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A parametric unified Apostol-type Bernoulli, Euler, Genocchi, Fubini polynomials and numbers

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Abstract. In recent years, mathemacians ([1], [3], [5], [22], [23]) introduced and investigated the Fubini Apostol-type numbers and polynomials. They gave some recurrence relations explicit properties and identities for these polynomials. In [12], author considered unified degenerate Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials and gave some relations and identities for these polynomials.

In this article, we consider a parametric unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials. By using the monomiality principle, we give some relations for the parametric unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials. Furthermore, we give summation formula for these polynomials.

1. Introduction, Definitions and Notations

Special polynomials and numbers have significant roles in various branches of mathematics and theorical physics. The Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and numbers investigated by Luo ([14]-[16]). Also, he established several elementary properties and gave some explicit relationships and derived various explicit series representations for these polynomials. Srivastava in ([28], [29], [31]) investigated Apostol-Bernoulli, Euler and Genocchi polynomials including explicit representations in terms of a certain generalized Hurwitz-Lerch zeta function. Also, Srivastava et. al. ([23], [30]) considered a parametric type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials.

Furthermore, Srivastava et. al. in [30] introduced and proved some relations for the three-variable unified Apostol-type q-polynomials. Masjed-Jamai et. al. ([6], [17], [18]) introduced and investigated a parametric Bernoulli, Euler and Genocchi polynomials. Ozarslan [19] presented a unified study of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and obtained the explicit relations. Dattali et al. [2] considered monomiality princple for the Appell polynomials. Using monomialty princple, Khan et al. in ([8]-[10]) presented certain result for the 2-variable Apostol-type and related polynomials. Acala [1] introduced a unification of the generalized multiparameter Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials of higher order. The authors [12] considered and investigated unified degenerate Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials.

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As usual notations, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ for the set of natural integers, integers, real numbers and complex numbers, respectively. Also, we let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{Z}^- = \{-1, -2, \cdots\}$. We begin by introducing the following definition and notation (see also [13]-[24]). The generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{N}_0$, the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{N}_0$ and the generalized Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{N}_0$ are defined by the following generating function ([13]-[16]), respectively

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt},$$

$$(|t| < 2\pi \text{ when } \lambda = 1;\ |t| < |\ln \lambda| \text{ when } \lambda \neq 1), \tag{1}$$

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt},$$

$$(|t| < \pi \text{ when } \lambda = 1;\ |t| < |\ln(-\lambda)| \text{ when } \lambda \neq 1) \tag{2}$$

and

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt},$$

$$(|t| < \pi \text{ when } \lambda = 1;\ |t| < |\ln(-\lambda)| \text{ when } \lambda \neq 1). \tag{3}$$

When $x = 0$ in (1), (2) and (3), we have the Apostol-Bernoulli numbers $\mathcal{B}_n^{(\alpha)}(0;\lambda) := \mathcal{B}_n^{(\alpha)}(\lambda)$, the Apostol-Euler numbers $\mathcal{E}_n^{(\alpha)}(0;\lambda) := \mathcal{E}_n^{(\alpha)}(\lambda)$ and the Apostol-Genocchi numbers $\mathcal{G}_n^{(\alpha)}(0;\lambda) := \mathcal{G}_n^{(\alpha)}(\lambda)$, respectively. Two variable Fubini polynomials of order α are defined in ([7], [12], [23]) as follows

$$\frac{e^{xt}}{[1 - y(e^t - 1)]^{\alpha}} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x,y)\frac{t^n}{n!}. \tag{4}$$

When $\alpha = 1$, $\mathcal{F}_n^{(1)}(x,y) = F_n(x,y)$ the two-variable Fubini polynomials. Moreover setting $x = 0$ and $y = 1$ in (4), we obtain the Fubini numbers $\mathcal{F}_n^{(\alpha)}(1) := \mathcal{F}_n^{(\alpha)}$ of order $\alpha \in \mathbb{N}$.

Acala [1] defined the generalized multiparameter Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials as follows

$$\sum_{n=0}^{\infty} \mathcal{F}_{n,k}^{(\alpha)}(x,y;a,b,c,\lambda)\frac{t^n}{n!} = \left(\frac{a^{-t}t^k}{1 - y\left(\lambda\left(\frac{b}{a}\right)^t - 1\right)}\right)^{\alpha} c^{xt} \tag{5}$$

$$\left(\left|t\log\left(\frac{b}{a}\right) + \ln\left(\frac{\lambda y}{y+1}\right)\right| < 2\pi;\ \alpha \in \mathbb{C};\ a, b, c \in \mathbb{R}^+,\ x, y \in \mathbb{R},\ k \in \mathbb{N}_0,\ 1^{\alpha} := 1\right).$$

The Stirling numbers of the second kind were defined by Srivastava et al. [25] as follows

$$\sum_{n=0}^{\infty} S(n,\nu,a,b,\beta)\frac{t^n}{n!} = \frac{\left(\beta^b e^t - a^b\right)^{\nu}}{\nu!} \tag{6}$$

where $\nu \in \mathbb{N}$, $a, b, \beta \in \mathbb{R}$, $a \neq b$.

Author [12] defined the higher-order unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials $\mathcal{F}_n^{(\alpha)}(x,y;a,b,\beta,k)$ of order $\alpha \in \mathbb{N}$, by means of the following generating function

$$\sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x,y;a,b,\beta,k)\frac{t^n}{n!} = \left(\frac{t^k}{1 - y\left(\beta^b e^t - a^b\right)}\right)^{\alpha} e^{xt} \tag{7}$$

$$\left(\left|t+\ln\left(\frac{y\beta^b}{1+ya^b}\right)\right|<2\pi; a, b\in\mathbb{R}^+, x, y\in\mathbb{R}, k, \alpha\in\mathbb{N}_0\right).$$

For x, $y \in \mathbb{R}$, the Taylor-Maclaurin expansions of the two functions $e^{xt}\cos(yt)$ and $e^{xt}\sin(yt)$ are given, respectively, by (see [24], [30])

$$e^{xt}\cos(yt)=\sum_{n=0}^{\infty}C_n(x,y)\frac{t^n}{n!} \tag{8}$$

and

$$e^{xt}\sin(yt)=\sum_{n=0}^{\infty}S_n(x,y)\frac{t^n}{n!} \tag{9}$$

where the functions $C_n(x,y)$ and $S_n(x,y)$ are defined as

$$C_n(x,y)=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{2k}x^{n-2k}y^{2k} \tag{10}$$

and

$$S_n(x,y)=\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}x^{n-2k-1}y^{2k+1}. \tag{11}$$

In [24], Srivastava H. M., Masjed-Jamei M. and Beyki M. R. introduced two parameters the Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order $\alpha\in\mathbb{N}_0$, the Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order $\alpha\in\mathbb{N}_0$ and the Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ of order $\alpha\in\mathbb{N}_0$ in [23] Srivastava and Kizilates defined the parametric kinds of above functions as

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}e^{xt}\cos(yt)=\sum_{n=0}^{\infty}\mathcal{B}_n^{(c,\alpha)}(x,y;\lambda)\frac{t^n}{n!} \tag{12}$$

and

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}e^{xt}\sin(yt)=\sum_{n=0}^{\infty}\mathcal{B}_n^{(s,\alpha)}(x,y;\lambda)\frac{t^n}{n!}; \tag{13}$$

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}e^{xt}\cos(yt)=\sum_{n=0}^{\infty}\mathcal{E}_n^{(c,\alpha)}(x,y;\lambda)\frac{t^n}{n!} \tag{14}$$

and

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}e^{xt}\sin(yt)=\sum_{n=0}^{\infty}\mathcal{E}_n^{(s,\alpha)}(x,y;\lambda)\frac{t^n}{n!}; \tag{15}$$

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}e^{xt}\cos(yt)=\sum_{n=0}^{\infty}\mathcal{G}_n^{(c,\alpha)}(x,y;\lambda)\frac{t^n}{n!} \tag{16}$$

and

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}e^{xt}\sin(yt)=\sum_{n=0}^{\infty}\mathcal{G}_n^{(s,\alpha)}(x,y;\lambda)\frac{t^n}{n!}, \tag{17}$$

respectively.

From the equations (12)-(17), we define two parametric kinds of the unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials as

$$\left(\frac{t^k}{1-y\left(\beta^b e^t - a^b\right)}\right)^{\alpha} e^{xt}\cos(ut) = \sum_{n=0}^{\infty} \mathcal{F}_n^{(c,\alpha)}\left(x,y,u;a,b,\beta,k\right)\frac{t^n}{n!} \tag{18}$$

and

$$\left(\frac{t^k}{1-y\left(\beta^b e^t - a^b\right)}\right)^{\alpha} e^{xt}\sin(ut) = \sum_{n=0}^{\infty} \mathcal{F}_n^{(s,\alpha)}\left(x,y,u;a,b,\beta,k\right)\frac{t^n}{n!}. \tag{19}$$

Some general polynomials have been described with the help of monomiality principle Khan et al. ([8], [9]) defined the 2-variable general polynomials ($2VGP$) $G_n(x,y)$ are defined by

$$e^{xt}\varphi(y,t) = \sum_{n=0}^{\infty} G_n(x,y)\frac{t^n}{n!}, \ G_0(x,y) \neq 1. \tag{20}$$

The $2VGP$ $G_n(x,y)$ are quasi-monimal ([8], [9]) with respect to the following multiplicative and derivative operators;

$$\widehat{M} = x + \frac{\varphi'\left(y,D_x\right)}{\varphi\left(y,D_x\right)}, \ \left(D_x := \frac{d}{dx}; \varphi'\left(y,t\right) = \frac{d}{dt}\varphi\left(y,t\right)\right) \tag{21}$$

and

$$\widehat{P_G} = D_x. \tag{22}$$

According to monomiality principle, the polynomials $G_n(x,y)$ satisfy the following relations

$$\widehat{M}_G\left\{G_n(x,y)\right\} = G_{n+1}(x,y), \tag{23}$$

$$\widehat{P_G}\left\{G_n(x,y)\right\} = nG_{n-1}(x,y), \tag{24}$$

$$\widehat{M}_G\,\widehat{P_G}\left\{G_n(x,y)\right\} = nG_n(x,y) \tag{25}$$

and

$$\exp\left(t\,\widehat{M}_G\right)\{1\} = \sum_{n=0}^{\infty} G_n(x,y)\frac{t^n}{n!}, \ |t| < \infty. \tag{26}$$

2. The 3-Variable Unified Apostol-type Bernoulli, Euler, Genocchi and Fubini Polynomials

In this section, we consider the 3-variable unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials ($3VUABEGFP$). Using the quasi-monomiality principle, we give some identities and related relations for these polynomials.

We define the 3-variable generalized unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials $\mathcal{F}_n^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)$ of order $\alpha \in \mathbb{N}_0$ as

$$\sum_{n=0}^{\infty} {}_G\mathcal{F}_n^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)\frac{t^n}{n!} = \left(\frac{t^k}{1-y\left(\beta^b e^t - a^b\right)}\right)^{\alpha} e^{xt}\varphi\left(z,t\right) \tag{27}$$

$$\left(\left|t + \ln\left(\frac{y\beta^b}{1+ya^b}\right)\right| < 2\pi; a, b \in \mathbb{R}^+, x, y \in \mathbb{R}, k, \alpha \in \mathbb{N}, \beta \in \mathbb{R}\right).$$

For $\alpha = 1$, we simply denote ${}_G\mathcal{F}_n^{(1)}\left(x,y,z;a,b,\beta,k\right)$ by ${}_G\mathcal{F}_n\left(x,y,z;a,b,\beta,k\right)$.

Remark 2.1. *Setting $k=\alpha=1$, $y=-2$, $a=b=1$ and $\beta=\frac{\lambda}{2}$ in (27), we get*

$$ {}_G\mathcal{F}_n^{(1)}\left(x,-2,z;1,1,\frac{\lambda}{2},1\right)=\mathcal{B}_n\left(x,z;\lambda\right). $$

Remark 2.2. *Choosing $k=0$, $\alpha=1$, $y=-\frac{1}{2}$, $a=b=1$ and $\beta=\lambda$ in (27), we get*

$$ {}_G\mathcal{F}_n^{(1)}\left(x,-\frac{1}{2},z;1,1,\lambda,0\right)=\mathcal{E}_n\left(x,z;\lambda\right). $$

Remark 2.3. *Letting $k=1$, $\alpha=1$, $y=-\frac{1}{2}$, $a=b=1$ and $\beta=\lambda$ in (27), we get*

$$ {}_G\mathcal{F}_n^{(1)}\left(x,-\frac{1}{2},z;1,1,\lambda,1\right)=\mathcal{G}_n\left(x,z;\lambda\right). $$

Remark 2.4. *Setting $k=0$, $\alpha=1$, $a=b=1$ and $\beta=\lambda$ in (27), we get*

$$ {}_G\mathcal{F}_n^{(1)}\left(x,y,z;1,1,\lambda,0\right)=\mathcal{F}_n\left(x,y,z;\lambda\right). $$

Theorem 2.5. *The 3-variable generalized unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials are quasi-monomial with respect to the following multiplicative and derivative operators*

$$ \widehat{M}_{{}_G\mathcal{F}_n^{(\alpha)}}=x+\alpha\left(\frac{k}{D_x}-\frac{y\beta^b e^{D_x}}{1-y\left(\beta^b e^{D_x}-a^b\right)}\right)+\frac{\varphi^{\prime}\left(z,\ D_x\right)}{\varphi\left(z,D_x\right)} \tag{28} $$

and

$$ \widehat{P}_{{}_G\mathcal{F}_n^{(\alpha)}}=D_x. \tag{29} $$

Proof. Differentiating Eq. (27) partially with respect to t gives

$$ \begin{aligned} &\sum_{n=0}^{\infty}{}_G\mathcal{F}_{n+1}^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\left\{x+\alpha\left(\frac{k}{t}+\frac{y\beta^b e^t}{1-y\left(\beta^b e^t-a^b\right)}\right)+\frac{\varphi^{\prime}\left(z,D_x\right)}{\varphi\left(z,D_x\right)}\right\}{}_G\mathcal{F}_n^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)\frac{t^n}{n!}. \end{aligned} \tag{30} $$

If $\varphi\left(z,t\right)$ is an invertible series and $\frac{\varphi^{\prime}(z,t)}{\varphi(z,t)}$ has Taylor series expansion in powers of t, we write as

$$ D_x\left(e^{xt}\varphi\left(z,t\right)\right)=t\left(e^{xt}\varphi\left(z,t\right)\right). \tag{31} $$

From (28) and (29), we can write the equation (30) as

$$ \begin{aligned} &\sum_{n=0}^{\infty}{}_G\mathcal{F}_{n+1}^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\left\{x+\alpha\left(\frac{k}{D_x}+\frac{y\beta^b e^{D_x}}{1-y\left(\beta^b e^{D_x}-a^b\right)}\right)+\frac{\varphi^{\prime}\left(z,D_x\right)}{\varphi\left(z,D_x\right)}\right\}{}_G\mathcal{F}_n^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)\frac{t^n}{n!}. \end{aligned} $$

From the last equaations, we have result.

By using (31), differentiating Eq. (27) partially with respect to x gives

$$ \begin{aligned} &D_x\left\{\sum_{n=0}^{\infty}{}_G\mathcal{F}_n^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)\frac{t^n}{n!}\right\}\\ =&\sum_{n=1}^{\infty}{}_G\mathcal{F}_{n-1}^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)\frac{t^n}{(n-1)!}. \end{aligned} \tag{32} $$

Using (24) and (32), we get (29). □

Theorem 2.6. *The (3VGUABEGFP) ${}_G\mathcal{F}_n^{(\alpha)}(x,y,z;a,b,\beta,k)$ satisfy the following differential equation*

$$\left(x\,D_x+\alpha\left(\frac{k}{D_x}-\frac{y\beta^b e^{D_x}}{1-y\left(\beta^b e^{D_x}-a^b\right)}\right)D_x+\frac{\varphi'(z,D_x)}{\varphi(z,D_x)}D_x-n\right){}_G\mathcal{F}_n^{(\alpha)}(x,y,z;a,b,\beta,k)=0. \tag{33}$$

Proof. Using the operators (28) and (29) and (25), we get (33). □

Theorem 2.7. *The following relation holds true:*

$$\begin{aligned}&{}_G\mathcal{F}_n^{(\alpha)}(x+u,y,z;a,b,\beta,k)\\ =&\sum_{m=0}^{n}\binom{n}{m}{}_G\mathcal{F}_{n-m}^{(\alpha)}(x,y,z;a,b,\beta,k)\ u^m.\end{aligned} \tag{34}$$

The proof of this theorem is easy prove from (27).

Theorem 2.8. *There is the following relation between the (3VGUABEGFP) and the generalized Stirling numbers of the second kinds*

$$\begin{aligned}&{}_G\mathcal{F}_n^{(\alpha)}(x,y,z;a,b,\beta,k)\\ =&\sum_{r=0}^{n-k\alpha}\binom{n-k\alpha}{r}(n+1-k\alpha)(n+2-k\alpha)\cdots n\sum_{l=0}^{\infty}\binom{l+\alpha-1}{l}\\ &\times l!y^l S(r,l,a,b,\beta)\ G_{n-k\alpha-r}(x,z),\end{aligned} \tag{35}$$

where $n\geq k\alpha$.

Proof. From (6), (20) and (27), we write as

$$\begin{aligned}\sum_{n=0}^{\infty}{}_G\mathcal{F}_n^{(\alpha)}(x,y,z;a,b,\beta,k)\frac{t^n}{n!}&=t^{\alpha k}\left[1-y\left(\beta^b e^t-a^b\right)\right]^{(-\alpha)}e^{xt}\varphi(z,t)\\ &=t^{\alpha k}\sum_{l=0}^{\infty}\binom{l+\alpha-1}{l}l!y^l\sum_{r=0}^{\infty}S(r,l,a,b,\beta)\frac{t^r}{r!}\sum_{m=0}^{\infty}G_m(x,z)\frac{t^m}{m!}\\ &=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\binom{l+\alpha-1}{l}l!y^l\sum_{r=0}^{n}\binom{n}{r}S(r,l,a,b,\beta)G_{n-r}(x,z)\frac{t^{n+r}}{n!}\end{aligned}$$

$$\begin{aligned}=&\sum_{n=k\alpha}^{\infty}\sum_{r=0}^{n-k\alpha}\binom{n-k\alpha}{r}(n+1-k\alpha)(n+2-k\alpha)\cdots n\sum_{l=0}^{\infty}\binom{l+\alpha-1}{l}\\ &\times l!S(r,l,a,b,\beta)G_{n-k\alpha-r}(x,z)\frac{t^n}{n!}.\end{aligned} \tag{36}$$

Comparing the coefficients of the both sides of equation (36), we have (35). □

Theorem 2.9. *The (3VGUABEGFP) ${}_G\mathcal{F}_n^{(\alpha)}(x,y,z;a,b,\beta,k)$ satisfy the following summation formulas*

$$\begin{aligned}&{}_G\mathcal{F}_{n+m}^{(\alpha)}(\nu,y,z;a,b,\beta,k)\\ =&\sum_{p=0}^{n}\sum_{q=0}^{m}\binom{n}{p}\binom{m}{q}(\nu-x)^{p+q}\ {}_G\mathcal{F}_{n+m-p-q}^{(\alpha)}(x,y,z;a,b,\beta,k).\end{aligned} \tag{37}$$

Proof. We first replace t by $t+u$ in (27). We then rewrite the generating function (27) as follows

$$\begin{aligned} & \left(\frac{(t+u)^k}{1-y\left(\beta^b e^{t+u}-a^b\right)}\right)^{\alpha} e^{x(t+u)}\varphi\left(z,t+u\right) \\ = & \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_G\mathcal{F}_{n+m}^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)\frac{t^n}{n!}\frac{u^m}{m!}. \end{aligned}$$

Upon replacing x by ν in the above equation, we get

$$\begin{aligned} & \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_G\mathcal{F}_{n+m}^{(\alpha)}\left(\nu,y,z;a,b,\beta,k\right)\frac{t^n}{n!}\frac{u^m}{m!} \\ = & e^{(\nu-x)(t+u)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_G\mathcal{F}_{n+m}^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)\frac{t^n}{n!}\frac{u^m}{m!}. \end{aligned}$$

Which, upon expanding the exponential equation, yields

$$\begin{aligned} & \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_G\mathcal{F}_{n+m}^{(\alpha)}\left(\nu,y,z;a,b,\beta,k\right)\frac{t^n}{n!}\frac{u^m}{m!} \\ = & \sum_{N=0}^{\infty}\frac{\left[(\nu-x)(t+u)\right]^N}{N!}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_G\mathcal{F}_{n+m}^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)\frac{t^n}{n!}\frac{u^m}{m!}. \end{aligned} \tag{38}$$

Now, by applying the following known series identity ([26],p. 52, Eq. 1.6 (2))

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \tag{39}$$

in the right hand side of (38), we get

$$\begin{aligned} & \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(\nu-x)^{p+q}\frac{t^p}{p!}\frac{u^q}{q!}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_G\mathcal{F}_{n+m}^{(\alpha)}\left(x,y,z;a,b,\beta,k\right)\frac{t^n}{n!}\frac{u^m}{m!} \\ = & \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_G\mathcal{F}_{n+m}^{(\alpha)}\left(\nu,y,z;a,b,\beta,k\right)\frac{t^n}{n!}\frac{u^m}{m!}. \end{aligned} \tag{40}$$

Finally, upon first replacing n by $n-p$ and m by $m-q$ and then applying a known result ([26, p. 100, Lemma 1, Eq. 2.1 (1)]) by using Cauchy product in the left-hand side of the above equation (40) and comparing the coefficients of $\frac{t^n}{n!}$ and $\frac{u^m}{m!}$ on both sides of the resulting equation. We complete our demonstration of the assertion (37) Theorem 5. □

3. A Parametric Generalized Unified Apostol-type Bernoulli, Euler, Genocchi and Fubini Polynomials

In this section, by using monomiality principle and operational methods. We introduce and investigate the parametric generalized unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials. We give some identities and relations for these polynomials.

Theorem 3.1. *The generating functions for the parametric unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials* ${}_G\mathcal{F}_n^{(c,\alpha)}\left(x,y,z,u;a,b,\beta,k\right)$ *and* ${}_G\mathcal{F}_n^{(s,\alpha)}\left(x,y,z,u;a,b,\beta,k\right)$ *are given as follows*

$$\sum_{n=0}^{\infty} {}_G\mathcal{F}_n^{(c,\alpha)}\left(x,y,z,u;a,b,\beta,k\right)\frac{t^n}{n!}=\left(\frac{t^k}{1-y\left(\beta^b e^t-a^b\right)}\right)^{\alpha} e^{xt}\varphi\left(z,t\right)\cos\left(ut\right) \tag{41}$$

and

$$\sum_{n=0}^{\infty} {}_G\mathcal{F}_n^{(s,\alpha)}(x,y,z,u;a,b,\beta,k)\frac{t^n}{n!} = \left(\frac{t^k}{1-y\left(\beta^b e^t - a^b\right)}\right)^{\alpha} e^{xt}\varphi(z,t)\sin(ut), \tag{42}$$

$$\left(\left|t+\ln\left(\frac{y\beta^b}{1+ya^b}\right)\right| < 2\pi,\, a,\, b\in\mathbb{R}^+,\, x,\, y\in\mathbb{R},\, k,\, \alpha\in\mathbb{N},\, \beta\in\mathbb{R}\right)$$

respectively.

Proof. In Eq. (18) replacing x and u by the multiplicative operator $\widehat{M}_G$ of the $2VGP$ $G_n(x,y)$ and u respectively, gives

$$\sum_{n=0}^{\infty} \mathcal{F}_n^{(c,\nu)}\left(x+\frac{\varphi'(z,D_x)}{\varphi(z,D_x)},u\right)\frac{t^n}{n!} = \left(\frac{t^k}{1-y\left(\beta^b e^t - a^b\right)}\right)^{\alpha} e^{xt}\varphi(z,t)\cos(ut).$$

Using the equation (41), we have

$${}_G\mathcal{F}_n^{(c,\nu)}(x,y,z,u;a,b,\beta,k) = \mathcal{F}_n^{(c,\nu)}\left(x+\frac{\varphi'(z,D_x)}{\varphi(z,D_x)},u\right). \tag{43}$$

Similiarly, we can prove the assertion in Eq. (42). □

Theorem 3.2. *The parametric unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials* ${}_G\mathcal{F}_n^{(c,\alpha)}(x,y,z,u;a,b,\beta,k)$ *and* ${}_G\mathcal{F}_n^{(s,\alpha)}(x,y,z,u;a,b,\beta,k)$ *are quasi-monomial with respect to the following multiplicative and derivate operators:*

$$\widehat{M}\,{}_G\mathcal{F}_n^{(c,\alpha)} = x+\alpha\left[\frac{k}{D_x}+\frac{y\beta^b e^{D_x}}{1-y\left(\beta^b e^{D_x}-a^b\right)}\right]+\frac{\varphi'(z,D_x)}{\varphi(z,D_x)}-u\tan(uD_x) \tag{44}$$

and

$$\widehat{P}\,{}_G\mathcal{F}_n^{(c)} = D_x; \tag{45}$$

$$\widehat{M}\,{}_G\mathcal{F}_n^{(s,\alpha)} = x+\alpha\left[\frac{k}{D_x}+\frac{y\beta^b e^{D_x}}{1-y\left(\beta^b e^{D_x}-a^b\right)}\right]+\frac{\varphi'(z,D_x)}{\varphi(z,D_x)}+u\cot(uD_x) \tag{46}$$

and

$$\widehat{P}\,{}_G\mathcal{F}_n^{(s)} = D_x \tag{47}$$

respectively.

Proof. Differentiating Eq. (41) partially with respect to t gives to

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_G\mathcal{F}_{n+1}^{(c,\alpha)}(x,y,z,u;a,b,\beta,k)\frac{t^n}{n!}\\ =\ &\sum_{n=0}^{\infty}\left\{x+\alpha\left(\frac{k}{t}+\frac{y\beta^b e^t}{1-y\left(\beta^b e^t-a^b\right)}\right)+\frac{\varphi'(z,t)}{\varphi(z,t)}-u\tan(ut)\right\}\\ &\times {}_G\mathcal{F}_n^{(c,\alpha)}(x,y,z,u;a,b,\beta,k)\frac{t^n}{n!}.\end{aligned} \tag{48}$$

Applying the identity

$$D_x\left(e^{xt}\varphi(z,t)\right) = t\left(e^{xt}\varphi(z,t)\right) \tag{49}$$

in Eq. (48) and comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get

$$\begin{aligned}
& {}_G\mathcal{F}_{n+1}^{(c,\alpha)}(x,y,z,u;a,b,\beta,k) \\
= & \left\{ x+\alpha\left(\frac{k}{D_x}+\frac{y\beta^b e^{D_x}}{1-y\left(\beta^b e^{D_x}-a^b\right)}\right)+\frac{\varphi^{\prime}(z,D_x)}{\varphi(z,D_x)}-u\tan(uD_x)\right\} \\
& \times {}_G\mathcal{F}_{n}^{(c,\alpha)}(x,y,z,u;a,b,\beta,k).
\end{aligned}$$

Which in view of the monomiality principle given in Eq. (23) mean the assertion in the equation (44). Differentiating on both sides of the equation (41) with respect to x gives

$$\begin{aligned}
& D_x\left\{\sum_{n=0}^{\infty} {}_G\mathcal{F}_{n}^{(c,\nu)}(x,y,z,u;a,b,\beta,k)\frac{t^n}{n!}\right\} \\
= & \sum_{n=1}^{\infty} {}_G\mathcal{F}_{n-1}^{(c,\nu)}(x,y,z,u;a,b,\beta,k)\frac{t^n}{(n-1)!}.
\end{aligned} \tag{50}$$

When comparing the coefficients of t^n on both sides, we obtain the monomiality principle property given in Eq. (24) for the equation (45).

Also, using similiar arguments as the proof of Eq. (44) and (45). We can prove equations (46) and (47). □

Theorem 3.3. *The parametric unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials* ${}_G\mathcal{F}_{n}^{(c,\alpha)}(x,y,z,u;a,b,\beta,k)$ *and* ${}_G\mathcal{F}_{n}^{(s,\alpha)}(x,y,z,u;a,b,\beta,k)$ *satisfy the following differential equations*

$$\begin{aligned}
& \left\{xD_x+\alpha\left(\frac{k}{D_x}+\frac{y\beta^b e^{D_x}}{1-y\left(\beta^b e^{D_x}-a^b\right)}\right)D_x+\frac{\varphi^{\prime}(z,D_x)}{\varphi(z,D_x)}D_x-u\tan(uD_x)D_x-n\right\} \\
& \times {}_G\mathcal{F}_{n}^{(c,\nu)}(x,y,z,u;a,b,\beta,k) \\
= & 0
\end{aligned} \tag{51}$$

and

$$\begin{aligned}
& \left\{xD_x+\alpha\left(\frac{k}{D_x}+\frac{y\beta^b e^{D_x}}{1-y\left(\beta^b e^{D_x}-a^b\right)}\right)D_x+\frac{\varphi^{\prime}(z,D_x)}{\varphi(z,D_x)}D_x+u\cot(uD_x)D_x-n\right\} \\
& \times {}_G\mathcal{F}_{n}^{(s,\nu)}(x,y,z,u;a,b,\beta,k) \\
= & 0
\end{aligned} \tag{52}$$

respectively.

Proof. Using operators (44) and (45) and employing the monomialty principle in the equation (25).

We get (51), similiarly, we can prove equation (52). □

Theorem 3.4. *The following relations hold true:*

$$\begin{aligned}
& {}_G\mathcal{F}_{n}^{(c,\alpha)}(x+x_1,y,z,u+u_1;a,b,\beta,k) \\
= & \sum_{r=0}^{n}\binom{n}{r}\left\{{}_G\mathcal{F}_{n-r}^{(c,\alpha)}(x,y,z,u;a,b,\beta,k)C_r(x_1,u_1)\right. \\
& \left. -{}_G\mathcal{F}_{n-r}^{(s,\alpha)}(x_1,y,z,u_1;a,b,\beta,k)S_r(x,u)\right\}
\end{aligned} \tag{53}$$

and

$${}_G\mathcal{F}_{n}^{(s,\alpha)}(x+x_1,y,z,u+u_1;a,b,\beta,k)$$

$$= \sum_{r=0}^{n} \binom{n}{r} \Big\{ {}_G\mathcal{F}_{n-r}^{(s,\alpha)}(x,y,z,u;a,b,\beta,k)\, C_r(x_1,u_1) + {}_G\mathcal{F}_{n-r}^{(c,\alpha)}(x,y,z,u;a,b,\beta,k)\, S_r(x_1,u_1) \Big\}. \quad (54)$$

Proof. From (41), (8) and (9), we write as

$$\sum_{n=0}^{\infty} {}_G\mathcal{F}_n^{(c,\alpha)}(x+x_1,y,z,u+u_1;a,b,\beta,k)\frac{t^n}{n!}$$

$$= \sum_{m=0}^{\infty} {}_G\mathcal{F}_m^{(c,\alpha)}(x,y,z,u;a,b,\beta,k)\frac{t^m}{m!}\sum_{r=0}^{\infty} C_r(x_1,u_1)\frac{t^r}{r!} - \sum_{m=0}^{\infty} {}_G\mathcal{F}_m^{(s,\alpha)}(x_1,y,z,u_1;a,b,\beta,k)\frac{t^m}{m!}\sum_{r=0}^{\infty} S_r(x,u)\frac{t^r}{r!}.$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$, we have (53).
Similarly, from (42), (8) and (9), we have (54). □

Corollary 3.5. *For $x = x_1$ and $u = u_1$ in (53) and (54), we have*

$$ {}_G\mathcal{F}_n^{(c,\alpha)}(2x,y,z,2u;a,b,\beta,k) = \sum_{r=0}^{n} \binom{n}{r} \Big\{ {}_G\mathcal{F}_{n-r}^{(c,\alpha)}(x,y,z,u;a,b,\beta,k)\, C_r(x,u) - {}_G\mathcal{F}_{n-r}^{(s,\alpha)}(x,y,z,u;a,b,\beta,k)\, S_r(x,u) \Big\}$$

and

$$ {}_G\mathcal{F}_n^{(s,\alpha)}(2x,y,z,2u;a,b,\beta,k) = \sum_{r=0}^{n} \binom{n}{r} \Big\{ {}_G\mathcal{F}_{n-r}^{(s,\alpha)}(x,y,z,u;a,b,\beta,k)\, C_r(x,u) + {}_G\mathcal{F}_{n-r}^{(c,\alpha)}(x,y,z,u;a,b,\beta,k)\, S_r(x,u) \Big\}.$$

4. Conclusion

We introduced the generalized unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials and gave some relations by using the monomiality principle. We proved summation formula for these polynomials. Also, we gave some recurrence relation for the parametric generalized unified Apostol-type Bernoulli, Euler, Genocchi and Fubini polynomials.

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