Filomat 37:22 (2023), 7467–7478
https://doi.org/10.2298/FIL2322467C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

EP elements of $\mathbb{Z}[x]/(x^2+x)$

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Abstract. In this paper, we first show that the quotient ring $\mathbb{Z}[x]/(x^2+x)$ is an involution-ring with the involution $*$ given by $(a_1+a_2x)^* = a_1-a_2-a_2x$, where $a_1, a_2 \in \mathbb{Z}$. Then, we determine explicitly all invertible elements, regular elements, MP-inverses, group invertible elements, EP elements and SEP elements of $\mathbb{Z}[x]/(x^2+x)$. Furthermore, we give a new characterization for Abel rings.

1. Introduction

The studies of generalized inverses are popular in many branches of mathematics [6–8, 12], such as matrix theory, operator theory and rings with involution, etc. For instances, the studies of EP, normal and Hermitian matrices can be seen in [1, 2, 9, 24]. In [3–5], Djordjević and Koliha considered the EP, normal and Hermitian operators. The researches of EP elements and MP-inverses in C^*-algebras can be found in [6, 10, 11]. Mosić and Djordjević considered the EP elements, MP-inverses, partial isometries, etc in rings with involution, see [12–17]. In recent years, the third author in this paper and his cooperators gave some new characterizations of EP elements, SEP elements, normal elements, partial isometrices, etc in rings with involution [19–23].

However, there are a few works on the concrete $*$-rings. In this paper, we consider the quotient ring $\mathbb{Z}[x]/(x^2+x)$. First, we prove that $\mathbb{Z}[x]/(x^2+x)$ is a $*$-ring with the involution $*$ given by $(a_1+a_2x)^* = a_1-a_2-a_2x$, where $a_1, a_2 \in \mathbb{Z}$. We then determine explicitly the invertible elements, regular elements, MP-inverses, group invertible elements, EP elements and SEP elements of $\mathbb{Z}[x]/(x^2+x)$. This paper is organized as follows. In Section 2, we recall some basic definitions and results, and make preparations for the rest of the paper. In Section 3, we study the invertible elements, regular elements, MP-inverses, group invertible elements, EP elements and SEP elements of $\mathbb{Z}[x]/(x^2+x)$. Moreover, we give a new characterization of Abel rings.

2. Preliminaries

Throughout, the letter $\mathbb{Z}$ stands for the ring of integers. In this section, we first recall the definitions of involution rings, MP-inverses, EP elements, SEP elements, etc. Then, we review the concept of quotient ring $\mathbb{Z}[x]/(x^2+x)$.

2020 *Mathematics Subject Classification*. 13B25, 16U90.
Keywords. Moore-Penrose inverse; Group invertible element; EP element; Polynomial ring; Abel ring.
Received: 26 November 2022; Accepted: 29 March 2023
Communicated by Dijana Mosić
The first author was supported by Scientific Research and Innovation Project of Graduate Students in Jiangsu Province (Grant No. KYCX22-3448) and National Natural Science Foundation of China (Grant No. 11871063).
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Let A be a ring and $* : A \to A$ be a bijective map. Then $*$ is called an involution of A provided that the followings hold:

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*. \tag{1}$$

Definition 2.1. *A ring A with an involution $*$ is called an involution ring or a $*$-ring [21].*

Definition 2.2. *An element a in a $*$-ring A is called an Hermitian element [12] if $a^* = a$.*

The set of all Hermitian elements of A is denoted by A^{Her}. It is clear that $\{aa^*, a^*a, a + a^*\} \subseteq A^{Her}$ for any $a \in A$.

Definition 2.3. *An element a in a $*$-ring A is called Moore-Penrose invertible (MP-invertible) (see [24]) if there exists $b \in A$ such that*

$$aba = a,\ bab = b,\ (ab)^* = ab,\ (ba)^* = ba, \tag{2}$$

and b is called the Moore-Penrose inverse of a, which is unique if it exists and is always written by a^+.

The set of all MP-invertible elements in A is denoted by A^+.

Definition 2.4. *An element a in a ring A is said to be a von Neumann regular element if there exists $b \in A$ such that*

$$a = aba, \tag{3}$$

and b is called an inner inverse of a.

The set of all regular elements in A is denoted by A^{reg}. In general, if $a \in A^{reg}$, the inner inverse of a is not unique. We denote the set of all inner inverses of a by $a\{1\}$, and a^- denotes some fixed inner inverse of a.

An element $e \in A$ is called idempotent if $e^2 = e$, and the set of all idempotent elements of A is denoted by $E(A)$.

Definition 2.5. *An element $e \in E(A)$ is called left (resp. right) semicentral idempotent if $ae = eae$ (resp. $ea = eae$) for each $a \in A$. If e is both left and right semicentral idempotent, then e is called central idempotent.*

Clearly, $e \in E(A)$ is left semicentral if and only if $1 - e$ is right semicentral.

A is called an Abel ring if each element $e \in E(A)$ is central. It is easy to see that A is Abel if and only if each $e \in E(A)$ is left semicentral.

Definition 2.6. *An element $e \in E(A)$ is called a projection if $e^* = e$.*

The set of all projections of A is denoted by A^{proj}. Clearly, $e \in A^{proj}$ if and only if $e = ee^*$ if and only if $e = e^*e$.

Definition 2.7. *An element a in a ring A is called a group invertible element if there exists $x \in A$ such that*

$$a = axa,\ x = xax,\ ax = xa,$$

where x is called the group inverse of a, which is unique if it exists and is denoted by $a^\#$.

The set of all group invertible elements of A is denoted by $A^\#$. Furthermore, $U(A) \subseteq A^\#$ and $E(A) \subseteq A^\#$, where $U(A)$ is the set of invertible elements of A. Moreover $A^\# = A^{reg}$ when A is commutative.

Definition 2.8. *An element $a \in A^\# \cap A^+$ is said to be EP if $a^\# = a^+$ [12].*

The set of all EP elements of A is denoted by A^{EP}.

Definition 2.9. *An element $a \in A^{EP}$ is called strongly EP (or SEP) [12, 20, 21, 23] if $a^+ = a^*$.*

The set of all SEP elements of A is denoted by A^{SEP}.

Lemma 2.10. *Let A be a $*$-ring. Then the followings hold:*

(1) Let $a \in A^{reg}$. Then $a\{1\} = \{a^- + b - a^-abaa^- | b \in A\}$, where a^- is some fixed inner inverse of a.

(1) $U(A) \subseteq A^\# \cap A^+ \subseteq A^{reg}$. Moreover, if $a \in U(A)$, then $a^\# = a^+ = a^{-1}$ and $a\{1\} = \{a^{-1}\}$.

(3) $A^{proj} = E(A) \cap A^{Her}$.

(4) A is an Abel ring if and only if for each $a \in A^{reg}$, $aa^- = a^-a$.

Proof. (1)-(3) is obvious, here we only prove (4).

"$\Rightarrow$" Assume that A is an Abel ring and $a \in A^{reg}$, then $aa^-, a^-a \in E(A) \subseteq C(A)$, the center of A. Hence $aa^- = (aa^-a)a^- = (a^-a)(aa^-) = a^-(aa^-)a = a^-a$.

"$\Leftarrow$" For any $e \in E(A)$, $x \in A$, let $g = e - ex(1-e)$, then $eg = g$, $ge = e$, $g^2 = g$. Note that $geg = g$, hence $ge = eg$ by hypothesis, and so $e = g$, i.e., $ex(1-e) = 0$. By the arbitrariness of x, we have $eA(1-e) = 0$, which implies that A is an Abel ring. □

Next, we review the concept of the quotient ring $\mathbb{Z}[x]/(x^2+x)$. It is easy to see that $\mathbb{Z}[x]/(x^2+x)$ has a $\mathbb{Z}$-basis $\{1, x\}$. Let $*$ be the map given by $(a_1 + a_2x)^* = a_1 - a_2 - a_2x$ of $\mathbb{Z}[x]/(x^2+x)$, where $a_1, a_2 \in \mathbb{Z}$. Then we have the following result.

Lemma 2.11. *The quotient ring $\mathbb{Z}[x]/(x^2+x)$ is a $*$-ring with $*$ determined by $(a_1 + a_2x)^* = a_1 - a_2 - a_2x$, where $a_1, a_2 \in \mathbb{Z}$.*

Proof. Let $a = a_1 + a_2x, b = b_1 + b_2x \in \mathbb{Z}[x]/(x^2+x)$, $a_i, b_i \in \mathbb{Z}$, $i = 1, 2$. Then

$$\begin{aligned}(a^*)^* &= ((a_1 + a_2x)^*)^* \\ &= ((a_1 - a_2) - a_2x)^* \\ &= (a_1 - a_2) + a_2 + a_2x \\ &= a_1 + a_2x \\ &= a,\end{aligned}$$

$$\begin{aligned}(a+b)^* &= ((a_1 + a_2x) + (b_1 + b_2x))^* \\ &= ((a_1 + b_1) + (a_2 + b_2)x)^* \\ &= a_1 + b_1 - a_2 - b_2 - (a_2 + b_2)x \\ &= (a_1 - a_2 - a_2x) + (b_1 - b_2 - b_2x) \\ &= a^* + b^*,\end{aligned}$$

and

$$\begin{aligned}(ab)^* &= ((a_1 + a_2x)(b_1 + b_2x))^* \\ &= (a_1b_1 + a_1b_2x + a_2b_1x + a_2b_2x^2)^* \\ &= (a_1b_1 + (a_1b_2 + a_2b_1 - a_2b_2)x)^* \\ &= (a_1b_1 - a_1b_2 - a_2b_1 + a_2b_2) - (a_1b_2 + a_2b_1 - a_2b_2)x \\ &= (a_1 - a_2 - a_2x)(b_1 - b_2 - b_2x) \\ &= (b_1 - b_2 - b_2x)(a_1 - a_2 - a_2x) \\ &= (b_1 + b_2x)^*(a_1 + a_2x)^* \\ &= b^*a^*.\end{aligned}$$

Thus, by (1), $\mathbb{Z}[x]/(x^2+x)$ is a $*$-ring. □

3. EP elements of $\mathbb{Z}[x]/(x^2+x)$

The aim of this section is to determine explicitly all invertible elements, projections, regular elements, group invertible elements, MP-inverses, EP elements and SEP elements of $\mathbb{Z}[x]/(x^2+x)$.

Lemma 3.1. $(\mathbb{Z}[x]/(x^2+x))^{Her} = \mathbb{Z}$.

Proof. Assume that $a = a_1 + a_2x \in (\mathbb{Z}[x]/(x^2+x))^{Her}$, $a_1, a_2 \in \mathbb{Z}$, then

$$a^* = (a_1 + a_2x)^* = a_1 - a_2 - a_2x = a_1 + a_2x.$$

So

$$\begin{cases} a_1 - a_2 = a_1, \\ -a_2 = a_2. \end{cases} \tag{4}$$

It is easy to see that $(k, 0)$, $k \in \mathbb{Z}$ are all solutions of (4). It follows that $(\mathbb{Z}[x]/(x^2+x))^{Her} = \mathbb{Z}$. □

Lemma 3.2. $E(\mathbb{Z}[x]/(x^2+x)) = \{0, 1, -x, 1+x\}$.

Proof. Assume that $a = a_1 + a_2x \in E(\mathbb{Z}[x]/(x^2+x))$, $a_1, a_2 \in \mathbb{Z}$, then

$$\begin{aligned} a^2 &= (a_1 + a_2x)^2 \\ &= a_1^2 + 2a_1a_2x + a_2^2x^2 \\ &= a_1^2 + (2a_1a_2 - a_2^2)x \\ &= a_1 + a_2x \\ &= a, \end{aligned}$$

which implies that

$$\begin{cases} a_1^2 = a_1, \\ 2a_1a_2 - a_2^2 = a_2. \end{cases}$$

Case I: $a_1 = 0$, then

$$-a_2^2 = a_2 \Rightarrow a_2 = 0 \text{ or } a_2 = -1.$$

So, $a = 0$ or $a = -x$.

Case II: $a_1 = 1$, then

$$a_2^2 = a_2 \Rightarrow a_2 = 0 \text{ or } a_2 = 1.$$

Hence $a = 1$ or $a = 1 + x$. By above, we have

$$E(\mathbb{Z}[x]/(x^2+x)) = \{0, 1, -x, 1+x\}.$$

□

Proposition 3.3. $(\mathbb{Z}[x]/(x^2+x))^{proj} = \{0, 1\}$.

Proof. By Lemmas 2.10 (3), 3.1 and 3.2, $(\mathbb{Z}[x]/(x^2+x))^{proj} = E(\mathbb{Z}[x]/(x^2+x)) \cap (\mathbb{Z}[x]/(x^2+x))^{Her} = \{0, 1, -x, 1+x\} \cap \mathbb{Z} = \{0, 1\}$. □

Theorem 3.4. $(\mathbb{Z}[x]/(x^2+x))^{reg} = \{0, 1, -1, x, -x, 1+x, -1-x, 1+2x, -1-2x\}$.

Proof. Let $a = a_1+a_2x \in (\mathbb{Z}[x]/(x^2+x))^{reg}$, where $a_1, a_2 \in \mathbb{Z}$. Then by (2), there exists $b = b_1+b_2x \in \mathbb{Z}[x]/(x^2+x)$, $b_1, b_2 \in \mathbb{Z}$, such that $a = aba$. That is,

$$\begin{aligned} aba &= a^2b \\ &= (a_1 + a_2x)^2(b_1 + b_2x) \\ &= ((a_1^2 + (2a_1a_2 - a_2^2)x)(b_1 + b_2x) \\ &= a_1^2b_1 + (a_1^2b_2 + 2a_1a_2b_1 - a_2^2b_1 - 2a_1a_2b_2 + a_2^2b_2)x \\ &= a_1 + a_2x \\ &= a, \end{aligned}$$

which shows that

$$\begin{cases} a_1^2b_1 = a_1, \\ a_1^2b_2 + 2a_1a_2b_1 - a_2^2b_1 - 2a_1a_2b_2 + a_2^2b_2 = a_2. \end{cases} \tag{5}$$

Case I: $a_1 = 0$. Then

$$a_2^2(b_2 - b_1) = a_2.$$

If $a_2 = 0$, then $a = 0$. It is obvious that $0 \in (\mathbb{Z}[x]/(x^2 + x))^{reg}$ and $0\{1\} = \mathbb{Z}[x]/(x^2 + x)$. If $a_2 \neq 0$, then

$$a_2(b_2 - b_1) = 1.$$

Since $a_2, b_1, b_2 \in \mathbb{Z}$, $a_2 = b_2 - b_1 = 1$ or $a_2 = b_2 - b_1 = -1$. In case $a_2 = b_2 - b_1 = 1$, $a = x$ and $b = b_1 + (1 + b_1)x$, $b_1 \in \mathbb{Z}$, then

$$\begin{aligned} a^2b &= x^2(b_1 + (1 + b_1)x) \\ &= -x(b_1 + (1 + b_1)x) \\ &= -b_1x + (1 + b_1)(-x^2) \\ &= -b_1x + (1 + b_1)x \\ &= x \\ &= a. \end{aligned}$$

It follows that $x \in (\mathbb{Z}[x]/(x^2 + x))^{reg}$ and $x\{1\} = \{k + (1 + k)x | k \in \mathbb{Z}\}$. If $a_2 = b_2 - b_1 = -1$, then $a = -x$ and $b = b_1 - (1 - b_1)x$, $b_1 \in \mathbb{Z}$. In this case,

$$\begin{aligned} a^2b &= (-x)^2(b_1 - (1 - b_1)x) \\ &= -x(b_1 - (1 - b_1)x) \\ &= -b_1x - (1 - b_1)(-x^2) \\ &= -b_1x - (1 - b_1)x \\ &= -x \\ &= a, \end{aligned}$$

which shows that $-x \in (\mathbb{Z}[x]/(x^2 + x))^{reg}$ and $-x\{1\} = \{k - (1 - k)x | k \in \mathbb{Z}\}$.

Case II: $a_1 \neq 0$, then $a_1b_1 = 1$. Since $a_1, b_1 \in \mathbb{Z}$, $a_1 = b_1 = 1$ or $a_1 = b_1 = -1$. If $a_1 = b_1 = 1$, then (5) becomes

$$b_2 + 2a_2 - a_2^2 - 2a_2b_2 + a_2^2b_2 = a_2,$$

i.e.,

$$b_2(a_2 - 1)^2 = a_2(a_2 - 1).$$

If $a_2 = 0$, then $a = 1$. Obviously, $1 \in (\mathbb{Z}[x]/(x^2+x))^{reg}$ and $1\{1\} = \{1\}$. If $a_2 = 1$, then $a = 1 + x$ and $b = 1 + b_2x$, $b_2 \in \mathbb{Z}$. In this case,

$$\begin{aligned} a^2b &= (1+x)^2(1+b_2x) \\ &= (1+x)(1+b_2x) \\ &= 1 + b_2x + x + b_2x^2 \\ &= 1 + x \\ &= a. \end{aligned}$$

It follows that $1 + x \in (\mathbb{Z}[x]/(x^2+x))^{reg}$ and $1 + x\{1\} = \{1 + kx | k \in \mathbb{Z}\}$. In case $a_2 \neq 0$ and $a_2 \neq 1$, then

$$b_2(a_2 - 1) = a_2,$$

i.e.,

$$(a_2 - 1)(b_2 - 1) = 1.$$

Note that $a_2, b_2 \in \mathbb{Z}$, so $a_2 - 1 = b_2 - 1 = 1$ or $a_2 - 1 = b_2 - 1 = -1$. When $a_2 = b_2 = 2$, then $a = 1 + 2x \in (\mathbb{Z}[x]/(x^2+x))^{reg}$ and $1 + 2x\{1\} = \{1 + 2x\}$. In case $a_2 = b_2 = 0$, then $a = 1$, which we consider above.

If $a_1 = b_1 = -1$, then (5) becomes

$$b_2 + 2a_2 + a_2^2 + 2a_2b_2 + a_2^2b_2 = a_2,$$

i.e.,

$$b_2(a_2 + 1)^2 = -a_2(a_2 + 1).$$

If $a_2 = 0$, then $a = 1$, which we study before. If $a_2 = -1$, then $a = -1 - x$ and $b = -1 + b_2x$, $b_2 \in \mathbb{Z}$. In this case,

$$\begin{aligned} a^2b &= (-1-x)^2(-1+b_2x) \\ &= (1+x)(-1+b_2x) \\ &= -1 - x \\ &= a, \end{aligned}$$

which shows that $a = -1 - x \in (\mathbb{Z}[x]/(x^2+x))^{reg}$ and $-1 - x\{1\} = \{-1 + kx | k \in \mathbb{Z}\}$. In case $a_2 \neq 0$ and $a_2 \neq -1$, then

$$b_2(a_2 + 1) = -a_2,$$

i.e.,

$$(a_2 + 1)(b_2 + 1) = 1.$$

Notice that $a_2, b_2 \in \mathbb{Z}$, so $a_2 + 1 = b_2 + 1 = 1$ or $a_2 + 1 = b_2 + 1 = -1$. When $a_2 = b_2 = 0$, then $a = -1 \in (\mathbb{Z}[x]/(x^2+x))^{reg}$ and $-1\{1\} = \{-1\}$. In case $a_2 = b_2 = -2$, then $a = -1 - 2x \in (\mathbb{Z}[x]/(x^2+x))^{reg}$ and $-1 - 2x\{1\} = \{-1 - 2x\}$. Summarizing the discussion above, we have

$$(\mathbb{Z}[x]/(x^2+x))^{reg} = \{0, 1, -1, x, -x, 1+x, -1-x, 1+2x, -1-2x\},$$

and $0\{1\} = \mathbb{Z}[x]/(x^2+x)$, $1\{1\} = \{1\}$, $-1\{1\}$, $x\{1\} = \{k + (1+k)x | k \in \mathbb{Z}\}$, $-x\{1\} = \{k - (1-k)x | k \in \mathbb{Z}\}$, $1 + x\{1\} = \{1 + kx | k \in \mathbb{Z}\}$, $-1 - x\{1\} = \{-1 + kx | k \in \mathbb{Z}\}$, $1 + 2x\{1\} = \{1 + 2x\}$ and $-1 - 2x\{1\} = \{-1 - 2x\}$. □

Corollary 3.5. $U(\mathbb{Z}[x]/(x^2+x)) = \{1, -1, 1+2x, -1-2x\}$.

Proof. By Lemma 2.10 (1), $U(\mathbb{Z}[x]/(x^2+x)) \subseteq (\mathbb{Z}[x]/(x^2+x))^{reg}$, and one can easily prove that the element 0 is not invertible, and $1^{-1} = 1$, $-1^{-1} = -1$, $(1+2x)^{-1} = 1+2x$ and $(-1-2x)^{-1} = -1-2x$. Now, it remains to show that $x, -x, 1+x, -1-x \notin U(\mathbb{Z}[x]/(x^2+x))$. In fact, for any $a = a_1 + a_2 \in \mathbb{Z}[x]/(x^2+x)$, $a_1, a_2 \in \mathbb{Z}$,

$$\begin{aligned} xa &= x(a_1 + a_2x) \\ &= a_1x + a_2x^2 \\ &= (a_1 - a_2)x \\ &\neq 1, \end{aligned}$$

and

$$\begin{aligned} (1+x)a &= (1+x)(a_1 + a_2x) \\ &= a_1 + a_2x + a_1x + a_2x^2 \\ &= a_1 + a_1x \\ &= a_1(1+x) \\ &\neq 1. \end{aligned}$$

Hence $x, 1+x \notin U(\mathbb{Z}[x]/(x^2+x))$. Similarly, one can prove that $-x, -1-x \notin U(\mathbb{Z}[x]/(x^2+x))$. □

Proposition 3.6. *The followings hold:*
(1) $(\mathbb{Z}[x]/(x^2+x))^+ = \{0, 1, -1, 1+2x, -1-2x\}$.
(2) $(\mathbb{Z}[x]/(x^2+x))^\# = \{0, 1, -1, x, -x, 1+x, -1-x, 1+2x, -1-2x\}$.

Proof. We first show (2). Since $\mathbb{Z}[x]/(x^2+x)$ is commutative, $(\mathbb{Z}[x]/(x^2+x))^\# = (\mathbb{Z}[x]/(x^2+x))^{reg}$. It is easy to see that $0^\# = 0$. By Lemma 2.10 (1), we have that for any $a \in U(\mathbb{Z}[x]/(x^2+x)) \subseteq (\mathbb{Z}[x]/(x^2+x))^{reg}$, $a^\# = a^{-1}$. Next we consider $x^\#, (-x)^\#, (1+x)^\#, (-1-x)^\#$. First, we consider $x^\#$. By the proof of Theorem 3.4, we know that $x^\#$ has the form $k + (1+k)x$, $k \in \mathbb{Z}$. Then by a straightforward computation, we have

$$\begin{aligned} (k+(1+k)x)x(k+(1+k)x) &= (kx + (1+k)x^2)(k+(1+k)x) \\ &= -x(k+(1+k)x) \\ &= -(-x) \\ &= x. \end{aligned}$$

Hence $k + (1+k)x = x \Rightarrow k = 0$, i.e., $x^\# = x$. Similarly, one can prove that if $a(-x)a = a$, then a must be $-x$, i.e., $(-x)^\# = -x$. For $1+x$, we know that $(1+x)^\#$ has the form $1+kx$, $k \in \mathbb{Z}$. By a direct computation,

$$\begin{aligned} (1+kx)(1+x)(1+kx) &= (1+kx)^2(1+x) \\ &= (1+kx)^2(1+x) \\ &= (1+(2k-k^2)x)(1+x) \\ &= 1 + x + (2k-k^2)x + (2k-k^2)x^2 \\ &= 1+x. \end{aligned}$$

Hence $1+kx = 1+x \Rightarrow k = 0$, i.e., $(1+x)^\# = 1+x$. Similarly, one can check that if $a(-1-x)a = a$, then a must be $-1-x$, i.e., $(-1-x)^\# = -1-x$. Thus (2) follows.

Now we prove (1). By Lemma 2.10 (1), $\{1, -1, 1+2x, -1-2x\} = U(\mathbb{Z}[x]/(x^2+x)) \subseteq (\mathbb{Z}[x]/(x^2+x))^+$. It is obvious that $0^+ = 0$. By the proof of (2), we know that in order to prove that $(\mathbb{Z}[x]/(x^2+x))^+ = \{0, 1, -1, 1+2x, -1-2x\}$, it is sufficient to show that $(x^2)^* \neq x^2$ and $((1+x)^2)^* \neq (1+x)^2$. In fact,

$$(x^2)^* = (-x)^* = 1 + x \neq -x = x^2,$$

and

$$((1+x)^2)^* = (1+x)^* = -x \neq 1+x = (1+x)^2.$$

Thus, the proof is finished. □

Proposition 3.7. *The following statements hold:*
(1) $(\mathbb{Z}[x]/(x^2+x))^{EP} = \{0, 1, -1, 1+2x, -1-2x\}$.
(2) $(\mathbb{Z}[x]/(x^2+x))^{SEP} = \{0, 1, -1\}$.

Proof. (1) follows form Proposition 3.6. (2) follows from the fact that

$$(1+2x)^+ = 1+2x \neq -1-2x = (1+2x)^*, \quad (-1-2x)^+ = -1-2x \neq 1+2x = (-1-2x)^*.$$

□

Corollary 3.8. $\mathbb{Z}[x]/(x^2+x)$ *is an Abel ring.*

Proof. It follows the fact that $\mathbb{Z}[x]/(x^2+x)$ is commutative and Lemma 2.10 (4). □

Let R be a ring with unit 1 and

$$T_2^{(e)}(R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R \right\},$$

where $e \in E(R)$. For any

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix} \in T_2^{(e)}(R),$$

where $a_1, a_2, b_1, b_2 \in R$, define the addition and multiplication of $T_2^{(e)}(R)$ respectively given by

$$A + B = \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ 0 & a_1+b_1 \end{pmatrix}, \quad AB = \begin{pmatrix} a_1b_1 & a_1b_2 + a_2b_1 - ea_2b_2 \\ 0 & a_1b_1 \end{pmatrix}.$$

Then we have the following results.

Theorem 3.9. $e \in E(R)$ *is central if and only if* $T_2^{(e)}(R)$ *is a ring.*

Proof. "⇒" Assume that $e \in E(R)$ is central, then in order to show that $T_2^{(e)}(R)$ is a ring, it is sufficient to prove that for any $A, B, C \in T_2^{(e)}(R)$, $(AB)C = A(BC), A(B+C) = AB + AC, (B+C)A = BA + CA$. Assume that

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix},$$

then by a straightforward computation, we have

$$\begin{aligned} (AB)C &= \begin{pmatrix} a_1b_1 & a_1b_2 + a_2b_1 - ea_2b_2 \\ 0 & a_1b_1 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_1c_1 & a_1b_1c_2 + (a_1b_2 + a_2b_1 - ea_2b_2)c_1 - e(a_1b_2 + a_2b_1 - ea_2b_2)c_2 \\ 0 & a_1b_1c_1 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_1c_1 & a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1 - ea_2b_2c_1 - ea_1b_2c_2 - ea_2b_1c_2 + e^2a_2b_2c_2 \\ 0 & a_1b_1c_1 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_1c_1 & a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1 - ea_2b_2c_1 - ea_1b_2c_2 - ea_2b_1c_2 + ea_2b_2c_2 \\ 0 & a_1b_1c_1 \end{pmatrix}, \end{aligned} \tag{6}$$

$$
\begin{aligned}
A(BC) &= \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} b_1c_1 & b_1c_2 + b_2c_1 - eb_2c_2 \\ 0 & b_1c_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1b_1c_1 & a_1(b_1c_2 + b_2c_1 - eb_2c_2) + a_2b_1c_1 - ea_2(b_1c_2 + b_2c_1 - eb_2c_2) \\ 0 & a_1b_1c_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1b_1c_1 & a_1b_1c_2 + a_1b_2c_1 - ea_1b_2c_2 + a_2b_1c_1 - ea_2b_1c_2 - ea_2b_2c_1 + e^2a_2b_2c_2 \\ 0 & a_1b_1c_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1b_1c_1 & a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1 - ea_2b_2c_1 - ea_1b_2c_2 - ea_2b_1c_2 + ea_2b_2c_2 \\ 0 & a_1b_1c_1 \end{pmatrix},
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
A(B+C) &= \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} b_1 + c_1 & b_2 + c_2 \\ 0 & b_1 + c_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1(b_1 + c_1) & a_1(b_2 + c_2) + a_2(b_1 + c_1) - ea_2(b_2 + c_2) \\ 0 & a_1(b_1 + c_1) \end{pmatrix} \\
&= \begin{pmatrix} a_1b_1 + a_1c_1 & a_1b_2 + a_1c_2 + a_2b_1 + a_2c_1 - ea_2b_2 - ea_2c_2 \\ 0 & a_1b_1 + a_1c_1 \end{pmatrix},
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
AB + AC &= \begin{pmatrix} a_1b_1 & a_1b_2 + a_2b_1 - ea_2b_2 \\ 0 & a_1b_1 \end{pmatrix} + \begin{pmatrix} a_1c_1 & a_1c_2 + a_2c_1 - ea_2c_2 \\ 0 & a_1c_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1b_1 + a_1c_1 & a_1b_2 + a_1c_2 + a_2b_1 + a_2c_1 - ea_2b_2 - ea_2c_2 \\ 0 & a_1b_1 + a_1c_1 \end{pmatrix},
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
(B+C)A &= \begin{pmatrix} b_1 + c_1 & b_2 + c_2 \\ 0 & b_1 + c_1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix} \\
&= \begin{pmatrix} (b_1 + c_1)a_1 & (b_1 + c_1)a_2 + (b_2 + c_2)a_1 - e(b_2 + c_2)a_2 \\ 0 & (b_1 + c_1)a_1 \end{pmatrix} \\
&= \begin{pmatrix} b_1a_1 + c_1a_1 & b_1a_2 + c_1a_2 + b_2a_1 + c_2a_1 - eb_2a_2 - ec_2a_2 \\ 0 & b_1a_1 + c_1a_1 \end{pmatrix},
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
BA + CA &= \begin{pmatrix} b_1a_1 & b_1a_2 + b_2a_1 - eb_2a_2 \\ 0 & b_1a_1 \end{pmatrix} + \begin{pmatrix} c_1a_1 & c_1a_2 + c_2a_1 - ec_2a_2 \\ 0 & c_1a_1 \end{pmatrix} \\
&= \begin{pmatrix} b_1a_1 + c_1a_1 & b_1a_2 + c_1a_2 + b_2a_1 + c_2a_1 - eb_2a_2 - ec_2a_2 \\ 0 & b_1a_1 + c_1a_1 \end{pmatrix}.
\end{aligned} \tag{11}
$$

Then $(AB)C = A(BC)$ by (6) and (7), $A(B+C) = AB + AC$ by (8) and (9), $(B+C)A = BA + CA$ by (10) and (11). Hence $T_2^{(e)}(R)$ is a ring.

"$\Leftarrow$" Assume that $T_2^{(e)}(R)$ is a ring, then for any $a \in R$, let

$$A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in T_2^{(e)}(R).$$

Then $(AB)C = A(BC)$, that is

$$\begin{pmatrix} 0 & a - ea \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a - ae \\ 0 & 0 \end{pmatrix}.$$

Hence $a - ea = a - ae$, it follows that $ae = ea$. Hence e is central. □

Theorem 3.10. *R is an Abel ring if and only if for each $e \in E(R)$, $T_2^{(e)}(R)$ is a ring.*

Proof. It follows from Theorem 3.9. □

Now let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in T_2^{(e)}(R)$. Define

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^* = \begin{pmatrix} a & (-1+2e)b \\ 0 & a \end{pmatrix}, \tag{12}$$

where $a, b \in R$. Then we have the following.

Theorem 3.11. *Let R be a commutative ring and $e \in E(R)$. Then $T_2^{(e)}(R)$ is a $*$-ring, where $*$ is defined as in (12).*

Proof. Assume that

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix} \in T_2^{(e)}(R),$$

then by a straightforward computation, we have

$$\begin{aligned}
(A^*)^* &= \begin{pmatrix} a_1 & (-1+2e)a_2 \\ 0 & a_1 \end{pmatrix}^* \\
&= \begin{pmatrix} a_1 & (-1+2e)^2 a_2 \\ 0 & a_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix} \\
&= A,
\end{aligned}$$

$$\begin{aligned}
(A+B)^* &= \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ 0 & a_1+b_1 \end{pmatrix}^* \\
&= \begin{pmatrix} a_1+b_1 & (-1+2e)(a_2+b_2) \\ 0 & a_1+b_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & (-1+2e)a_2 \\ 0 & a_1 \end{pmatrix} + \begin{pmatrix} b_1 & (-1+2e)b_2 \\ 0 & b_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}^* + \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix}^* \\
&= A^* + B^*,
\end{aligned}$$

$$\begin{aligned}
(AB)^* &= \begin{pmatrix} a_1b_1 & a_1b_2 + a_2b_1 - ea_2b_2 \\ 0 & a_1b_1 \end{pmatrix}^* \\
&= \begin{pmatrix} a_1b_1 & (-1+2e)(a_1b_2 + a_2b_1 - ea_2b_2) \\ 0 & a_1b_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1b_1 & (-1+2e)(a_1b_2 + a_2b_1) - ea_2b_2 \\ 0 & a_1b_1 \end{pmatrix},
\end{aligned}$$

and

$$B^*A^* = \begin{pmatrix} b_1 & (-1+2e)b_2 \\ 0 & b_1 \end{pmatrix}\begin{pmatrix} a_1 & (-1+2e)a_2 \\ 0 & a_1 \end{pmatrix}$$
$$= \begin{pmatrix} b_1a_1 & b_1(-1+2e)a_2 + (-1+2e)b_2a_1 - e(-1+2e)^2a_2b_2 \\ 0 & b_1a_1 \end{pmatrix}$$
$$= \begin{pmatrix} a_1b_1 & (-1+2e)(a_1b_2 + a_2b_1) - ea_2b_2 \\ 0 & a_1b_1 \end{pmatrix}.$$

Thus, by (1), $T_2^{(e)}(R)$ is a $*$-ring. □

Theorem 3.12. $\mathbb{Z}[x]/(x^2+x) \cong T_2^{(1)}(\mathbb{Z})$.

Proof. Consider the map $f : \mathbb{Z}[x]/(x^2+x) \to T_2^{(1)}(\mathbb{Z})$,

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

i.e.,

$$a_1 + a_2x \mapsto \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix},$$

where $a_1, a_2 \in \mathbb{Z}$. Then for any $a_1 + a_2x, b_1 + b_2x \in \mathbb{Z}[x]/(x^2+x)$, we have

$$f(a_1 + a_2x + b_1 + b_2x) = f(a_1 + b_1 + (a_2 + b_2)x)$$
$$= \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ 0 & a_1 + b_1 \end{pmatrix}$$
$$= \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix}$$
$$= f(a_1 + a_2x) + f(b_1 + b_2x),$$

and

$$f((a_1 + a_2x)(b_1 + b_2x)) = f(a_1b_1 + (a_1b_2 + a_2b_1 - a_2b_2)x)$$
$$= \begin{pmatrix} a_1b_1 & a_1b_2 + a_2b_1 - a_2b_2 \\ 0 & a_1b_1 \end{pmatrix}$$
$$= \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix}$$
$$= f(a_1 + a_2x)f(b_1 + b_2x).$$

Thus, f is a ring homomorphism. It is easy to see that f is injective and surjective. This completes the proof. □

Acknowledgement

The authors thank the anonymous referee for numerous suggestions that helped improve our paper substantially.

Conflict of Interest

The authors declared that they have no conflict of interest.

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