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Convergence of series and almost sure convergence for weighted random variables under sub-linear expectations

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Abstract. In this paper, convergence of series and almost sure convergence are established for weighted random variables under a sub-linear expectation space. Our results are very extensive versions which contain the related convergence of series and almost sure convergence for sequences of random variables and so on, and are extensions and improvements of classical convergence of series and almost sure convergence from the traditional probability space to the sub-linear expectation space.

1. Introduction

The classical convergence of series and almost sure convergence are the basic limit theorems in the theory of probability and statistics. They have played an effective role in the development of probability theory and its applications. However, many uncertain phenomena can not be well modeled by using additive probabilities and additive expectations. Therefore, Peng (2006 [4]) proposed the concept of sub-linear expectation space. It is a natural extension of the classical linear expectation.

Due to the fact that sub-linear expectation provides a very flexible framework for the modeling of sub-linear probability problems, it has attracted the attention and research of many probability and statistics scholars. Peng (2006 [4], 2008 [5], 2009[6]) constructed the basic framework, basic properties and the central limit theorem. Since Zhang (2016a [12], 2016b [13]) and Tang et al. (2019 [7]) established exponential inequalities, Rosenthal's inequalities, and some inequalities for sub-linear expectation and capacity of partial sums, which provide some powerful tools for the study of limit theory of sub-linear expectation space. The limit theory under sub-linear expectation space has been developed rapidly and many basic theorems have been obtained. For example, Zhang (2016a [12], 2016b [13]) and Hu (2016 [2]) obtained Kolmogorov's strong law of larger numbers and Hartman-Wintner's law of iterated logarithm, Wu et al. (2018 [10]) studied the asymptotic approximation of inverse moment, Wu and Jiang (2018 [8]), Wu and Lu (2020 [9]) studied Chover's law of iterated logarithm, Deng and Wang (2020 [1]) obtained the complete convergence, and so on. However, because sub-linear expectation and capacity do not have the additivity of expectation and probability in traditional probability space, many powerful tools and common methods

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of linear expectation and probability are no longer effective, so the study of limit theorem under sub-linear expectation becomes much more complex and difficult. There is still a big gap between the limit theory of sub-linear expectation space and the traditional probability space. There are still many problems of limit theory of sub-linear expectation space which need further study.

It is well known that in the traditional probability space, if the sequence of norming constants monotonically increases to infinity, according to Kronecker's lemma, almost sure convergence can be derived from the convergence of series. Therefore, the convergence of series is a stronger result of limit theory than almost sure convergence, and it is also one of the tools to study almost sure convergence. The three series theorem is a powerful tool to study the convergence of series of random variables. The case of sub-linear expectation space is similar. In this paper, we study and obtain the convergence of series of weighted random variables by using the three series theorem under the sub-linear expectation space obtained by Xu and Zhang (2019 [11]), so as to obtain the almost sure convergence of the sums of weighted random variables. Our results not only extend the corresponding results of the traditional probability space to the sub-linear expected space, but also extend the sequence $\{X_n; n \geq 1\}$ to the weighted sequence $\{b_n X_n; n \geq 1\}$. In addition, our results are very extensive results including the famous the Marcinkiewicz strong law of large numbers and so on, and we can obtain various forms convergence of series and almost sure convergence for weighted random variables by taking different forms of weight sequence $\{b_n\}$, norming constant sequence $\{a_n\}$ and function sequence $\{g_n(\cdot)\}$. The corresponding results of Hu (2016 [2], 2020 [3]) are special cases of our results in this paper.

In the next section, we introduce the basic concepts and properties of sub-linear expectation spaces. In Section 3, convergence of series and almost sure convergence for weighted independence random variables under sub-linear expectation space are established and proven.

2. Preliminaries

The general framework of the sub-linear expectation in a general function space was introduced by Peng (2006 [4], 2008 [5]). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq c(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $c > 0, m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case we denote $X \in \mathcal{H}$.

Definition 2.1. *A function $\hat{\mathbb{E}} : \mathcal{H} \to [-\infty, \infty]$ is said to be a sub-linear expectation if it satisfies for all $X, Y \in \mathcal{H}$,*
(a) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}X \geq \hat{\mathbb{E}}Y$;
(b) Constant preserving: $\hat{\mathbb{E}}c = c$, $\forall c \in \mathbb{R}$;
(c) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$;
(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}X, \forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space. The conjugate expectation of $\hat{\mathbb{E}}$ is defined by

$$\hat{\varepsilon}X := -\hat{\mathbb{E}}(-X), \quad \forall X \in \mathcal{H}.$$

From the definition, obviously, for all $X, Y \in \mathcal{H}$,

$$\hat{\varepsilon}X \leq \hat{\mathbb{E}}X, \quad \hat{\mathbb{E}}(X + c) = \hat{\mathbb{E}}X + c, \quad |\hat{\mathbb{E}}(X - Y)| \leq \hat{\mathbb{E}}|X - Y|. \tag{1}$$

If $\hat{\mathbb{E}}Y = \hat{\varepsilon}Y$, then $\hat{\mathbb{E}}(X + aY) = \hat{\mathbb{E}}X + a\hat{\mathbb{E}}Y$ for any $a \in \mathbb{R}$.

Definition 2.2. *A function $V : \mathcal{F} \to [0, 1]$ is called a capacity if*

$$V(\emptyset) = 0, \quad V(\Omega) = 1 \text{ and } V(A) \leq V(B) \text{ for } \forall A \subseteq B, A, B \in \mathcal{F}.$$

It is called to be sub-additive if $V(A \bigcup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{F}$.

A sub-linear expectation $\hat{\mathbb{E}}$ could generate a pair of capacity denoted by

$$\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}\xi; I(A) \leq \xi, \xi \in \mathcal{H}\}, \quad \nu(A) := 1 - \mathbb{V}(A^c), \forall A \in \mathcal{F},$$

where A^c is the complement set of A.

By definition of $\mathbb{V}$ and ν, it is obvious that $\mathbb{V}$ is sub-additive, and

$$\nu(A) \leq \mathbb{V}(A), \quad \forall A \in \mathcal{F},$$

$$\hat{\mathbb{E}}f \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}g, \hat{\varepsilon}f \leq \nu(A) \leq \hat{\varepsilon}g, \quad \text{if} \quad f \leq I(A) \leq g, \quad f, g \in \mathcal{H}. \tag{2}$$

Further, if X is not in $\mathcal{H}$, we define $\hat{\mathbb{E}}X$ by

$$\hat{\mathbb{E}}X := \inf\{\hat{\mathbb{E}}Y; X \leq Y, Y \in \mathcal{H}\}.$$

Then

$$\mathbb{V}(A) = \hat{\mathbb{E}}I(A), \quad \text{for any} \quad A \in \mathcal{F}.$$

(2) implies Markov inequality: $\forall X \in \mathcal{H}$,

$$\mathbb{V}(|X| \geq x) \leq \hat{\mathbb{E}}(|X|^p)/x^p, \quad \forall \quad x > 0, p > 0 \tag{3}$$

from $I(|X| \geq x) \leq |X|^p/x^p \in \mathcal{H}$. By Lemma 4.1 in Zhang (2016b [13]), we have Hölder inequality: $\forall X, Y \in \mathcal{H}, p, q > 1$ satisfying $p^{-1} + q^{-1} = 1$,

$$\hat{\mathbb{E}}(|XY|) \leq \left(\hat{\mathbb{E}}(|X|^p)\right)^{1/p} \left(\hat{\mathbb{E}}(|Y|^q)\right)^{1/q},$$

particularly, Jensen inequality: $\forall X \in \mathcal{H}$,

$$\left(\hat{\mathbb{E}}(|X|^r)\right)^{1/r} \leq \left(\hat{\mathbb{E}}(|X|^s)\right)^{1/s} \quad \text{for} \quad 0 < r \leq s. \tag{4}$$

Definition 2.3. *(i) A sub-linear expectation $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ is called to be countably sub-additive if it satisfies*

$$\hat{\mathbb{E}}(X) \leq \sum_{n=1}^{\infty} \hat{\mathbb{E}}(X_n), \quad \text{whenever} \quad X \leq \sum_{n=1}^{\infty} X_n, \quad 0 \leq X, X_n \in \mathcal{H}, \quad .$$

(ii) A function $V : \mathcal{F} \to [0, 1]$ is called to be countably sub-additive if

$$V\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} V(A_n), \quad \forall A_n \in \mathcal{F}.$$

In the sub-linear expectation space, the concepts of independence and identical distribution are different from the traditional probability space. We adopt the following notion of independence and identical distribution for sub-linear expectation which is initiated by Peng(2006 [4], 2008 [5]).

Definition 2.4. *(Peng (2006 [4], 2008 [5]))*

(i) (Identical distribution) Let X_1 and X_2 be two random variables defined respectively in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed if

$$\hat{\mathbb{E}}_1(\varphi(X_1)) = \hat{\mathbb{E}}_2(\varphi(X_2)), \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}),$$

whenever the sub-expectations are finite.

(ii) (Independence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$*, a random vector* $\mathbf{Y} = (Y_1, \ldots, Y_n)$*,* $Y_i \in \mathcal{H}$ *is said to be independent to another random vector* $\mathbf{X} = (X_1, \ldots, X_m), X_i \in \mathcal{H}$ *under* $\hat{\mathbb{E}}$ *if for each test function* $\varphi \in C_{l,Lip}(\mathbb{R}_m \times \mathbb{R}_n)$ *we have* $\hat{\mathbb{E}}(\varphi(\mathbf{X}, \mathbf{Y})) = \hat{\mathbb{E}}[\hat{\mathbb{E}}(\varphi(\mathbf{x}, \mathbf{Y}))|_{\mathbf{x}=\mathbf{X}}]$*, whenever* $\bar{\varphi}(\mathbf{x}) := \hat{\mathbb{E}}\left(|\varphi(\mathbf{x}, \mathbf{Y})|\right) < \infty$ *for all* $\mathbf{x}$ *and* $\hat{\mathbb{E}}\left(|\bar{\varphi}(\mathbf{X})|\right) < \infty$*.*

(iii) (Independent random variables) A sequence of random variables $\{X_n; n \geq 1\}$ *is said to be independent, if* X_{i+1} *is independent to* $(X_1, \ldots, X_i)$ *for each* $i \geq 1$*.*

3. Main Results and Proofs

In the sub-linear expectation space, the almost sure convergence of a sequence of random variables is different from the traditional probability space. We first give the concept of almost sure in the sub-linear expected space.

Definition 3.1. *For arbitrary event* $A \in \mathcal{F}$*, it is said* A *almost surely* V *(denoted by* A *a.s.* V*), if* $V(A^c) = 0$*, where* A^c *is the complement set of* A*.*

In particular, a sequence of random variables $\{X_n; n \geq 1\}$ *is said to converge to* X *almost surely* V*, denoted by* $X_n \to X$ *a.s.* V *as* $n \to \infty$ *if,* $V(X_n \nrightarrow X) = 0$*.*

V can be replaced by $\mathbb{V}$ and ν respectively. By $\nu(A) \leq \mathbb{V}(A)$ and $\nu(A) + \mathbb{V}(A^c) = 1$ for any $A \in \mathcal{F}$, it is obvious that $X_n \to X$ a.s. $\mathbb{V}$ implies $X_n \to X$ a.s. ν. However, we must point out that $X_n \to X$ a.s. ν does not imply $X_n \to X$ a.s. $\mathbb{V}$. Wu and Lu (2020 [9]) gave a counter example of this as follows.

Example 3.2. *(Wu and Lu, Example 3.3 (2020 [9])) Let* X_n *be independent G-normal random variables with* $X_n \sim \mathcal{N}(0, [1/4^{2n}, 1])$ *in a sub-linear expectation space* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$*.* $\hat{\mathbb{E}}$ *and* $\mathbb{V}$ *are continuous. Then* $X_n \to 0$ *a.s.* ν*; but not* $X_n \to 0$ *a.s.*$\mathbb{V}$*.*

Therefore, in sub-linear expectations space, the almost sure convergence is essentially different from the ordinary probability space, and its study is much more complex and difficult.

In the following, let $\{X_n; n \geq 1\}$ be a sequence of random variables in a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $\{a_n; n \geq 1\}$ and $\{b_n; n \geq 1\}$ be two sequences of numbers and $a_n > 0$. The symbol c stands for a generic positive constant which may differ from one place to another.

Theorem 3.3. *Let* $\{X_n; n \geq 1\}$ *be a sequence of independent random variables in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *and* $\mathbb{V}$ *is countably sub-additive. Let* $\{g_n(x); n \geq 1\}$ *be a sequence of even functions, positive and non-decreasing in the interval* $x > 0$*. Suppose that for every* n *one or other of the following conditions is satisfied:*

(i) $x/g_n(x)$ *does not decrease in the interval* $x > 0$*;*

(ii) $x/g_n(x)$ *and* $g_n(x)/x^2$ *do not increase in the interval* $x > 0$*, and also* $\hat{\mathbb{E}}X_n = \hat{\varepsilon}X_n = 0$*.*

If

$$\sum_{n=1}^{\infty} \frac{\hat{\mathbb{E}}(g_n(b_n X_n))}{g_n(a_n)} < \infty, \tag{5}$$

then the series

$$\sum_{n=1}^{\infty} \frac{b_n X_n}{a_n} \ \text{ converges a.s. } \mathbb{V}. \tag{6}$$

Further, if $a_n \uparrow \infty$, then

$$\frac{\sum\limits_{i=1}^{n} b_i X_i}{a_n} \to 0 \text{ a.s. } \mathbb{V} \quad \text{as} \quad n \to \infty. \tag{7}$$

Remark 3.4. *Theorem 3.3 not only extends the corresponding results of the traditional probability space to the sub-linear expected space, but also extends the sequence $\{X_n; n \geq 1\}$ to the weighted sequence $\{b_n X_n; n \geq 1\}$. In addition, Theorem 3.3 is a very extensive result, we can obtain various forms convergence of series and almost sure convergence for weighted random variables by taking different forms of weight sequence $\{b_n\}$, normal constant sequence $\{a_n\}$ and function sequence $\{g_n(\cdot)\}$.*

The following results need to introduce additional notation. We denote by Ψ_c the set of functions $\psi(x)$ such that

(i) $\psi(x)$ is positive and non-decreasing in the interval $x \geq x_0$ for some x_0, and

(ii) $\sum \frac{1}{n\psi(n)} < \infty$.

Here, and in the following, $\sum f(n)$ denotes the summation over all positive integers n for which $f(n)$ is defined and non-negative. In the definition of the sets Ψ_c the value of x_0 need not be the same for different functions ψ.

For example, functions $x^\alpha, \ln^{1+\alpha} x$ and $\ln x (\ln\ln x)^{1+\alpha}$ for every $\alpha > 0$ all belong to Ψ_c.

Taking $g_n(x) = |x|^p \psi(|x|)$ for $0 \leq p < 2$ and a slowly varying function $\psi(x) \in \Psi_c$ and $g_n(x) = |x|^p, 0 < p \leq 2$ in Theorem 3.3 we can immediately obtain the following important Corollaries 3.5 and 3.6.

Corollary 3.5. *Let $\{X_n; n \geq 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\mathbb{V}$ is countably sub-additive. For $0 \leq p < 2$ and some slowly varying function $\psi(x) \in \Psi_c$, if*

$$\sum \frac{\hat{\mathbb{E}}\left(|b_n X_n|^p \psi(|b_n X_n|)\right)}{a_n^p \psi(a_n)} < \infty,$$

and also $\hat{\mathbb{E}} X_n = \hat{\varepsilon} X_n = 0$ for $1 \leq p < 2$, then the series $\sum_{n=1}^{\infty} (b_n X_n)/a_n$ converges a.s. $\mathbb{V}$. Further, if $a_n \uparrow \infty$, then $a_n^{-1} \sum_{i=1}^{n} b_i X_i \to 0$ a.s. $\mathbb{V}$, as $n \to \infty$.

Corollary 3.6. *Let $\{X_n; n \geq 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\mathbb{V}$ is countably sub-additive. For $0 < p \leq 2$, if*

$$\sum_{n=1}^{\infty} \frac{\hat{\mathbb{E}}(|b_n X_n|^p)}{a_n^p} < \infty,$$

and also $\hat{\mathbb{E}} X_n = \hat{\varepsilon} X_n = 0$ for $1 < p \leq 2$, then the series $\sum_{n=1}^{\infty} (b_n X_n)/a_n$ converges a.s. $\mathbb{V}$. Further, if $a_n \uparrow \infty$, then $a_n^{-1} \sum_{i=1}^{n} b_i X_i \to 0$ a.s. $\mathbb{V}$, as $n \to \infty$.

Notice that for any $\alpha > 0$,

$$\sum \frac{1}{n\psi(n^\alpha)} < \infty \Longleftrightarrow \sum \frac{1}{n\psi(n)} < \infty$$

from $\int_c^\infty \frac{\mathrm{d}x}{x\psi(x^\alpha)} = \frac{1}{\alpha} \int_{c^\alpha}^\infty \frac{\mathrm{d}x}{x\psi(x)}$ for any $c > 0$.

Taking $a_n = n^{1/p}$ in Corollary 3.5, if $\sup_{n\geq 1} \hat{\mathbb{E}}(|b_n X_n|^p \psi(|b_n X_n|)) < \infty$ for some slowly varying function $\psi \in \Psi_c$, then

$$\sum \frac{\hat{\mathbb{E}}(|b_n X_n|^p \psi(|b_n X_n|))}{a_n^p \psi(a_n)} \leq c \sum \frac{1}{n\psi(n^{1/p})} < \infty.$$

Therefore, we can immediately obtain the following corollary.

Corollary 3.7. *Let $\{X_n; n \geq 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\mathbb{V}$ is countably sub-additive. For $0 \leq p < 2$, if $\sup_{n\geq 1} \hat{\mathbb{E}}(|b_n X_n|^p \psi(|b_n X_n|)) < \infty$ for some slowly varying function $\psi \in \Psi_c$, and also $\hat{\mathbb{E}} X_n = \hat{\varepsilon} X_n = 0$ for $1 \leq p < 2$, then the series $\sum_{n=1}^{\infty} (b_n X_n)/n^{1/p}$ converges a.s. $\mathbb{V}$ and $n^{-1/p} \sum_{i=1}^{n} b_i X_i \to 0$ a.s. $\mathbb{V}$ as $n \to \infty$.*

Remark 3.8.

(i) By Lemma 3.6 in Zhang (2016b [13]), if $\mathbb{V}$ is continuous from below, then it is countably sub-additive.

(ii) Theorem 3.1 in Hu (2016 [2]) is a special case of Corollary 3.7 when $b_n \equiv 1$ and $p = 1$.

(iii) Theorem 3 in Hu (2020 [3]) is a special case of Corollary 3.7 when $b_n \equiv 1$, and Corollary 3.7 weakens the conditions that $\sup_{n\geq 1} \hat{\mathbb{E}}(|X_n|^p \ln^{p-1}(1 + |X_n|)\psi(|X_n|)) < \infty$ for $1 \leq p < 2$, $\sup_{n\geq 1} \hat{\mathbb{E}}(|X_n|^p \ln^p(1 + |X_n|)\psi(|X_n|)) < \infty$ for $0 < p < 1$ to that $\sup_{n\geq 1} \hat{\mathbb{E}}(|X_n|^p \psi(|X_n|)) < \infty$ for $0 < p \leq 2$.

Therefore, Theorem 3.3, Corollary 3.5 and Corollary 3.7 generalize and improve the corresponding results of Hu (2016 [2]) and Hu (2020 [3]).

(iv) For $b_n \equiv 1$ and $p = 1$ in Corollary 3.7, Hu (2016 [2]) gave Example 4.1 to show that if $\sum 1/(n\psi(n)) < \infty$ is replaced by $\sum 1/(n\psi(n)) = \infty$, then the Corollary 3.7 is not true. Therefore, the moment condition: $\sup_{n\geq 1} \hat{\mathbb{E}}(|b_n X_n|^p \psi(|b_n X_n|)) < \infty$ for some $\psi \in \Psi_c$ is the weakest moment condition to ensure the validity of Corollary 3.7.

(v) Theorem 1 in Zhang and Lin (2018 [14]): If $\{X_n; n \geq 1\}$ is i.i.d., $\mathbb{V}$ is continuous, $\hat{\mathbb{E}} X_n = \hat{\varepsilon} X_n = 0$, for $1 \leq p < 2$, $\lim_{c\to\infty} \hat{\mathbb{E}}(|X_1|^p - c)^+ = 0$, and $C_{\mathbb{V}}(|X_1|^p) < \infty$, then $n^{-1/p} \sum_{i=1}^{n} X_i \to 0$ a.s. $\mathbb{V}$. Theorem 1 of Zhang and Lin (2018 [14]) and Corollary 3.7 of this paper do not contain each other.

Taking $a_n = (n\psi(n))^{1/p}$, $\psi(n) \in \Psi_c$ in Corollary 3.6 we can immediately obtain the following corollary.

Corollary 3.9. *Let $\{X_n; n \geq 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\mathbb{V}$ is countably sub-additive. For $0 < p \leq 2$, if $\sup_{n\geq 1} \hat{\mathbb{E}}(|b_n X_n|^p) < \infty$, and also $\hat{\mathbb{E}} X_n = \hat{\varepsilon} X_n = 0$ for $1 < p \leq 2$, then for every function $\psi(n) \in \Psi_c$, the series $\sum (b_n X_n)/(n\psi(n))^{1/p}$ converges a.s. $\mathbb{V}$ and $\frac{\sum_{i=1}^{n} b_i X_i}{(n\psi(n))^{1/p}} \to 0$ a.s. $\mathbb{V}$ as $n \to \infty$.*

The following theorem directly uses a function of the sums of sub-linear expectations of weighted random variables as a norming sequence $\{a_n\}$.

Theorem 3.10. *Let $\{X_n; n \geq 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\mathbb{V}$ is countably sub-additive. Let $g(x)$ be an even continuous function, positive and strictly increasing in the interval $x > 0$, and such that $g(x) \to \infty$ as $x \to \infty$. Suppose that at least one of the two following conditions is satisfied:*

(i) *$x/g(x)$ is not-decreasing in the interval $x > 0$;*

(ii) *$x/g(x)$ and $g(x)/x^2$ do not increase in the interval $x > 0$, and also $\hat{\mathbb{E}} X_n = \hat{\varepsilon} X_n = 0$ for every n.*

Suppose further that $\hat{\mathbb{E}}(g(b_n X_n)) < \infty$ for every n and

$$M_n := \sum_{k=1}^{n} \hat{\mathbb{E}}(g(b_k X_k)) \to \infty \quad \text{as} \quad n \to \infty. \tag{8}$$

Then for every function $\psi(x) \in \Psi_c$, the series

$$\sum \frac{b_n X_n}{g^{-1}(M_n \psi(M_n))} \text{ converges a.s. } \mathbb{V}, \tag{9}$$

and

$$\frac{\sum\limits_{i=1}^{n} b_i X_i}{g^{-1}(M_n \psi(M_n))} \to 0 \text{ a.s. } \mathbb{V} \quad \text{as} \quad n \to \infty, \tag{10}$$

where g^{-1} is the inverse of g.

Theorem 3.10 has much simpler consequences in the case when $g(x) = |x|^p, 0 < p \le 2$.

Corollary 3.11. *Let $\{X_n; n \ge 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\mathbb{V}$ is countably sub-additive. For $0 < p \le 2$, if $\hat{\mathbb{E}}(|X_n|^p) < \infty$ for every n, and also $\hat{\mathbb{E}}X_n = \hat{\varepsilon}X_n = 0$ for $1 < p \le 2$. Suppose further that*

$$M_n := \sum_{k=1}^{n} \hat{\mathbb{E}}(|b_k X_k|^p) \to \infty \quad \text{as} \quad n \to \infty.$$

Then for every function $\psi(x) \in \Psi_c$ the series $\sum (b_n X_n)/(M_n \psi(M_n))^{1/p}$ converges a.s. $\mathbb{V}$ and $(M_n\psi(M_n))^{-1/p} \sum_{i=1}^{n} (b_i X_i) \to 0$ a.s. $\mathbb{V}$ as $n \to \infty$.

Corollary 3.11 for $p = 2$ is useful in studying the applicability of the strong law of large numbers with the simplest normalization, namely of the form

$$\frac{\sum_{i=1}^{n} b_i X_i}{n} \to 0 \ \text{ a.s. } \mathbb{V} \quad \text{as} \quad n \to \infty. \tag{11}$$

We shall apply Corollary 3.11 for $p = 2$ to the strong law of large numbers (11).

Theorem 3.12. *Let $\{X_n; n \ge 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\mathbb{V}$ is countably sub-additive. Suppose $\hat{\mathbb{E}}(X_n^2) < \infty$ and $\hat{\mathbb{E}}X_n = \hat{\varepsilon}X_n = 0$ for every n. If*

$$B_n := \sum_{k=1}^{n} \hat{\mathbb{E}}(b_k X_k)^2 = O(n^2/\psi(n)) \tag{12}$$

for some function $\psi(x) \in \Psi_c$, then (11) holds.

Remark 3.13. *Taking $b_n \equiv 1$ in all the above results, we get the corresponding results of a sequence of random variables $\{X_n; n \ge 1\}$ respectively.*

Here are two examples of our results. For this, we need to do some preparatory work.
From Peng (2009 [6]), if $\xi \sim \mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2])$ under $\hat{\mathbb{E}}$, then for each convex function φ,

$$\hat{\mathbb{E}}(\varphi(\xi)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\bar{\sigma}x) \mathrm{e}^{-x^2/2} \mathrm{d}x, \tag{13}$$

but if φ is a concave function, the above $\bar{\sigma}$ must be replaced by $\underline{\sigma}$. If $\sigma = \bar{\sigma} = \underline{\sigma}$, then $\mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2]) = \mathcal{N}(0, \sigma^2)$ which is a classical normal distribution.

In particular, notice that $\varphi(x) = |x|^p, p \ge 1$ is a convex function and $\varphi(x) = |x|^p, 0 < p < 1$ is a concave function, taking $\varphi(x) = |x|^p, p > 0$ in (13), we get

$$\hat{\mathbb{E}}(|\xi|^p) = \frac{2\underline{\sigma}^p}{\sqrt{2\pi}} \int_{0}^{\infty} x^p \mathrm{e}^{-x^2/2} \mathrm{d}x = c_p \underline{\sigma}^p \ \text{ for } \ 0 < p < 1, \tag{14}$$

and

$$\hat{\mathbb{E}}(|\xi|^p) = c_p \bar{\sigma}^p \ \text{ for } \ p \ge 1, \tag{15}$$

where c_p is a constant only related to p.

Example 3.14. *Let $\{X_n; n \geq 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $X_n \sim \mathcal{N}(0, [\underline{\sigma}_n^2, \bar{\sigma}_n^2])$ under $\hat{\mathbb{E}}$, where $0 \leq \underline{\sigma}_n^2 \leq \bar{\sigma}_n^2 < \infty$. Suppose that $\mathbb{V}$ is countably sub-additive. If*

$$\sum_{n=1}^{\infty} \frac{|b_n|^p \underline{\sigma}_n^p}{a_n^p} < \infty \quad \text{for} \quad 0 < p < 1, \tag{16}$$

and

$$\sum_{n=1}^{\infty} \frac{|b_n|^p \bar{\sigma}_n^p}{a_n^p} < \infty \quad \text{for} \quad 1 \leq p \leq 2, \tag{17}$$

then the series $\sum_{n=1}^{\infty}(b_n X_n)/a_n$ converges a.s. $\mathbb{V}$. Further, if $a_n \uparrow \infty$, then $a_n^{-1}\sum_{i=1}^{n} b_i X_i \to 0$ a.s. $\mathbb{V}$, as $n \to \infty$.

Proof By (14) and (15),

$$\hat{\mathbb{E}}(|X_n|^p) = c_p \underline{\sigma}_n^p \quad \text{for} \quad 0 < p < 1,$$

and

$$\hat{\mathbb{E}}(|X_n|^p) = c_p \bar{\sigma}_n^p \quad \text{for} \quad p \geq 1.$$

Hence, by (16), (17) and Corollary 3.6, we immediately obtain the conclusion of Example 3.14.

From Theorem 3.12, we immediately obtain the following Example 3.15.

Example 3.15. *Let $\{X_n; n \geq 1\}$ be a sequence of independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $X_n \sim \mathcal{N}(0, [\underline{\sigma}_n^2, \bar{\sigma}_n^2])$ under $\hat{\mathbb{E}}$, where $0 \leq \underline{\sigma}_n^2 \leq \bar{\sigma}_n^2 < \infty$. Suppose that $\mathbb{V}$ is countably sub-additive. If*

$$B_n = \sum_{k=1}^{n} b_k^2 \bar{\sigma}_k^2 = O\left(\frac{n^2}{\psi(n)}\right) \quad \text{for some} \quad \psi \in \Psi_c$$

then

$$\frac{\sum_{i=1}^{n} b_i X_i}{n} \to 0 \ \text{a.s.} \ \mathbb{V} \ \text{as} \ n \to \infty.$$

In particular,

(i) if $0 \leq \underline{\sigma}_n^2 \leq n$, $X_n \sim \mathcal{N}(0, [\underline{\sigma}_n^2, n])$ and $b_k = 1/\ln^{\alpha} k$ for $\alpha > 1/2$, then $B_n = O(n^2/\ln^{2\alpha} n)$.

(ii) if $0 \leq \underline{\sigma}_n^2 \leq 1$, $X_n' \sim \mathcal{N}(0, [\underline{\sigma}_n^2, 1])$ and $b_k = k^{1/2}/\ln^{\alpha} k$ for $\alpha > 1/2$, then $B_n = O(n^2/\ln^{2\alpha} n)$.

Therefore,

$$\frac{\sum_{k=1}^{n} \frac{X_k}{\ln^{\alpha} k}}{n} \to 0 \ \text{a.s.} \ \mathbb{V} \ \text{as} \ n \to \infty,$$

and

$$\frac{\sum_{k=1}^{n} \frac{k^{1/2} X_k'}{\ln^{\alpha} k}}{n} \to 0 \ \text{a.s.} \ \mathbb{V} \ \text{as} \ n \to \infty.$$

To prove our results, we need the following two lemmas.

Lemma 3.16. *(Xu and Zhang 2019 [11], Theorem 3.3) Let* $\{X_n; n \geq 1\}$ *be a sequence of independent random variables in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *and* $\mathbb{V}$ *is countably sub-additive. Suppose that the following three conditions hold for some* $c > 0$,
(i) $\sum_{n=1}^{\infty} \mathbb{V}(|X_n| > c) < \infty$,
(ii) $\sum_{n=1}^{\infty} \hat{\mathbb{E}}(X_n^{(c)})$ *and* $\sum_{n=1}^{\infty} \hat{\varepsilon}(X_n^{(c)})$ *both converge,*
(iii) $\sum_{n=1}^{\infty} \hat{\mathbb{E}}(|X_n^{(c)}|^p) < \infty$ *for some* $1 \leq p \leq 2$,
where $X_n^{(c)} := -cI(X_n < -c) + X_n I(|X_n| \leq c) + cI(X_n > c)$ *and* $I(\cdot)$ *denotes an indicator function. Then* $\sum_{n=1}^{\infty} X_n$ *converge a.s.* $\mathbb{V}$.

Lemma 3.17. *(Xu and Zhang 2019 [11], Theorem 3.4) Assume that* $\{X_n; n \geq 1\}$ *is a sequence of independent random variables in the sub-linear expectation space* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, *if* $\mathbb{V}$ *is countably sub-additive,* $\sum_{n=1}^{\infty} \hat{\mathbb{E}}(X_n)$ *and* $\sum_{n=1}^{\infty} \hat{\varepsilon}(X_n)$ *both converge, and there exists* $1 \leq p \leq 2$, *such that* $\sum_{n=1}^{\infty} \hat{\mathbb{E}}(|X_n|^p) < \infty$. *Then* $\sum_{n=1}^{\infty} X_n$ *converges a.s.* $\mathbb{V}$.

Proof of Theorem 3.3 By Lemma 3.16, in order to prove (6) it suffices to prove convergence of the corresponding series for the sequence of the random variables $\{b_n X_n / a_n; n \geq 1\}$ and $c = 1$.

For each $n \geq 1$,

$$\left(\frac{b_n X_n}{a_n}\right)^{(1)} = -I\left(\frac{b_n X_n}{a_n} < -1\right) + \frac{b_n X_n}{a_n} I\left(\left|\frac{b_n X_n}{a_n}\right| \leq 1\right) + I\left(\frac{b_n X_n}{a_n} > 1\right).$$

Since the functions $g_n(x)$ are even and non-decreasing in the interval $x > 0$. Therefore, (5) and the Markov inequality: (3) imply that

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\left|\frac{b_n X_n}{a_n}\right| > 1\right) \leq \sum_{n=1}^{\infty} \mathbb{V}(g_n(b_n X_n) \geq g_n(a_n)) \leq \sum_{n=1}^{\infty} \frac{\hat{\mathbb{E}}(g_n(b_n X_n))}{g_n(a_n)} < \infty. \tag{18}$$

Suppose that the function $g_n(x)$ satisfies condition (i). Then in the interval $|b_n x| \leq a_n$, we have $\frac{|b_n x|}{a_n} \leq \frac{g_n(b_n x)}{g_n(a_n)} \leq 1$. Therefore

$$\frac{|b_n x|^2}{a_n^2} \leq \frac{g_n^2(b_n x)}{g_n^2(a_n)} \leq \frac{g_n(b_n x)}{g_n(a_n)}.$$

If the function $g_n(x)$ satisfies condition (ii), then in the same interval $|b_n x| \leq a_n$, we obtain $\frac{g_n(a_n)}{a_n^2} \leq \frac{g_n(b_n x)}{|b_n x|^2}$, i.e. $\frac{|b_n x|^2}{a_n^2} \leq \frac{g_n(b_n x)}{g_n(a_n)}$. Therefore in both cases, we have

$$\frac{|b_n x|^2}{a_n^2} \leq \frac{g_n(b_n x)}{g_n(a_n)}, \quad \text{for } |b_n x| \leq a_n.$$

That, in conjunction with (5) and (18), yields

$$\begin{aligned}
\sum_{n=1}^{\infty} \hat{\mathbb{E}}\left(\left(\frac{b_n X_n}{a_n}\right)^{(1)}\right)^2 &\leq \sum_{n=1}^{\infty} \mathbb{V}\left(\left|\frac{b_n X_n}{a_n}\right| > 1\right) + \sum_{n=1}^{\infty} \hat{\mathbb{E}}\left(\frac{b_n^2 X_n^2}{a_n^2}\right) I_{(|b_n X_n| \leq a_n)} \\
&\leq 2 \sum_{n=1}^{\infty} \frac{\hat{\mathbb{E}}(g_n(b_n X_n))}{g_n(a_n)} < \infty.
\end{aligned} \tag{19}$$

Furthermore,

$$\begin{aligned}
\hat{\mathbb{E}}\left(\frac{b_n X_n}{a_n}\right)^{(1)} &\leq \hat{\mathbb{E}}\left(\frac{b_n X_n}{a_n}\right) I_{(|b_n X_n| \leq a_n)} + \hat{\mathbb{E}}\left(I\left(\frac{b_n X_n}{a_n} > 1\right) - I\left(\frac{b_n X_n}{a_n} < -1\right)\right) \\
&\leq \left|\hat{\mathbb{E}}\left(\frac{b_n X_n}{a_n}\right) I_{(|b_n X_n| \leq a_n)}\right| + \mathbb{V}\left(\left|\frac{b_n X_n}{a_n}\right| > 1\right).
\end{aligned}$$

On the other hand, by $\hat{\mathbb{E}}(X-Y) \geq \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y$,

$$\begin{aligned}\hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}\right)^{(1)} &\geq \hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}\right)I_{(|b_nX_n|\leq a_n)} - \hat{\mathbb{E}}\left(I\left(\frac{b_nX_n}{a_n}<-1\right)-I\left(\frac{b_nX_n}{a_n}>1\right)\right)\\ &\geq -\left|\hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}\right)I_{(|b_nX_n|\leq a_n)}\right| - \mathbb{V}\left(\left|\frac{b_nX_n}{a_n}\right|>1\right).\end{aligned}$$

Therefore,

$$\left|\hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}\right)^{(1)}\right| \leq \mathbb{V}\left(\left|\frac{b_nX_n}{a_n}\right|>1\right) + \left|\hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}\right)I_{(|b_nX_n|\leq a_n)}\right|. \tag{20}$$

If condition (i) is satisfied, then $|b_nX_n|/g_n(b_nX_n) \leq a_n/g_n(a_n)$ for $|b_nX_n| \leq a_n$ and

$$\left|\hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}\right)^{(1)}\right| \leq \mathbb{V}\left(\left|\frac{b_nX_n}{a_n}\right|>1\right) + \hat{\mathbb{E}}\left(\frac{|b_nX_n|}{a_n}I_{(|b_nX_n|\leq a_n)}\right) \leq 2\frac{\hat{\mathbb{E}}g_n(b_nX_n)}{g_n(a_n)}.$$

If condition (ii) is satisfied, then

$$\hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}\right) = \begin{cases}\frac{b_n\hat{\mathbb{E}}(X_n)}{a_n} & \text{if} \quad b_n \geq 0,\\ \frac{b_n\hat{\varepsilon}(X_n)}{a_n} & \text{if} \quad b_n < 0\end{cases} = 0$$

from $\hat{\mathbb{E}}(X_n) = \hat{\varepsilon}(X_n) = 0$.

Therefore, combining (1): $|\hat{\mathbb{E}}(X-Y)| \leq \hat{\mathbb{E}}|X-Y|$ for any $X, Y \in \mathcal{H}$,

$$\begin{aligned}\left|\hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}\right)I_{(|b_nX_n|\leq a_n)}\right| &= \left|\hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}\right) - \hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}I_{(|b_nX_n|\leq a_n)}\right)\right|\\ &\leq \hat{\mathbb{E}}\left|\frac{b_nX_n}{a_n} - \frac{b_nX_n}{a_n}I_{(|b_nX_n|\leq a_n)}\right|\\ &\leq \hat{\mathbb{E}}\left(\left|\frac{b_nX_n}{a_n}\right|I_{(|b_nX_n|>a_n)}\right) \leq \frac{\hat{\mathbb{E}}(g_n(b_nX_n))}{g_n(a_n)}.\end{aligned}$$

Therefore in both cases, by (5) and (20), we have

$$\sum_{n=1}^{\infty}\left|\hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}\right)^{(1)}\right| \leq 2\sum_{n=1}^{\infty}\frac{\hat{\mathbb{E}}(g_n(b_nX_n))}{g_n(a_n)} < \infty. \tag{21}$$

Note that $\hat{\mathbb{E}}(-X_n) = -\hat{\varepsilon}(X_n)$ and $\hat{\varepsilon}(-X_n) = -\hat{\mathbb{E}}(X_n)$, we get that if $\hat{\mathbb{E}}(X_n) = \hat{\varepsilon}(X_n) = 0$, then $\hat{\mathbb{E}}(-X_n) = \hat{\varepsilon}(-X_n) = 0$. Hence, $\{-X_n; n \geq 1\}$ also satisfies the conditions of Theorem 3.3. Considering $\{-X_n; n \geq 1\}$ instead of $\{X_n; n \geq 1\}$ in (21), we can obtain

$$\sum_{n=1}^{\infty}\left|\hat{\mathbb{E}}\left(\frac{-b_nX_n}{a_n}\right)^{(1)}\right| < \infty.$$

This implies

$$\sum_{n=1}^{\infty}\left|\hat{\varepsilon}\left(\frac{b_nX_n}{a_n}\right)^{(1)}\right| < \infty.$$

This is combined with (21) to obtain $\sum\limits_{n=1}^{\infty} \hat{\mathbb{E}}\left(\frac{b_nX_n}{a_n}\right)^{(1)}$ and $\sum\limits_{n=1}^{\infty} \hat{\varepsilon}\left(\frac{b_nX_n}{a_n}\right)^{(1)}$ both converge. Together with (18) and (19), from Lemma 3.16 we know that (6) holds.

Further, if $a_n \uparrow \infty$, by the Kronecker's lemma and (6), (7) holds.

Proof of Theorem 3.10 Suppose $\psi(x) \in \Psi_c$. By the hypotheses of the Theorem 3.10, there exists a positive inverse $g^{-1}(x)$ in the interval $x > 0$. We put $a_n = g^{-1}(M_n\psi(M_n))$. Then $a_n \uparrow \infty$ from (8).

Let n_0 be such that $M_{n_0} > 0$ and $\psi(M_{n_0}) > 0$. The series $\sum \frac{1}{n\psi(n)}$ converges, and, therefore, so does the series

$$\begin{aligned}\sum_{n=n_0+1}^{\infty} \frac{\hat{\mathbb{E}}(g(b_nX_n))}{g(a_n)} &= \sum_{n=n_0+1}^{\infty} \frac{M_n - M_{n-1}}{M_n\psi(M_n)} = \sum_{n=n_0+1}^{\infty} \int_{M_{n-1}}^{M_n} \frac{\mathrm{d}x}{M_n\psi(M_n)} \\ &\le \sum_{n=n_0+1}^{\infty} \int_{M_{n-1}}^{M_n} \frac{\mathrm{d}x}{x\psi(x)} = \int_{M_{n_0}}^{\infty} \frac{\mathrm{d}x}{x\psi(x)} \\ &< \infty.\end{aligned}$$

By Theorem 3.3 and Kronecker's lemma, (9) and (10) hold.

Proof of Theorem 3.12

(i) If $B_n \nrightarrow \infty$, then

$$\sum_{k=1}^{\infty} \hat{\mathbb{E}}(b_kX_k)^2 < \infty. \tag{22}$$

And, consequently,

$$\sum_{k=1}^{\infty} \frac{\hat{\mathbb{E}}(b_kX_k)^2}{k^2} < \infty. \tag{23}$$

By the Hölder inequality, Jensen inequality: (4) and (22),

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{|\hat{\mathbb{E}}(b_kX_k)|}{k} &\le \left(\sum_{k=1}^{\infty} (\hat{\mathbb{E}}(b_kX_k))^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{1/2} \\ &\le \left(\sum_{k=1}^{\infty} \hat{\mathbb{E}}(b_kX_k)^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{1/2} \\ &< \infty.\end{aligned} \tag{24}$$

Obviously, $\{-X_k; k \ge 1\}$ also satisfies the conditions of Theorem 3.12. Considering $\{-X_k; k \ge 1\}$ instead of $\{X_k; k \ge 1\}$ in (24), we can obtain

$$\sum_{k=1}^{\infty} \frac{|-\hat{\varepsilon}(b_kX_k)|}{k} = \sum_{k=1}^{\infty} \frac{|\hat{\mathbb{E}}(-b_kX_k)|}{k} < \infty. \tag{25}$$

(24) and (25) imply series $\sum_{k=1}^{\infty} \frac{\hat{\mathbb{E}}(b_kX_k)}{k}$ and $\sum_{k=1}^{\infty} \frac{\hat{\varepsilon}(b_kX_k)}{k}$ both converge. In combination with (23) and Lemma 3.17, the series $\sum_{k=1}^{\infty} \frac{b_kX_k}{k}$ converges a.s. $\mathbb{V}$. Further, by the Kronecker's lemma, (11) holds.

(ii) If $B_n \to \infty$, then construct a function f as follows. We put $f(n^2) = \psi(n)$, and for values of n that are not squares of integers we choose $f(x)$ in such a way as to make it non-decreasing.

Without losing generality, it can be assumed $\psi(1) > 0$, we have

$$\sum_{n=1}^{\infty} \frac{1}{nf(n)} = \sum_{k=1}^{\infty} \sum_{k^2 \le n < (k+1)^2} \frac{1}{nf(n)} \le \sum_{k=1}^{\infty} \frac{(k+1)^2 - k^2}{k^2 f(k^2)} = \sum_{k=1}^{\infty} \frac{2k+1}{k^2 \psi(k)} < \infty.$$

This prove that $f(x) \in \Psi_c$. By the case $p = 2$ in Corollary 3.11, we have

$$\frac{\sum_{k=1}^{n} b_k X_k}{(B_n f(B_n))^{1/2}} \to 0 \text{ a.s. } \mathbb{V}.$$

Therefore, (12) implies that

$$\frac{\sum_{k=1}^{n} b_k X_k}{n\left(\frac{f(B_n)}{\psi(n)}\right)^{1/2}} \to 0 \text{ a.s. } \mathbb{V}.$$

Using (12) again, and using the fact that every function $\psi \in \Psi_c$ satisfies the condition $\psi(n) \uparrow \infty$, we conclude that

$$\frac{\sum_{k=1}^{n} b_k X_k}{n\left(\frac{f(n^2)}{\psi(n)}\right)^{1/2}} \to 0 \text{ a.s. } \mathbb{V}.$$

That (11) holds from $f(n^2) = \psi(n)$.

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