Filomat 37:22 (2023), 7455–7465
https://doi.org/10.2298/FIL2322455D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Analysis of distinguishability of linear descriptor control systems using Drazin inverse

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Abstract. This article aims to provide the equivalent criteria for the distinguishability of linear descriptor systems (LDS). Regularity of the matrix pencil, which, loosely speaking, guarantees the existence, and uniqueness of the solution of LDS for any inhomogeneity, is required in this article. A characterization of observability for LDS in terms of distinguishability is given. The Laplace transform together with the Cayley-Hamilton theorem exploited to derive Hautus-type criteria for the distinguishability. In addition, we present examples of distinguishable systems.

1. Introduction and Preliminaries

Let $\mathbb{F}^{n\times m}$ denote the set of all $n \times m$ matrices over the field $\mathbb{F}$ (real or complex). Let $\mathcal{N}(P)$ and rank(P) denote the null space and rank of $P \in \mathbb{F}^{n\times m}$, respectively. The smallest number $k \in \mathbb{Z}^+ \cup \{0\}$ is called the index of $F \in \mathbb{C}^{n\times n}$, denoted by ind$(F)$, if rank$\left(F^{k+1}\right)$ =rank$\left(F^{k}\right)$. Let $F \in \mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ such that satisfies $F^{k+1}X = F^k, XFX = X$ and $FX = XF$, is called the Drazin inverse (DI) of F and denoted by F^D. If ind$(F) \leq 1$, then the DI of F is called the group inverse and denoted by $F^{\#}$. The set of all k-time continuously differentiable vector-valued functions defined on the domain X is denoted by $C^k(X, \mathbb{R}^m)$. Let $C^\infty[a, b]$ denote the set of all analytical vector-valued functions on the domain $[a, b]$.

Consider the following differential algebraic system with a state variable $z(\cdot) \in \mathbb{R}^n$, a control input $u(\cdot) \in \mathbb{R}^m$

$$Fz'(t) = Pz(t) + Qu(t), \tag{1}$$

where $F, P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{n\times m}$ are constant matrices.

If F has nonzero entries, then derivatives of respective components of z are involved. If F has a zero row, then there does not involve any derivative and equation becomes purely algebraic. This justifies to call (1) a differential algebraic system. Assume that $\det(F) = 0$ and $\det(aF - P) \neq 0$ for some $a \in \mathbb{C}$, then (1) is known as LDS with regular pencil (F, P). The inspection of LDS have taken many attentions, see, for example, [1, 9, 10]. Since LDS can be analyzed though Jordan canonical form but this study seems to be complex as pointed out in [13]. Hence, an efficient tool to analyze the LDS is with the help of DI. There are applications of DI to solve LDS [2, 12]. The DI of any square matrix always exist and unique, see in [1].

2020 *Mathematics Subject Classification*. Primary 15A22, 15A21; Secondary 34A09, 34A30.
Keywords. Distinguishability; Linear descriptor system; Drazin inverse; Pencil.
Received: 04 August 2022; Accepted: 18 March 2023
Communicated by Dijana Mosić
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In [6], steps to find F^D of F are expressed. The following

$$F^D = E\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} E^{-1} \tag{2}$$

is DI in terms of the Jordan canonical form of $F = E\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} E^{-1}$, where N is nilpotent and J constitutes the Jordan blocks corresponding to non-zero eigenvalues.

An example for finding the DI is:

Example 1.1. *For any* $F = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $a \in \mathbb{Z} \backslash \{0\}$ *where* $\mathbb{Z}$ *is the set of integers, while* $\det F = 0$, *and* $rank(F) = 1$, F *is decomposed as* $F = WV$ *with* $W = \begin{bmatrix} a \\ 0 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 \end{bmatrix}$, *then* $F^D = W(VFW)^{-1} V = \begin{bmatrix} 1/a & 0 \\ 0 & 0 \end{bmatrix}$.

Theorem 1.2. *[2] Consider that* $FP = PF$ *and* $\mathcal{N}(F) \cap \mathcal{N}(P) = \{0\}$, *then there exists a unique solution of (1) with* $z(0) = z_0$, *if and only if*

$$z_0 = Hq + (H - I)\sum_{n=0}^{k-1} \left(FP^D\right)^n P^D u^{(n)}(0),$$

where $k = ind(F)$, $H = F^D F$ *and* q *is an arbitrary constant vector.*

Pre-multiplying (1) by $(aF - P)^{-1}$ to overcome the commutative condition of Theorem 1.2, as a result, we get

$$\widehat{F}z'(t) = \widehat{P}z(t) + \widehat{Q}u(t), \tag{3}$$

with

$$\widehat{F} = [aF - P]^{-1}F,\ \widehat{P} = [aF - P]^{-1}P, \tag{4}$$

and $\widehat{Q} = [aF - P]^{-1}Q$.

The following Lemma is related to some properties of $\widehat{F}$ and $\widehat{P}$.

Lemma 1.3. *[2] From (4), we have*

1. $\mathcal{N}\left(\widehat{P}\right) \cap \mathcal{N}\left(\widehat{F}\right) = \{0\}$,
2. $\widehat{PF} = \widehat{FP}$, $\widehat{P}^D\widehat{F} = (\widehat{FP})^D$, $\widehat{F}^D\widehat{P} = (\widehat{PF})^D$ *and* $\widehat{P}^D\widehat{F}^D = \widehat{F}^D\widehat{P}^D$,
3. $(\widehat{FF}^D - I)\widehat{PP}^D = \left(\widehat{FF}^D - I\right)$ *and* $(I - \widehat{FF}^D)(\widehat{FP}^D)^k = 0$,
4. $\widehat{F} = S\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} S^{-1}$, $\widehat{F}^D = S\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$, $\det(S) \neq 0$, $N \in \mathbb{R}^{n_2 \times m_2}$, $J \in \mathbb{R}^{n_1 \times m_1}$, *$J$ is non-singular*, $N^k = 0$, *while* $N^{k-1} \neq 0$, $n_1 + n_2 = n$.

The following theorem states the solution of LDS (3) (and (1)):

Theorem 1.4. *[2] Consider* (F, P) *to be a regular pencil, then (3) (and (1)) has a unique solution*

$$z(t) = e^{\widehat{F}^D\widehat{P}t}\hat{H}q + \widehat{F}^D e^{\widehat{F}^D\widehat{P}t}\int_0^t e^{-\widehat{F}^D\widehat{P}s}\widehat{Q}u(s)ds + \left(\hat{H} - I\right)\sum_{i=0}^{k-1}\left(\widehat{FP}^D\right)^i \widehat{P}^D\widehat{Q}u^{(i)}(t),\ \textit{for all } t \geq 0, \tag{5}$$

if and only if

$$z_0 = \hat{H}q + \left(\widehat{F}^D\widehat{F} - I\right)\sum_{i=0}^{k-1}\left(\widehat{FP}^D\right)^i \widehat{P}^D\widehat{Q}u^{(i)}(0), \tag{6}$$

where q *is arbitrary constant vector,* $\widehat{F}^D\widehat{F} = \hat{H}$ *and* $k = ind(F)$.

Putting left hand side of (6) in (5) results

$$z(t) = e^{\widehat{F^D}\widehat{P}t}z_0 - e^{\widehat{F^D}\widehat{P}t}\left(\hat{H} - I\right)\sum_{i=0}^{k-1}\left(\widehat{FP^D}\right)^i \widehat{P^D}\widehat{Q}u^{(i)}(0) + \left(\hat{H} - I\right)\sum_{i=0}^{k-1}\left(\widehat{FP^D}\right)^i \widehat{P^D}\widehat{Q}u^{(i)}(t)$$
$$+\widehat{F^D}e^{\widehat{F^D}\widehat{P}t}\int_0^t e^{-\widehat{F^D}\widehat{P}s}\widehat{Q}u(s)ds, \text{for all } t \geq 0. \tag{7}$$

If ind$(F) = 1$, (7) becomes

$$z(t) = e^{\widehat{F^D}\widehat{P}t}\left(x_0 - \left(\hat{H} - I\right)\widehat{P^D}\widehat{Q}u(0)\right) + \widehat{F^D}e^{\widehat{F^D}\widehat{P}t}\int_0^t e^{-\widehat{F^D}\widehat{P}s}\widehat{Q}u(s)ds + (\hat{H} - I)\widehat{P^D}\widehat{Q}u(t), \text{ for all } t \geq 0. \tag{8}$$

Next, firstly recall that observability theory has been a long-run research interest by many investigators, see in [3, 11]. Following this work, the notion of distinguishability is used to define observability [7]. Roughly speaking about the distinguishability is that two modes are said to be distinguishable if for the non-zero state and control inputs, their outputs are distinct. Recently, easily verifiable distinguishability criteria for hybrid system are presented in [8]. This study motivates us to describe the idea of distinguishability in LDS and its characterization. Consider the following time-invariant LDS with $z(\cdot) \in \mathbb{R}^n$, $u(\cdot) \in \mathbb{R}^m$, $y(\cdot) \in \mathbb{R}^p$ as

$$T_i : \begin{cases} F_i z'(t) = P_i z(t) + Q_i u(t), \\ y(t) = R_i z(t), \ (i = 1, 2, \cdots, p), \end{cases} \tag{9}$$

where F_i, $P_i \in \mathbb{R}^{n\times n}$, $Q_i \in \mathbb{R}^{n\times m}$, $R_i \in \mathbb{R}^{p\times n}$ are constant matrices. There is no loss of generality in assuming (9) for $i = 1, 2$. Hereafter, pre-multiply (9) by $(a_i F - P)^{-1}$, for a closed form solution and consistent initial conditions, we have

$$\widehat{T}_i : \begin{cases} \widehat{F}_i z'(t) = \widehat{P}_i z(t) + \widehat{Q}_i u(t), \\ z(0) = z_{i0}, \\ y(t) = R_i z(t). \end{cases} \tag{10}$$

It is easy to see that system (9) is equivalent to (10).

For notational simplicity, hereafter, we will denote

$$F = \begin{bmatrix} \widehat{F}_1 & 0 \\ 0 & \widehat{F}_2 \end{bmatrix}, \ P = \begin{bmatrix} \widehat{P}_1 & 0 \\ 0 & \widehat{P}_2 \end{bmatrix}, \ Q = \begin{bmatrix} \widehat{Q}_1 \\ \widehat{Q}_2 \end{bmatrix} \tag{11}$$

$$R = \begin{bmatrix} R_1 & -R_2 \end{bmatrix}, \ Z_0 = \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix}, \ Y(\cdot) = y_1(\cdot) - y_2(\cdot). \tag{12}$$

Remark 1.5. **(i)** *From $\widehat{F}_i$ and $\widehat{P}_i$, it is easy to see that $FP = PF$ and $\mathcal{N}(F) \cap \mathcal{N}(P) = \{0\}$.*

(ii) *Pencils (F_1, P_1) and (F_2, P_2) are regular, if and only if the pencil (F, P) is regular.*

(iii) *If ind$(F_1) = k_1$, ind$(F_2) = k_2$, then ind$(F) = \max\{k_1, k_2\} \equiv k$.*

(iv) *From the definition of DI:*

$$P^D = \begin{bmatrix} \widehat{P}_1^D & 0 \\ 0 & \widehat{P}_2^D \end{bmatrix}, \ F^D = \begin{bmatrix} \widehat{F}_1^D & 0 \\ 0 & \widehat{F}_2^D \end{bmatrix}. \tag{13}$$

Let us recall the consistency space $\bar{\mathcal{V}}_i^*$ of inhomogeneous system for distinguishability of LDS, which is:

$$\bar{\mathcal{V}}_i^* := \left\{ \begin{array}{l} z_0 \in \mathbb{R}^n \text{ such that there exists } u \in C^\infty[0, T], \\ z \text{ is a solution of (9) with } z(0) = z_0 \end{array} \right\}.$$

Definition 1.6. *$\widehat{T}_1$ and $\widehat{T}_2$ are said to be distinguishable on $X := [0,T]$, if for any $0 \neq (z_{10}, z_{20}, u(\cdot)) \in \bar{\mathcal{V}}_1^* \times \bar{\mathcal{V}}_2^* \times C^\infty[0,T]$, the outputs $y_1(\cdot)$ and $y_2(\cdot)$ are not identical to each other on X.*

Definition 1.7. *Consider that $\mathcal{U} \subseteq C^\infty[0,T]$. We say that $\widehat{T}_1$ and $\widehat{T}_2$ are $\mathcal{U}$ input distinguishable on X if for $0 \neq (z_{10}, z_{20}, u(\cdot)) \in \bar{\mathcal{V}}_1^* \times \bar{\mathcal{V}}_2^* \times \mathcal{U}$, the outputs $y_1(\cdot)$ and $y_2(\cdot)$ are not identical to each other on X.*

Therefore, the distinguishability of $\widehat{T}_1$ and $\widehat{T}_2$ on X is equivalent to that for

$$W : \begin{cases} FZ'(t) = PZ(t) + Qu(t), \\ Z(0) = Z_0, \\ Y(t) = RZ(t), \end{cases} \tag{14}$$

$(Z_0, u(\cdot)) \neq 0$ implies that $Y(\cdot) \not\equiv 0$ on X. Thus this problem of distinguishability of LDS relates to the notion of nontrivial zero dynamics. From Remark 1.5, it follows that the closed form solution with consistent initial condition of the system (14) is

$$\begin{aligned} Z(t) = {} & e^{F^D P t}\left(Z_0 - F_p \sum_{i=0}^{k-1} \left(FP^D\right)^i P^D Q u^{(i)}(0)\right) + F_p \sum_{i=0}^{k-1} \left(FP^D\right)^i P^D Q u^{(i)}(t) \\ & + F^D e^{F^D P t} \int_0^t e^{-F^D P s} Q u(s) ds, \text{ for all } t \geq 0, \end{aligned} \tag{15}$$

where $F_p = F^D F - I$.

For the distinguishability, it is clear from equation (15) that the input $u(\cdot)$ have to smooth enough that is to avoid impulsive situation of the solution and to switch from one mode to other, at least $u(\cdot) \in C^{k-1}(X, \mathbb{R}^m)$, where $k = \text{ind}(F)$.

The organization of this paper is as: Section 2 has covered the certain consequences and equivalent criteria of distinguishability with the approach of DI. In Section 3, conclusion is given.

2. New Results

In this section, different criteria related to distinguishability are described. For our next results, let us state the following full column rank, block matrices $\hat{\mathcal{M}}$ and $\hat{\mathcal{M}}_N$, respectively as:

$$\hat{\mathcal{M}} = \begin{bmatrix} R & 0 & 0 & \cdots \\ RF^D P & RF^D P & \cdots & 0 \\ R\left(F^D P\right)^2 & R\left(F^D P\right)^2 & \cdots & RF_p\left(FP^D\right)^{k-1} P^D Q \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} \tag{16}$$

and

$$\hat{\mathcal{M}}_N = \begin{bmatrix} R & 0 & 0 & \cdots \\ RF^D P & RF^D Q & \cdots & 0 \\ R\left(F^D P\right)^2 & R\left(F^D\right)^2 PQ & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ R\left(F^D P\right)^{N+1} & R\left(F^D\right)^{N+1} P^N Q & \cdots & RF^D Q \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}, \tag{17}$$

for ind$(F) > 0$.

For characterization of polynomial input distinguishability, let us define the following polynomial classes:

$$\mathcal{P}(X) := \{f : X \to \mathbb{R}^m : f \text{ is polynomial function}\}.$$
$$\mathcal{P}_N(X) := \{f : X \to \mathbb{R}^m : f \in \mathcal{P}(X),\ \deg f \leq N\}.$$

Theorem 2.1. *Consider that ind(F) = k, $u \in \mathcal{P}(X)$, $\widehat{T}_1$ and $\widehat{T}_2$ are distinguishable, if and only if every sub-matrix constitutes of the left finite column vector of $\hat{\mathcal{M}}$ has full column rank (FCR) and it does not depend on T. Furthermore, for $u \in \mathcal{P}_N(X)$, equivalent distinguishability condition of $\widehat{T}_1$ and $\widehat{T}_2$ is that $\hat{\mathcal{M}}_N$ has FCR.*

Proof. For $u(\cdot) \in \mathbb{R}^m$, output (14) becomes:

$$Y(t) = Re^{F^D Pt}\left(Z_0 - F_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q u^{(i)}(0)\right) + RF_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q u^{(i)}(t) \\ +R\int_0^t e^{F^D P(t-s)} F^D Q u(s)\, ds, \text{ for all } 0 \le t. \tag{18}$$

Consider that $u \in \mathcal{P}_N(X)$ provided that

$$u(t) = \alpha_0 + \alpha_1 t + \cdots + \frac{\alpha_N}{N!}t^N, \ t \in X, \text{ and } \alpha_j \in \mathbb{R}^m.$$

Accordingly, we have

$$Y(t) = Re^{F^D Pt}\left(Z_0 - F_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q\alpha_i\right) + RF_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q \sum_{j=i}^{N} \frac{\alpha_j}{(j-i)!}t^{j-i} + R\int_0^t e^{F^D P(t-s)} F^D Q\left(\sum_{i=0}^{N}\frac{\alpha_i}{i!}s^i\right) ds.$$

From this, $Y(\cdot)$ is analytic which implies that $Y \equiv 0$, if and only if

$$Y^{(j)}(0) = 0, \text{ for all } j = 0, 1, 2, \cdots.$$

There is no loss of generality in assuming that ind$(F) \le N$, then we have

$$\begin{cases} RZ_0 = 0 \\ RF^D PZ_0 + \cdots + RF_p\left(FP^D\right)^{k-1} P^D Q\alpha_k = 0 \\ R\left(F^D P\right)^2 Z_0 + \cdots + RF_p\left(FP^D\right)^{k-1} P^D Q\alpha_{k+1} = 0 \\ \vdots \\ R\left(F^D P\right)^{N+1} Z_0 + \cdots + RF^D Q\alpha_N = 0 \\ \vdots \end{cases} \tag{19}$$

We acquire that (19) is equivalent to:

$$\hat{\mathcal{M}}_N [Z_0; \alpha_0; \cdots; \alpha_N] = 0. \tag{20}$$

Thus, it is N-th polynomial input distinguishable (ID) and does not depend on T which proves our theorem. □

Note that, it is evident that the observability of each mode is the necessary condition of the distinguishability. We can demonstrate it by a trivial example that two similar observable systems implies that they are clearly not distinguishable.

Corollary 2.2. *For $u \in \mathcal{P}(X)$, distinguishability of $\widehat{T}_1$ and $\widehat{T}_2$, implies $m \le 2n$.*

Next, we have the following necessary and sufficient criteria for analytical input distinguishability:

Theorem 2.3. *$\widehat{T}_1$ and $\widehat{T}_2$ are distinguishable for analytic input u, if and only if*

$$\hat{\mathcal{M}}[Z_0; \alpha_0; \alpha_1; \cdots] = 0, \tag{21}$$

which has only trivial solution provided that

$$u(t) = \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} t^j$$

converges in an interval which is open and including X.

Recall certain concepts from [7] to derive further results:

An infinite dimensional matrix having infinite rows and infinite columns is F-type if in every row of that matrix , there are only finite non-zero elements. A particular kind of F-type:

$$\mathcal{G} \equiv \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} & 0 & \cdots \\ A_{21} & A_{11} & A_{12} & \cdots & A_{1k} & 0 \\ A_{31} & A_{21} & A_{11} & A_{12} & \cdots & A_{1k} \\ A_{41} & A_{31} & A_{21} & A_{11} & A_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is $p \times m$ A-type matrix if $\{A_{ij}\}_{i=1}^{\infty} \in \mathcal{Q}_{p,m}$, where

$$\mathcal{Q}_{p,m} \equiv \left\{ \{Q_{ij}\}_{i=1}^{\infty} : Q_{ij} \in \mathbb{R}^{p\times m},\ M^i \geq \left\| Q_{ij} \right\|,\ \text{for some } M > 0 \right\},$$

for all $j = 1, \cdots, k$.

Henceforth, we consider that there are (**Type I-III**) three invertible transformations for $p \times m$ A-type matrices, [7].

Next, one of the consequence of [7, Lemma 4.6] is:

Theorem 2.4. *Consider that $RF_p \left(FP^D\right)^{k-1} P^D Q$ has FCR m, then the analytic ID does not depend on T. Indeed, it is equivalent to that (21) has only trivial solution.*

Proof. Consider that $\text{rank}\left(RF_p \left(FP^D\right)^{k-1} P^D Q\right) = m$, then $m \leq p$ and there exist $A \in \mathbb{C}^{m\times m}$, and $B \in \mathbb{C}^{p\times p}$ with $\det A \neq 0$, $\det B \neq 0$ such that $BRF_p \left(FP^D\right)^{k-1} P^D QA = I_m$ when $m = p$ or

$$BRF_p \left(FP^D\right)^{k-1} P^D QA = \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \tag{22}$$

when $m < p$. There is no loss of generality in assuming $m < p$ and consider that (22) holds. Then applying [7, Lemma 4.6] to

$$\mathcal{G}_1 \equiv \begin{bmatrix} RF_p \left(FP^D\right)^{k-1} P^D Q & 0 & 0 & \cdots \\ RF_p \left(FP^D\right)^{k-2} P^D Q & \ddots & 0 & \cdots \\ RF_p \left(FP^D\right)^{k-3} P^D Q & \ddots & RF_p \left(FP^D\right)^{k-1} P^D Q & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

implies

$$\begin{bmatrix} \tilde{D}_{11} & \cdots & \tilde{D}_{1k-1} & I_m & 0 & \cdots \\ \tilde{D}_{21} & \cdots & \tilde{D}_{1k-2} & 0 & I_m & \cdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} A^{-1} z_0 \\ A^{-1} z_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}, \tag{23}$$

where $\{\tilde{D}_{ij}\}_{i=1}^{\infty} \in Q_{m,m}$ with $j = 1, \cdots, k-1$. Hence, it is clear that $\{z_i\}_{i=1}^{\infty} \in Q_{m,1}$ and if (23) has non-trivial solutions, it must admits non-trivial solution provided that

$$\sum_{i=1}^{\infty} \frac{z_i}{i!} t^i \text{ converges in } (-\infty, +\infty).$$

Therefore, $\widehat{T_1}$ and $\widehat{T_2}$ are analytic ID on X, if and only if (23) has trivial solution. Accordingly, it does not depend on T. □

Beside that $RF_p \left(FP^D\right)^{k-1} P^D Q$ has m as FCR, by [7, Lemma 4.6], we have the following theorem:

Theorem 2.5. *Consider that $m > p$. Then $\widehat{T_1}$ and $\widehat{T_2}$ are not analytic ID.*

Next, we have another result related to analytic ID which is as follows:

Theorem 2.6. *$\widehat{T_1}$ and $\widehat{T_2}$ are analytic ID on X, if and only if (21) has only trivial solution. Accordingly, it does not depend on T.*

Proof. The proof is in the same way as in [7]. □

Theorem 2.7 implies that analytic and smooth ID are equivalent:

Theorem 2.7. *$\widehat{T_1}$ and $\widehat{T_2}$ are analytic ID, if and only if they are smooth ID.*

The above criteria are not easy to verify. Let us go further for an equivalent criteria which are relatively easy to derive.

If $\widehat{T_1}$ and $\widehat{T_2}$ are not analytic ID from Theorem 2.5, then there exists $(Z_0, u(\cdot))$ provided that

$$(Z_0, u(\cdot)) \neq 0; \qquad Y(t) = 0; \tag{24}$$

$$u(t) = \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} t^j \text{ for } t \in \mathbb{R}^+ \cup \{0\}, \tag{25}$$

with

$$\left|\alpha_j\right| \leq M^{j+1}, \text{ for all } j = 0, 1, 2, \cdots, \tag{26}$$

for some $0 < M$. It results

$$|u(t)| \leq M e^{Mt}, \text{ for all } t \in \mathbb{R}^+ \cup \{0\}$$

and Laplace transform $\mathcal{L}(u(\cdot))(s)$ can be defined for any $M < s$.

Remark 2.8. *Let $\lambda_k \in \mathbb{C}$ and $P_k(\cdot)$ be polynomial $(k = 1, 2, 3, \cdots)$. $g(\cdot)$ has the following form:*

$$g(t) = e^{\lambda_1 t} P_1(t) + e^{\lambda_2 t} P_2(t) + \cdots + e^{\lambda_n t} P_n(t),$$

if and only if $\mathcal{L}(g)$ is a proper rational function.

By using notations, $\mathcal{L}\left(Re^{F^DP}\right)(s) = \Phi(s)$,

$\Phi(s) F^D Q = \Psi(s)$, $\mathcal{L}(u(\cdot))(s) = U(s)$,

and suppose the Laplace transform of (24) which implies that $\mathcal{L}(Y(\cdot))(s)$ is

$$\begin{aligned} &\Phi(s)\left(Z_0 - F_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q u^{(i)}(0)\right) - RF_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q \sum_{j=0}^{i} s^{i-j} u^{(i-j)}(0) \\ &+\left(\Psi(s) + RF_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q s^i\right) U(s) = 0. \end{aligned} \tag{27}$$

Moreover

$$\begin{aligned} \left(\Psi(s) + RF_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q s^i\right) U(s) = &-\Phi(s)\left(Z_0 - F_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q u^{(i)}(0)\right) \\ &+RF_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q \sum_{j=0}^{i} s^{i-j} u^{(i-j)}(0). \end{aligned}$$

Assume that

$$\begin{aligned} r &= \operatorname{rank}\left(\Psi(s_0) + RF_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q s_0^i\right) \\ &= \max_{s\in\mathbb{R}^+\cup\{0\}}\left(\Psi(s) + RF_p \sum_{i=0}^{k-1}\left(FP^D\right)^i P^D Q s^i\right). \end{aligned}$$

The next consequences are the direct generalization of [8, Lemma 3.1 and Lemma 3.2]:

Lemma 2.9. *If $\widehat{T}_1$ and $\widehat{T}_2$ are not distinguishable, we can obtain a $\left(\widehat{Z}_0, \bar{u}(\cdot)\right)$ which satisfies (24) with*

$$\bar{u}(\cdot) = e^{\lambda_1 t} P_1(t) + e^{\lambda_2 t} P_2(t) + \cdots + e^{\lambda_q t} P_q(t),$$

where $P_i(\cdot)$, $(i = 1, 2, 3, \cdots, q)$ are vector-valued polynomials and $\lambda_i \in \mathbb{C}$.

Lemma 2.10. *If $\widehat{T}_1$ and $\widehat{T}_2$ are not distinguishable, we can find a $\left(\widetilde{Z}_0, \tilde{u}(\cdot)\right)$ satisfying (24) with*

$$\tilde{u}(\cdot) = e^{\lambda t}\varsigma,$$

where $\varsigma \in \mathbb{C}^m$ and $\lambda \in \mathbb{C}$.

From above, it is clear that the equivalent criteria for 0-th polynomial ID and the k-th polynomial ID are analogous. Hereafter, for any $0 \le N$,

$$\begin{bmatrix} R & 0 & 0 & \cdots \\ RF^DP & RF^DQ & \cdots & 0 \\ R\left(F^DP\right)^2 & R\left(F^D\right)^2 PQ & \cdots & \cdots \\ \vdots & \vdots & \cdots & \ddots \\ R\left(F^DP\right)^{N+1} & R\left(F^D\right)^{N+1} P^N Q & \cdots & RF^DQ \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

has FCR, if and only if

$$\begin{bmatrix} R & 0 \\ RF^DP & RF^DQ \\ R\left(F^DP\right)^2 & R\left(F^D\right)^2 PQ \\ \vdots & \vdots \\ R\left(F^DP\right)^{N+1} & R\left(F^D\right)^{N+1} P^NQ \\ \vdots & \vdots \end{bmatrix} \tag{28}$$

has FCR.

Example 2.11. *Consider* $F = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $F^D = \begin{bmatrix} 0.33 & 0 \\ 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ *corresponding to (14), where* $ind(F) = 1$, *we can determine (28) as* $\begin{bmatrix} 2.00 & 0 & 0 & 0 \\ 3.00 & 1.00 & 0 & 0 \\ 1.32 & 0 & 0.66 & 2.64 \\ 1.98 & 0 & 0.99 & 3.96 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$, *since it has FCR 4, so the corresponding system is polynomial ID.*

Furthermore, for the next results, keeping in view [4, Theorem 1] and [5, Theorem 9] for DI matrices and singular systems respectively, we have:

Corollary 2.12. *Suppose that* $FP = PF$ *and* $n > r = rank(F)$, *we have*

$$\sum_{i=0}^{r} a_i \left(F^DP\right)^i = 0, \tag{29}$$

if

$$\det(sF - P) = a_r s^r + a_{r-1}s^{r-1} + \cdots + a_1 s + a_0. \tag{30}$$

Hence, it follows from Corollary 2.12, (28) is equivalent to

$$\begin{bmatrix} R & 0 \\ RF^DP & RF^DP \\ R\left(F^DP\right)^2 & R\left(F^D\right)^2 PQ \\ \vdots & \vdots \\ R\left(F^DP\right)^r & R\left(F^D\right)^r P^{r-1}Q \end{bmatrix}$$

has a FCR.

Let us consider the following modified LDS:

$$\widetilde{T}_i : \begin{cases} \widehat{F}_i\tilde{z}'(t) = \left(\widehat{P}_i - \lambda\hat{F}_i\right)\tilde{z}(t) + \widehat{Q}_i\tilde{u}(t), \\ \tilde{z}_{i0}(0) = \tilde{z}_{i0}, \\ \tilde{y}(t) = R_i\tilde{z}(t), \end{cases}$$

for $\lambda \in \mathbb{C}$. Similar to previous notations (see e.g. (11) and (12)), let us consider the following system

$$\begin{cases} F\tilde{Z}'(t) = (P - \lambda F)\tilde{Z}(t) + Q\tilde{u}(t) \\ \tilde{Z}(0) = \tilde{Z}_0 \\ \tilde{Y}(t) = R\tilde{Z}(t), \end{cases} \tag{31}$$

where, for $\tilde{Z}_0 \in \mathbb{C}^{2n}$ and $\varsigma \in \mathbb{C}^m$, $\left(\tilde{Z}_0, \varsigma\right) \neq 0$.

We claim that the solution of (31) related to $\tilde{Z}_0$ and $\tilde{u}(t) \equiv \varsigma$ satisfying $\tilde{Y}(t) \not\equiv 0$ on $\mathbb{R}^+ \cup \{0\}$, analogously, $\widetilde{T}_1$ and $\widetilde{T}_2$ are 0-th polynomial ID. If it is not the situation, then $\left(\tilde{Z}_0, \varsigma\right) \neq 0$ provided that $\tilde{Y}(t) \equiv 0$. Consider that

$$Z(t) = e^{\lambda t}\tilde{Z}(t), \; Y(t) = e^{\lambda t}\tilde{Y}(t),$$

implies $\left(\tilde{Z}(\cdot), Y(\cdot)\right)$ solves (14) with the following

$$Z_0 = \tilde{Z}_0 \text{ and } u(t) = e^{\lambda t}\tilde{u}(t).$$

Since $Y(t) = e^{\lambda t}\tilde{Y}(t) = 0$. By Lemma 2.10, $\widehat{T}_1$ and $\widehat{T}_2$ are not distinguishable. This is a contradiction. Thus, from Theorem 2.1

$$\hat{\mathcal{M}}_\lambda := \begin{bmatrix} R & 0 \\ RF^D(P-\lambda F) & RF^DQ \\ R\left(F^D(P-\lambda F)\right)^2 & R\left(F^D\right)^2(P-\lambda F)Q \\ \vdots & \vdots \\ R\left(F^D(P-\lambda F)\right)^r & R\left(F^D\right)^r(P-\lambda F)^{r-1}Q \end{bmatrix} \tag{32}$$

has FCR. While, if $\widehat{T}_1$ and $\widehat{T}_2$ are not distinguishable, then Lemma 2.10 implies that we can determine $\left(\tilde{Z}_0, \tilde{u}(\cdot)\right)$ satisfies (24) with

$$\tilde{u}(\cdot) = e^{\lambda t}\varsigma,$$

where $\lambda \in \mathbb{C}$ and $\varsigma \in \mathbb{C}^m$. It follows that $\widetilde{T}_1$ and $\widetilde{T}_2$ are not 0-th polynomial ID. As a consequence, $\hat{\mathcal{M}}_\lambda$ has not FCR.

Summing up the above arguments, we acquire our main result as:

Theorem 2.13. *$\widehat{T}_1$ and $\widehat{T}_2$ are analytic ID, if and only if for any $\lambda \in \mathbb{C}$, $\mathcal{M}_\lambda$ has a FCR.*

Example 2.14. *Consider the following in (31) as: $P = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and take $\lambda = 1$.*

We get that (32) is

$$\hat{\mathcal{M}}_\lambda = \begin{bmatrix} 5 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2.5 & 5 & 7.5 & 2.5 \\ -0.5 & 1 & 1.5 & 0.5 \end{bmatrix}.$$

Since, $\hat{\mathcal{M}}_\lambda$ has FCR 4, so the corresponding system is analytic ID.

3. Conclusion

This paper has examined the observability of time-invariant descriptor systems. We have given characterizations of different distinguishability criteria in terms of rank conditions. Some rank conditions and also the equivalent criteria related to polynomial input distinguishability, analytic and smooth input distinguishability has been developed. We see these criteria are somehow difficult to verify. To overcome this, with the help of Laplace transform and Cayley-Hamilton theorem, we obtained a more simple Hautus-type condition for distinguishability (see, Theorem 2.13).

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