_n$ , (44) and (52), we get

$$\begin{aligned} |\mathfrak{Q}_n(u; x) - u(x)| &\leq \mathfrak{Q}_n(|u(t) - u(x)|; x) \\ &\leq \left( 1 + \frac{1}{\zeta} \mathfrak{Q}_n(|t-x|; x) \right) \omega(u; \zeta). \end{aligned}$$

According to the Cauchy-Schwarz inequality for  $\mathfrak{Q}_n(|t-x|; x)$ , we obtain that

$$|\mathfrak{Q}_n(u; x) - u(x)| \leq \left( 1 + \frac{1}{\zeta} \sqrt{\mathfrak{Q}_n((t-x)^2; x)} \right) \omega(u; \zeta).$$

If we choose  $\zeta = \zeta_n(x) = \sqrt{\mathfrak{Q}_n((t-x)^2; x)}$ , we get the result.  $\square$

Let

$$C_B^2[0, \infty) = \{v \in C_B[0, \infty) : v', v'' \in C_B[0, \infty)\}, \quad (54)$$

with the norm

$$\|v\|_{C_B^2[0, \infty)} = \|v\|_{C_B[0, \infty)} + \|v'\|_{C_B[0, \infty)} + \|v''\|_{C_B[0, \infty)} \quad (55)$$

also

$$\|v\|_{C_B[0, \infty)} = \sup_{x \in [0, \infty)} |v(x)|, \quad (56)$$

(see [44]).

**Theorem 5.6.** For any  $v \in C_B^2[0, \infty)$ , we have

$$|\mathfrak{Q}_n(v; x) - v(x)| \leq \frac{1}{2} \zeta_n (2 + \zeta_n) \|v\|_{C_B^2[0, \infty)},$$

where

$$\zeta_n = \left[ \frac{1}{n^2} \left( 4yh \left( \frac{y}{h} - 1 \right) + 12yz + 9zh \left( \frac{z}{h} - 1 \right) + (1+h)(2y+6z) + (1+h)(2y+3z) \right) + \frac{x}{n} \right]^{\frac{1}{2}}.$$

*Proof.* From the Taylor's series expansion of the function  $v \in C_B^2[0, \infty)$ , we have

$$v(t) = v(x) + (t-x)v'(x) + \frac{(t-x)^2}{2!}v''(u),$$

where  $u$  between  $x$  and  $t$ , from which it follows

$$|v(t) - v(x)| \leq M_1 |t-x| + \frac{1}{2} M_2 (t-x)^2,$$

where

$$\begin{aligned} M_1 &= \sup_{x \in [0, \infty)} |v'(x)| = \|v'\|_{C_B[0, \infty)} \leq \|v\|_{C_B^2[0, \infty)}, \\ M_2 &= \sup_{x \in [0, \infty)} |v''(x)| = \|v''\|_{C_B[0, \infty)} \leq \|v\|_{C_B^2[0, \infty)} \end{aligned}$$

because of (55). Thus,

$$|v(t) - v(x)| \leq \left( |t-x| + \frac{1}{2} (t-x)^2 \right) \|v\|_{C_B^2[0, \infty)}.$$

Since

$$|\mathfrak{Q}_n(v; x) - v(x)| = |\mathfrak{Q}_n(v(t) - v(x); x)| \leq \mathfrak{Q}_n(|v(t) - v(x)|; x),$$

and  $\mathfrak{Q}_n(|t-x|; x) \leq \left( \mathfrak{Q}_n((t-x)^2; x) \right)^{\frac{1}{2}} = \zeta_n$  we get

$$\begin{aligned} |\mathfrak{Q}_n(v; x) - v(x)| &\leq \left( \mathfrak{Q}_n(|t-x|; x) + \frac{1}{2} \mathfrak{Q}_n((t-x)^2; x) \right) \|v\|_{C_B^2[0, \infty)} \\ &\leq \frac{1}{2} \zeta_n (2 + \zeta_n) \|v\|_{C_B^2[0, \infty)}. \end{aligned}$$

This complete the proof.  $\square$

In the proof of the following theorem in the use relation between Peetre’s  $K$ -functional and the second of the modulus continuity, is defined by

$$\omega_2(u; \tau) := \sup_{0 < t \leq \tau} \|u(\cdot + 2t) - 2u(\cdot + t) + u(\cdot)\|_{C_B[0, \infty)}$$

for  $u \in C_B[0, \infty)$ , which is

$$K(u; \tau) \leq C \left\{ \omega_2(u; \sqrt{\tau}) + \min(1, \tau) \|u\|_{C_B} \right\}, \tag{57}$$

see [44].

**Theorem 5.7.** *If  $u \in C_B[0, \infty)$ , then we obtain*

$$|\mathfrak{Q}_n(u; x) - u(x)| \leq 2M \left\{ \omega_2(u; \sqrt{\tau}) + \min(1, \tau) \|u\|_{C_B} \right\}, \tag{58}$$

where

$$\tau = \tau_n = \frac{1}{2} \zeta_n \tag{59}$$

with  $\zeta$  is same as Theorem 5.6 and  $M > 0$  is a constant which is independent of the function  $u$  and  $\tau$ .

*Proof.* We prove this by using Theorem 5.6. Let  $v \in C_B^2[0, \infty)$ . Since

$$|\mathfrak{Q}_n(u; x) - u(x)| \leq |\mathfrak{Q}_n(u - v; x) - (u - v)(x)| + |\mathfrak{Q}_n(v; x) - v(x)|. \tag{60}$$

For  $|\mathfrak{Q}_n(u - v; x) - (u - v)(x)|$ , we have

$$\begin{aligned} |\mathfrak{Q}_n(u - v; x) - (u - v)(x)| &\leq |\mathfrak{Q}_n(u - v; x)| + |(u - v)(x)| \\ &\leq 2 \|u - v\|_{C_B}. \end{aligned} \tag{61}$$

Using (61) and write into (60), we get

$$\begin{aligned} |\mathfrak{Q}_n(u; x) - u(x)| &\leq 2 \|u - v\|_{C_B} + \frac{1}{2} \zeta_n (2 + \zeta_n) \|v\|_{C_B^2[0, \infty)} \\ &\leq 2 \left( \|u - v\|_{C_B} + \frac{1}{4} \zeta_n (2 + \zeta_n) \|v\|_{C_B^2[0, \infty)} \right) \\ &\leq 2 \inf_{v \in C_B^2[0, \infty)} \left\{ \|u - v\|_{C_B} + \tau \|v\|_{C_B^2} \right\} \\ &\leq 2K(u; \tau), \end{aligned}$$

where  $K(u; \delta)$  is Peetre’s  $K$ -functional defined by (57). Then we have

$$|\mathfrak{Q}_n(u; x) - u(x)| \leq 2M \left\{ \omega_2(f; \sqrt{\tau}) + \min(1, \delta) \|f\|_{C_B} \right\}.$$

□

## 6. Conclusion

$\Delta_h$ -GHAP were defined in [38] and interesting properties of these polynomials containing special polynomials were obtained. In this paper, we studied  $3D$ - $\Delta_h$ -Hermite Appell polynomials. We give some of their properties such as recurrence relation, determinantal form, shift operators, etc. As special cases, we introduce  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Charlier polynomials,  $3D$ - $\Delta_h$ -Hermite degenerate Apostol-Bernoulli polynomials,  $3D$ - $\Delta_h$ -Hermite degenerate Apostol-Euler polynomials and  $3D$ - $\Delta_h$ -Hermite  $\lambda$ -Boole polynomials. We exhibit certain of their properties such as explicit and determinantal forms, recurrence relation, raising and lowering operators and difference equation.

It can be noted that the  $r$ -variable case can be defined via generating relation

$$\vartheta(t) \prod_{i=1}^r (1 + ht^i)^{\frac{x_i}{h}} = \sum_{j=0}^{\infty} \mathcal{A}_j(x_1, x_2, \dots, x_r) \frac{t^j}{j!}, \quad |ht^i| < 1, \quad (i = 0, 1, \dots, r). \quad (62)$$

Starting from this definition and their properties, it is possible to derive families with  $r$  variables. Its properties such as explicit representation, determinantal form, recurrence relation, summation formulas and shift operators can be investigated. In addition to these, more  $r$ -variable polynomials can be defined as special cases of  $\mathcal{A}_j(x_1, x_2, \dots, x_r)$  which may have some potential applications in number theory, special function theory and approximation theory.

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