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Harmonic trigonometrically convexity

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Abstract. In this study, we introduce and study the concept of harmonic trigonometrically convex functions and their some algebric properties. We prove two Hermite-Hadamard type inequalities for the newly introduced class of functions. We also obtain some refinements of the Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is harmonic trigonometrically convex.

1. Introduction

Throughout the paper I is a non-empty interval in $\mathbb{R}$.

A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. See articles [4, 8–10, 12–18] and the references therein.

Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

for all $a, b \in I$ with $a < b$. This double inequality is well known as the Hermite-Hadamard inequality (for more information, see [5]). Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [3, 20]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if the function f is concave.

The main purpose of this paper is to introduce the concept of harmonic trigonometrically convex functions and establish some results connected with the right-hand side of new inequalities similar to the Hermite-hadamard inequality for these classes of functions.

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Definition 1.1 ([4]). *A non-negative function $f : I \to \mathbb{R}$ is said to be a P-function if the inequality*

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. The set of P-functions on the interval I is denoted by $P(I)$.

Definition 1.2. *[19] Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$. It is clear that, if we choose $h(\alpha) = \alpha$ and $h(\alpha) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

In [11], Kadakal gave the concept of trigonometrically convex function and Hermite-Hadamard type inequalities as follows:

Definition 1.3 ([11]). *A non-negative function $f : I \to \mathbb{R}$ is called trigonometrically convex if for every $x, y \in I$ and $t \in [0,1]$,*

$$f(tx + (1-t)y) \leq \left(\sin\frac{\pi t}{2}\right) f(x) + \left(\cos\frac{\pi t}{2}\right) f(y). \tag{1}$$

The class of all trigonometrically convex functions is denoted by $TC(I)$ on interval I. We note that, every trigonometrically convex function is a h-convex function for $h(t) = \sin\frac{\pi t}{2}$. Morever, if $f(x)$ is a nonnegative function, then every trigonometric convex function is a P-function.

Theorem 1.4 ([11]). *Let $f : [a,b] \to \mathbb{R}$ be a trigonometrically convex function. If $a < b <$ and $f \in L[a,b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{2}{\pi}\left[f(a) + f(b)\right].$$

Let the function $f : [a,b] \to \mathbb{R}$,be a trigonometrically convex function. If $a < b$ and $f \in L[a,b]$, then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\sqrt{2}}{b-a}\int_a^b f(x)dx.$$

In [1], Bekar gave the concept of trigonometrically P-function and Hermite-Hadamard type inequalities as follows:

Definition 1.5. *A non-negative function $f : I \to \mathbb{R}$ is called trigonometrically P-functions if for every $x, y \in I$ and $t \in [0,1]$,*

$$f(tx + (1-t)y) \leq \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right)\left[f(x) + f(y)\right].$$

We will denote by $TP(I)$ the class of all trigonometrically P-functions on interval I.

Theorem 1.6. *Let $f : [a,b] \to \mathbb{R}$ be a trigonometrically P-function. If $a < b <$ and $f \in L[a,b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{4}{\pi}\left[f(a) + f(b)\right].$$

Theorem 1.7. *Let the function $f : [a, b] \to \mathbb{R}$,be a trigonometrically P-function. If $a < b$ and $f \in L[a, b]$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2\sqrt{2}}{b-a} \int_a^b f(x)dx.$$

In [6], Kadakal and Kadakal gave the concept of inverse trigonometrically convex functions and Hermite-Hadamard type inequalities as follows:

Definition 1.8 ([6]). *A non-negative function $f : I \to \mathbb{R}$ is called inverse trigonometrically convex function (or inverse trigonometrically convex) if for every $x, y \in I$ and $t \in [0, 1]$,*

$$f(tx + (1-t)y) \leq \left(\frac{2}{\pi} \arcsin t\right) f(x) + \left(\frac{2}{\pi} \arccos t\right) f(y). \tag{2}$$

We will denote by $IT(I)$ the class of all inverse trigonometrically convex functions on interval I.

Theorem 1.9 ([6]). *Let $f : [a, b] \to \mathbb{R}$ be an inverse trigonometrically convex function. If $a < b$ and $f \in L[a, b]$, then*

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \left(1 - \frac{2}{\pi}\right) f(a) + \frac{2}{\pi} f(b).$$

Theorem 1.10 ([6]). *Let the function $f : [a, b] \to \mathbb{R}$,be an inverse trigonometrically convex function. If $a < b$ and $f \in L[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx.$$

Definition 1.11 ([7]). *Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq t f(y) + (1-t) f(x)$$

for all $x, y \in I$ and $t \in [0, 1]$. If this inequality is reversed, then the function f is said to be harmonically concave.

Definition 1.12. *(Beta Function) The Beta function denoted by $\beta(a, b)$ is defined by*

$$\beta(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \ \ a, b > 0.$$

Definition 1.13. *(Incomplete Beta Function) The incomplete beta function is defined by*

$$\beta_x(p, q) = \int_0^x t^{p-1}(1-t)^{q-1}\, dt$$

with $Rep > 0$, $Req > 0$, $0 \leq x \leq 1$.

Definition 1.14. *The hypergeometric function defined by*

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} dt, \ c > b > 0, \ |z| < 1.$$

2. Main Results

In this section, we introduce a new concept, which is called harmonic trigonometrically convexity and we give by setting their some algebraic properties for the harmonic trigonometrically convex functions. Also, we discuss some connections between the class of harmonic trigonometrically convex functions and other classes of generalized convex functions, as follows:

Definition 2.1. *A non-negative function $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ is called harmonic trigonometrically convex function (or harmonic trigonometrically convex) if for every $x, y \in I$ and $t \in [0,1]$,*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \left(\sin\frac{\pi t}{2}\right) f(y) + \left(\cos\frac{\pi t}{2}\right) f(x). \tag{3}$$

If this inequality is reversed, then the function f is said to be harmonic trigonometrically concave.

The class of all harmonic trigonometrically convex functions is denoted by $HTC(I)$ on interval I.

Example 2.2. *Let $f : (0,\infty) \to \mathbb{R}$, $f(x) = x$ and $g : (-\infty, 0) \to \mathbb{R}$, $g(x) = x$, then f is a harmonic trigonometrically convex function and g is a harmonic trigonometrically concave function. Really, since $t \leq \sin\frac{\pi t}{2}$ and $1-t \leq \cos\frac{\pi t}{2}$ for $t \in [0,1]$, we can write*

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(b) + (1-t)f(a) \leq \left(\sin\frac{\pi t}{2}\right) f(b) + \left(\cos\frac{\pi t}{2}\right) f(a)$$

and similarly

$$g\left(\frac{ab}{ta+(1-t)b}\right) \geq tg(b) + (1-t)g(a) \geq \left(\sin\frac{\pi t}{2}\right) g(b) + \left(\cos\frac{\pi t}{2}\right) g(a).$$

The following proposition is obvious from this example:

Proposition 2.3. *Let $I \subset \mathbb{R}\setminus\{0\}$ be a real interval and $f : I \to \mathbb{R}$ is a function, then;*

i). if $I \subset (0,\infty)$ and f is trigonometrically convex and nondecreasing function then f is harmonic trigonometrically convex.

ii). if $I \subset (0,\infty)$ and f is harmonic trigonometrically convex and nonincreasing function then f is trigonometrically convex.

iii). if $I \subset (-\infty,0)$ and f is harmonic trigonometrically concave and nonincreasing function then f is trigonometrically concave.

iv). if $I \subset (-\infty,0)$ and f is trigonometrically concave and nondecreasing function then f is harmonic trigonometrically concave.

Proof. Since, $H_t \leq A_t$ for all $a, b \in (0,\infty)$ and $H_t \geq A_t$ for all $a, b \in (-\infty, 0)$, proofs are can be seen easily. □

Remark 2.4. *Clearly, if $f(x)$ is a nonnegative function, then every harmonic trigonometrically convex function is a P-function. Indeed, for every $x, y \in I$ and $t \in [0,1]$ we have*

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq \left(\sin\frac{\pi t}{2}\right) f(b) + \left(\cos\frac{\pi t}{2}\right) f(a) \leq f(x) + f(y).$$

Example 2.5. *Every constant function is a harmonic trigonometrically convex function.*

Theorem 2.6. *Let $f, g : [a,b] \subset \mathbb{R}/\{0\} \to \mathbb{R}$. If f and g are harmonic trigonometrically convex functions, then*
(i) $f+g$ is harmonic trigonometrically convex function,
(ii) For $c \in \mathbb{R}$ ($c \geq 0$) cf is harmonic trigonometrically convex functions.

Proof. (i) Let f, g be harmonic trigonometrically convex functions, then

$$\begin{aligned}(f+g)\left(\frac{xy}{tx+(1-t)y}\right) &= f\left(\frac{xy}{tx+(1-t)y}\right)+g\left(\frac{xy}{tx+(1-t)y}\right)\\ &\le \left(\sin\frac{\pi t}{2}\right)f(y)+\left(\cos\frac{\pi t}{2}\right)f(x)+\left(\sin\frac{\pi t}{2}\right)g(y)+\left(\cos\frac{\pi t}{2}\right)g(x)\\ &= \left(\sin\frac{\pi t}{2}\right)[f(y)+g(y)]+\left(\cos\frac{\pi t}{2}\right)[f(x)+g(x)]\\ &= \left(\sin\frac{\pi t}{2}\right)(f+g)(y)+\left(\cos\frac{\pi t}{2}\right)(f+g)(x).\end{aligned}$$

(ii) Let f be harmonic trigonometrically convex function and $c \in \mathbb{R}$ *($c \ge 0$), then*

$$\begin{aligned}(cf)\left(\frac{xy}{tx+(1-t)y}\right) &\le c\left[\left(\sin\frac{\pi t}{2}\right)f(y)+\left(\cos\frac{\pi t}{2}\right)f(x)\right]\\ &= \left(\sin\frac{\pi t}{2}\right)cf(y)+\left(\cos\frac{\pi t}{2}\right)cf(x)\\ &= \left(\sin\frac{\pi t}{2}\right)(cf)(y)+\left(\cos\frac{\pi t}{2}\right)(cf)(x).\end{aligned}$$

□

Theorem 2.7. *Let $f_\alpha : [a,b] \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be an arbitrary family of harmonic trigonometrically convex functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in [a,b] : f(u) < \infty\}$ is nonempty, then J is an interval and f is a harmonic trigonometrically convex function on J.*

Proof. Let $t \in [0,1]$ and $x, y \in J$ be arbitrary. Then

$$\begin{aligned}f\left(\frac{xy}{tx+(1-t)y}\right) &= \sup_\alpha f_\alpha\left(\frac{xy}{tx+(1-t)y}\right)\\ &\le \sup_\alpha\left[\left(\sin\frac{\pi t}{2}\right)f_\alpha(y)+\left(\cos\frac{\pi t}{2}\right)f_\alpha(x)\right]\\ &\le \left(\sin\frac{\pi t}{2}\right)\sup_\alpha f_\alpha(y)+\left(\cos\frac{\pi t}{2}\right)\sup_\alpha f_\alpha(x)\\ &= \left(\sin\frac{\pi t}{2}\right)f(y)+\left(\cos\frac{\pi t}{2}\right)f(x)\\ &< \infty.\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is a harmonic trigonometrically convex function on J. This completes the proof of theorem. □

3. Hermite-Hadamard inequality for harmonic trigonometrically convex functions

The goal of this section is to establish some inequalities of Hermite-Hadamard type for harmonic trigonometrically convex functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on $[a,b]$.

The following result of the Hermite-Hadamard type holds for harmonic trigonometrically convex function.

Theorem 3.1. *Let $f : [a,b] \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be a harmonic trigonometrically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

$$\frac{\sqrt{2}}{2}f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{4}{\pi}\frac{f(a)+f(b)}{2}.$$

Proof. Since $f : [a, b] \to \mathbb{R}$ is a harmonically convex function, we have, for all $x, y \in [a, b]$ (with $t = \frac{1}{2}$ in the inequality (3))

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{\sqrt{2}\,(f(y) + f(x))}{2}.$$

By choosing $x = \frac{ab}{ta+(1-t)b}$ and $y = \frac{ab}{tb+(1-t)a}$, we get

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\sqrt{2}\left[f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right)\right]}{2}.$$

Further, integrating for $t \in [0, 1]$, we have

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\sqrt{2}}{2}\left[\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt + \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt\right] \\ &= \frac{\sqrt{2}}{2}\left[\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx + \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right] \\ &= \frac{\sqrt{2}ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx, \end{aligned}$$

where

$$\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt = \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx.$$

The proof of the second inequality follows by using (3) with $x = a$ and $y = b$ and integrating with respect to t over $[0, 1]$. That is,

$$\begin{aligned} \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt &\leq \int_0^1 \left(\sin\frac{\pi t}{2}\right) f(b)dt + \int_0^1 \left(\cos\frac{\pi t}{2}\right) f(a)dt \\ \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx &\leq \frac{4}{\pi}\frac{f(a) + f(b)}{2}. \end{aligned}$$

This completes the proof of theorem. □

4. Some new inequalities for harmonic trigonometrically convex functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard type inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is harmonic trigonometrically convex function. İşcan [7] used the following lemma:

Lemma 4.1. *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a, b]$*

$$\frac{f(a) + f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx = \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt.$$

Theorem 4.2. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonic trigonometrically convex function on interval $[a, b]$ for $q \geq 1$, then the following inequality*

$$\begin{aligned} \left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{ab(b-a)^{1-\frac{1}{p}}}{2}\Big[A\left(L_{-2p}^{-2p}(A, a) - L_{-2p}^{-2p}(A, b)\right) \\ &\quad + L_{1-2p}^{1-2p}(A, b) - L_{1-2p}^{1-2p}(A, a)\Big]^{\frac{1}{p}} \left[\frac{4\left(\pi - 4\sqrt{2} + 4\right)}{\pi^2} A\left(|f'(a)|^q, |f'(b)|^q\right)\right]^{\frac{1}{q}}, \end{aligned}$$

holds for $t \in [0,1]$, where $\frac{1}{p}+\frac{1}{q}=1$, A is the arithmetic mean and L is the p-logarithmic mean.

Proof. From Lemma 4.1, the Hölder integral inequality and the inequality

$$\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q \le \left(\sin\frac{\pi t}{2}\right)\left|f'(a)\right|^q + \left(\cos\frac{\pi t}{2}\right)\left|f'(b)\right|^q, \tag{4}$$

we get

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{ab(b-a)}{2}\left|\int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt\right| \\ &\le \frac{ab(b-a)}{2}\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^{2p}}\right|dt\right)^{\frac{1}{p}}\left(\int_0^1 |1-2t|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Hence, by using the harmonic trigonometrically convexity of the function $\left|f'\right|^q$ on the interval $[a,b]$, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \le\ & \frac{ab(b-a)}{2}\left(\int_0^1 \frac{|1-2t|}{(tb+(1-t)a)^{2p}}dt\right)^{\frac{1}{p}}\left(\int_0^1 |1-2t|\left[\left(\sin\frac{\pi t}{2}\right)\left|f'(a)\right|^q+\left(\cos\frac{\pi t}{2}\right)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\ =\ & \frac{ab(b-a)}{2}\left(\int_0^1 \frac{|1-2t|}{(tb+(1-t)a)^{2p}}dt\right)^{\frac{1}{p}}\left(\left|f'(a)\right|^q\int_0^1|1-2t|\sin\frac{\pi t}{2}dt+\left|f'(b)\right|^q\int_0^1|1-2t|\cos\frac{\pi t}{2}dt\right)^{\frac{1}{q}} \\ =\ & \frac{ab(b-a)^{1-\frac{1}{p}}}{2}\left[(a+b)\left(L_{-2p}^{-2p}(A,a)-L_{-2p}^{-2p}(A,b)\right)+2\left(L_{1-2p}^{1-2p}(A,b)-L_{1-2p}^{1-2p}(A,a)\right)\right]^{\frac{1}{p}} \\ & \times\left[\frac{4\left(\pi-4\sqrt{2}+4\right)}{\pi^2}A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}\end{aligned}$$

where (by changing a variable as $u=tb+(1-t)a$)

$$\begin{aligned}\int_0^1\frac{|1-2t|}{(tb+(1-t)a)^{2p}}dt &= \int_0^{\frac{1}{2}}\frac{1-2t}{(tb+(1-t)a)^{2p}}dt-\int_{\frac{1}{2}}^1\frac{1-2t}{(tb+(1-t)a)^{2p}}dt \\ &= \frac{1}{(b-a)^2}\left[(a+b)\frac{u^{1-2p}}{1-2p}-2\frac{u^{2-2p}}{2-2p}\right]\Bigg|_a^{\frac{a+b}{2}}+\frac{1}{(b-a)^2}\left[(a+b)\frac{u^{1-2p}}{1-2p}-2\frac{u^{2-2p}}{2-2p}\right]\Bigg|_{\frac{a+b}{2}}^b \\ &= \frac{1}{b-a}\left[\frac{a+b}{2}\frac{\left(\frac{a+b}{2}\right)^{1-2p}-a^{1-2p}}{\left(\frac{b-a}{2}\right)(1-2p)}-\frac{\left(\frac{a+b}{2}\right)^{2-2p}-a^{2-2p}}{\left(\frac{b-a}{2}\right)(2-2p)}\right] \\ &\quad+\frac{1}{b-a}\left[\frac{a+b}{2}\frac{b^{1-2p}-\left(\frac{a+b}{2}\right)^{1-2p}}{\left(\frac{b-a}{2}\right)(1-2p)}-\frac{b^{2-2p}-\left(\frac{a+b}{2}\right)^{2-2p}}{\left(\frac{b-a}{2}\right)(2-2p)}\right] \\ &= \frac{1}{b-a}\left[A\left(L_{-2p}^{-2p}(A,a)-L_{-2p}^{-2p}(A,b)\right)+L_{1-2p}^{1-2p}(A,b)-L_{1-2p}^{1-2p}(A,a)\right] \\ \int_0^1|1-2t|\sin\frac{\pi t}{2}dt &= \int_0^1|1-2t|\cos\frac{\pi t}{2}dt=\frac{2\left(\pi-4\sqrt{2}+4\right)}{\pi^2},\end{aligned}$$

This completes the proof of the Theorem. □

Theorem 4.3. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonic trigonometrically convex function on interval $[a,b]$, then the following inequality holds for $t \in [0,1]$*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2ab}\left(\frac{4}{\pi}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)$$
$$\times\left(H^2(a,b)\frac{{}_2F_1\left(2p, p+1; p+2; \frac{b-a}{b+a}\right)}{2(p+1)} + a^2\frac{{}_2F_1\left(2p, 1; p+2; \frac{1}{2}\left(1-\frac{a}{b}\right)\right)}{2(p+1)}\right)$$

where $\frac{1}{p}+\frac{1}{q}=1$ and H is the harmonic mean and A is the arithmetic mean.

Proof. From Lemma 4.1, the Hölder inequality and the inequality (4), we get

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{ab(b-a)}{2}\left|\int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt\right| \\
&= \frac{ab(b-a)}{2}\int_0^1 \frac{|1-2t|}{(tb+(1-t)a)^2}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt \\
&\leq \frac{ab(b-a)}{2}\left[\left(\int_0^1 \frac{|1-2t|^p}{(tb+(1-t)a)^{2p}}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{ab(b-a)}{2}\left[\left(\int_0^{\frac{1}{2}} \frac{(1-2t)^p}{(tb+(1-t)a)^{2p}}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 \frac{(2t-1)^p}{(tb+(1-t)a)^{2p}}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

Hence, by using the harmonic trigonometrically convexity of the function $|f'|^q$ on the interval $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{ab(b-a)}{2}\left(\int_0^{\frac{1}{2}} \frac{(1-2t)^p}{(tb+(1-t)a)^{2p}}dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\sin\frac{\pi t}{2}\right)|f'(a)|^q + \left(\cos\frac{\pi t}{2}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
&+\frac{ab(b-a)}{2}\left(\int_{\frac{1}{2}}^1 \frac{(2t-1)^p}{(tb+(1-t)a)^{2p}}dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\sin\frac{\pi t}{2}\right)|f'(a)|^q + \left(\cos\frac{\pi t}{2}\right)|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2ab}H^2(a,b)\left[\frac{{}_2F_1\left(2p, p+1; p+2; \frac{b-a}{b+a}\right)}{2(p+1)}\right]^{\frac{1}{p}}\left(\frac{4}{\pi}A\left(|f'(a)|^q, |f'(b)|^q\right)\right)^{\frac{1}{q}} \\
&+\frac{a(b-a)}{2b}\left[\frac{{}_2F_1\left(2p, 1; p+2; \frac{1}{2}\left(1-\frac{a}{b}\right)\right)}{2(p+1)}\right]^{\frac{1}{p}}\left(\frac{4}{\pi}A\left(|f'(a)|^q, |f'(b)|^q\right)\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2ab}\left(\frac{4}{\pi}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\left(H^2(a,b)\frac{{}_2F_1\left(2p, p+1; p+2; \frac{b-a}{b+a}\right)}{2(p+1)} + a^2\frac{{}_2F_1\left(2p, 1; p+2; \frac{1}{2}\left(1-\frac{a}{b}\right)\right)}{2(p+1)}\right),
\end{aligned}$$

where

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{(1-2t)^p}{(tb+(1-t)a)^{2p}}dt &= \frac{1}{2}\int_0^1 \frac{(1-u)^p}{\left[b\frac{u}{2}+\left(1-\frac{u}{2}\right)a\right]^{2p}}du, \quad (u=2t) \\
&= \frac{1}{2}\left(\frac{a+b}{2}\right)^{-2p}\int_0^1 u^p\left[1-\left(\frac{b-a}{b+a}\right)u\right]^{-2p}du, \\
&= \frac{1}{2}\left(\frac{a+b}{2}\right)^{-2p}\frac{1}{\beta(p+1,1)}\cdot{}_2F_1\left(2p,p+1;p+2;\frac{b-a}{b+a}\right) \\
&= \left(\frac{a+b}{2}\right)^{-2p}\frac{1}{2(p+1)}\cdot{}_2F_1\left(2p,p+1;p+2;\frac{b-a}{b+a}\right),
\end{aligned}$$

$$\begin{aligned}
\int_{\frac{1}{2}}^1 \frac{(2t-1)^p}{(tb+(1-t)a)^{2p}}dt &= \int_0^{\frac{1}{2}} \frac{(1-2t)^p}{(ta+(1-t)b)^{2p}}dt \\
&= \frac{1}{2}\int_0^1 \frac{(1-u)^p}{\left[a\frac{u}{2}+\left(1-\frac{u}{2}\right)b\right]^{2p}}du \\
&= \frac{1}{2}\int_0^1 b^{-2p}(1-u)^p\left[1-\frac{1}{2}\left(1-\frac{a}{b}\right)u\right]^{-2p}du \\
&= b^{-2p}\frac{1}{\beta(1,p+1)}\cdot{}_2F_1\left(2p,1;p+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right) \\
&= b^{-2p}\frac{1}{2(p+1)}\cdot{}_2F_1\left(2p,1;p+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right), \\
\int_0^1 \sin\frac{\pi t}{2}dt &= \int_0^1 \cos\frac{\pi t}{2}dt = \frac{2}{\pi}.
\end{aligned}$$

This completes the proof of theorem. □

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