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Existence and well-posed results for nonclassical diffusion systems with nonlocal diffusion

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Abstract. Our goal in this paper is to present well-posed results for nonclassical diffusion systems which have applications in population dynamics. First, we establish the existence and uniqueness of a mild solution to the initial value problem. The asymptotic behavior of the mild solution is also considered when the parameter tends to zero. Second, we obtain a local well-posedness result for nonclassical diffusion systems with a nonlocal time condition. The main idea to obtain the above theoretical results is to use Banach's theorem and some techniques in Fourier series analysis. Some numerical tests are also presented to illustrate the theory.

1. Introduction

Let $\Omega \subset \mathbb{R}^N, N \geq 1$ be a bounded and connected domain with a smooth boundary $\partial\Omega$. In this paper, we consider the following problem

$$\begin{cases} u_t - k\Delta u_t - \mathcal{B}\left(l(u), l(v)\right)\Delta u = 0, & \text{in } \Omega \times (0,T), \\ v_t - k\Delta v_t - \mathcal{B}\left(l(u), l(v)\right)\Delta v = 0, & \text{in } \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where $\mathcal{B}$ is nonlocal term which is density dependent in diffusion and is given by

$$\begin{aligned} \mathcal{B}\left(l(u)(t), l(v)(t)\right) &= \mathcal{B}\left(\int_\Omega f(x)u(x,t)dx, \int_\Omega f(x)v(x,t)dx\right), \\ l(u)(t) &= \int_\Omega f(x)u(x,t)dx, \end{aligned} \tag{2}$$

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where (u, v) here is the density of population located at x at the time t, f is the external source and $\mathcal{B}$ is the diffusion rate. In a simple case of the system, that is, there is only one equation of the form

$$u_t - k\Delta u_t - \mathcal{B}\left(l(u)\right)\Delta u = f(u) + h(t) \tag{3}$$

was recently studied in [11]. The existence of a global attractor for the nonclassical diffusion equations with Kirchhoff type

$$u_t - \Delta u_t - \left(1 + \epsilon\|\nabla u\|^2\right)\Delta u = f(u) + h(t) \tag{4}$$

was obtained in [17].

Our paper extends the above equation to a system of equations and explores some related properties of the solution. Specially, our main method seems to be a slight modification of the method in [26]. We shall study both the initial value problem and the nonlocal problem with Kirchhoff type. Nonclassical diffusion equations arise in a variety of physical and biological applications, see e.g. [1–3, 11, 21, 27–29] and the references therein. In order to study the interaction of two or more biological species, systems of parabolic equations were proposed.

In the following, we outline some of the works that are related to our model. Regarding the system of nonlocal problems, we pay attention to recent work in [4, 5, 7–10, 12] where the authors investigated the system of two population densities as follows

$$\begin{aligned}
&u_t - \mathcal{D}_1\Big(\ell_1(u)(t), \ell_2(v)(t)\Big)\Delta u + \lambda_1|u|^{p-2}u = F(x,t), && (x,t) \in \Omega\times(0,T),\\
&v_t - \mathcal{D}_2\Big(\ell_3(u)(t), \ell_4(v)(t)\Big)\Delta v + \lambda_2|v|^{p-2}v = G(x,t), && (x,t) \in \Omega\times(0,T),\\
&u(x,t) = 0, \quad v(x,t) = 0, && (x,t) \in \partial\Omega\times(0,T),\\
&u(x,0) = u_0, \quad v(x,0) = v_0, && x \in \Omega,
\end{aligned}$$

and they obtained results on the existence, uniqueness of a smooth global solution for the above system. M. Chipot and B. Lovat [14] studied the nonlocal problem

$$\begin{cases}
u_t - \mathcal{D}(\ell(u)(t))\Delta u = f, & \text{in } \Omega\times(0,T),\\
u = 0, & \text{on } \partial\Omega\times(0,T),\\
u = u_0, & \text{in } \Omega\times\{0\},
\end{cases}$$

and similar models for nonlocal parabolic and parabolic with memory term were obtained in [13, 15, 16, 18–20, 22–25]. To the best of our knowledge, there has been no work concerning problem (1). The main contributions of this paper are:

- The existence of a unique solution to problem (1) with the initial data

  $$u(x,0) = u_0(x),\ v(x,0) = v_0(x).$$

  Also we study the asymptotic behaviour of the mild solution to this problem when $k \to 0$.

- The existence and uniqueness of a solution to problem (1) with a nonlocal in time condition.

The structure of the paper is as follows. In section 2, we introduce some preliminaries and in Section 3, we study the initial value problem for nonclassical diffusion equations. Section 4 gives a well-posedness result for a nonlocal problem and Section 5 gives numerical tests.

2. Preliminaries

For $s \geq 0$, we define the following space (as a type of Hilbert scale space)

$$\mathbf{H}^s(\Omega) \equiv D(\mathcal{A}^s) = \left\{ L^2(\Omega) \ni v = \sum_j \varphi_j(x) \left(v, \varphi_j\right) \quad \text{s.t.} \quad \sum_j \left(v, \varphi_j\right)^2 \lambda_j^{2s} < \infty \right\}, \tag{5}$$

which is endowed with the norm

$$\|v\|_{D(\mathcal{A}^s)} = \|v\|_{\mathbf{H}^s(\Omega)} = \left(\sum_j \left(v, \varphi_j\right)^2 \lambda_j^{2s} \right)^{\frac{1}{2}},$$

where λ_j and φ_j are the eigenvalues and complete orthonormal system of eigenfunctions forming an orthogonal basis, respectively, such that $-\Delta\varphi_j = \lambda_j \varphi_j$ and $\varphi_j\big|_{\partial\Omega} = 0$ for $j \in \mathbb{N}$.

For any $b > 0$, denote by $L_b^\infty(0, T; \mathbf{H}^s)$ the function space $L^\infty(0, T; \mathbf{H}^s(\Omega))$ associated with the norm

$$\|v\|_{L_b^\infty(0,T;\mathbf{H}^s(\Omega))} := \max_{0 \leq t \leq T} \left\| \exp(-bt) v(.,t) \right\|_{\mathbf{H}^s(\Omega)}, \quad \forall\, v \in L_b^\infty(0, T; \mathbf{H}^s(\Omega)).$$

We consider the following hypotheses:

($Hyp1$): The functions $\mathcal{B}$ are non-negative such that the mapping

$$v \mapsto \mathcal{B}(z_1, z_2), \tag{H1}$$

is continuous for $(z_1, z_2) \in \mathbb{R} \times \mathbb{R}$;

($Hyp2$): There are two real positive number μ_0, μ_1 such that

$$\mu_0 \leq \mathcal{B}(z_1, z_2) \leq \mu_1, \quad \text{for all} \quad (z_1, z_2) \in \mathbb{R} \times \mathbb{R}; \tag{H2}$$

($Hyp3$): Assume that $\mathcal{B}(0, 0) = 0$ and there exists a positive real number L such that $\forall z_1, z_2 \in \mathbb{R}$,

$$\left| \mathcal{B}(z_1, \bar{z}_1) - \mathcal{B}(z_2, \bar{z}_2) \right| \leq L \left(|z_1 - z_2| + |\bar{z}_1 - \bar{z}_2| \right). \tag{H3}$$

In the next lemma, we present some useful embeddings between the spaces mentioned above.

Lemma 2.1. *(See Theorem 6.7, [6]) For $N \geq 1$, and $-\frac{N}{4} < \beta \leq 0 \leq \alpha < \frac{N}{4}$, we have the following embeddings:*

$$\left.\begin{aligned} L^p(\Omega) &\hookrightarrow D(\mathcal{A}^\beta), \quad \text{if} \quad -\frac{N}{4} < \beta \leq 0, \quad p \geq \frac{2N}{N - 4\beta}, \\ D(\mathcal{A}^\alpha) &\hookrightarrow L^p(\Omega), \quad \text{if} \quad 0 \leq \alpha < \frac{N}{4}, \quad p \leq \frac{2N}{N - 4\alpha}. \end{aligned}\right\} \tag{6}$$

A couple (u, v) of functions $u(x, t), v(x, t) : \overline{Q}_T \to \mathbb{R}$, ($\overline{Q}_T = \overline{\Omega} \times [0, T]$) is called a function of two variables x, t

$$(u, v) : \overline{Q}_T \to \mathbb{R}^2$$
$$(u, v)(x, t) = (u(x, t), v(x, t)).$$

Note the norm of $(u, v) \in \mathbb{X} \times \mathbb{X}$ (for any space $\mathbb{X}$) is defined by

$$\|(u, v)\|_{(\mathbb{X})^2} = \|u\|_{\mathbb{X}} + \|v\|_{\mathbb{X}}.$$

Remark 2.2. *Let $f \in L^2(\Omega)$. For $v_1, v_2, \overline{v}_1, \overline{v}_2 \in L^\infty(0, T; L^2(\Omega))$, we obtain the following estimate*

$$\begin{aligned} &\left| \mathcal{B}(l(v_1), l(\overline{v}_1)) - \mathcal{B}(l(v_2), l(\overline{v}_2)) \right| \\ &\qquad \leq L\|f\|_{L^2(\Omega)} \left(\|v_1(.,t) - v_2(.,t)\|_{L^2(\Omega)} + \|\overline{v}_1(.,t) - \overline{v}_2(.,t)\|_{L^2(\Omega)} \right). \end{aligned} \tag{7}$$

3. The initial value problem

In this section, we are interested in the following initial value problem

$$\begin{cases} u_t - k\Delta u_t - \mathcal{B}\left(l(u), l(v)\right)\Delta u = 0, & \text{in } \Omega\times(0,T),\\ v_t - k\Delta v_t - \mathcal{B}\left(l(u), l(v)\right)\Delta v = 0, & \text{in } \Omega\times(0,T),\\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T),\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & \text{in } \Omega. \end{cases} \tag{8}$$

3.1. Existence and uniqueness

Theorem 3.1. *For $s \geq 0$, let $\mathbf{H}^s$ be given as in (5) and let the initial datum $(u_0, v_0) \in \mathbf{H}^s(\Omega)\times\mathbf{H}^s(\Omega)$. Then Problem (8) has a unique solution u which satisfies the nonlinear equation*

$$u(x,t) = \sum_j \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u)(r), l(v)(r)\right)dr\right)u_{0,j}\varphi_j,$$
$$v(x,t) = \sum_j \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u)(r), l(v)(r)\right)dr\right)v_{0,j}\varphi_j. \tag{9}$$

Proof. Let us consider the function u as the Fourier series $u(x,t) = \sum_j u_j(t)\varphi_j(x)$, with $u_j(t) = (u(\cdot,t), \varphi_j)$. Then we arrive at the following ordinary differential equations

$$\begin{cases} u_j'(t) + k\lambda_j u_j'(t) + \lambda_j\mathcal{B}\left(l(u)(t), l(v)(t)\right)u_j(t) = 0,\\ v_j'(t) + k\lambda_j v_j'(t) + \lambda_j\mathcal{B}\left(l(u)(t), l(v)(t)\right)v_j(t) = 0. \end{cases} \tag{10}$$

The equation is equivalent to the following equation

$$\frac{du_j}{dt} + \frac{\lambda_j\mathcal{B}\left(l(u)(t), l(v)(t)\right)}{1+k\lambda_j}u_j(t) = 0, \quad \frac{dv_j}{dt} + \frac{\lambda_j\mathcal{B}\left(l(u)(t), l(v)(t)\right)}{1+k\lambda_j}v_j(t) = 0. \tag{11}$$

Solving the latter equation, we get

$$u_j(t) = \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u)(r), l(v)(r)\right)dr\right)u_{0,j}, \ \ u_{0,j} = (u_0, \varphi_j), \tag{12}$$

and

$$v_j(t) = \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u)(r), l(v)(r)\right)dr\right)v_{0,j}, \ \ v_{0,j} = (v_0, \varphi_j). \tag{13}$$

Hence, the mild solution is given by the Fourier series as in (9). Now, we show that (9) has a unique mild solution. For any $b \geq 0$, denote by $\left(L_b^\infty(0,T;\mathbf{H}^s)\right)^2$ the function space $(L^\infty(0,T;\mathbf{H}^s))^2$ associated with the norm

$$\|(u,v)\|_{b,s} := \max_{0\leq t\leq T}\left\|\exp(-bt)u(\cdot,t)\right\|_{\mathbf{H}^s} + \max_{0\leq t\leq T}\left\|\exp(-bt)v(\cdot,t)\right\|_{\mathbf{H}^s}$$

for any $(u,v) \in (L^\infty(0,T;\mathbf{H}^s))^2$. Let us define the operator

$$\mathbf{T}(u,v)(t) = \left(\mathcal{T}_1(u,v)(t), \mathcal{T}_2(u,v)(t)\right), \tag{14}$$

where (u,v) is a solution of (9), and $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined by

$$\mathcal{T}_1(u,v)(t) = \sum_j \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u)(r), l(v)(r)\right) dr\right) u_{0,j}\varphi_j, \tag{15}$$

$$\mathcal{T}_2(u,v)(t) = \sum_j \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u)(r), l(v)(r)\right) dr\right) v_{0,j}\varphi_j. \tag{16}$$

We show that $\mathbf{T}$ has a fixed point u in the space $\left(L_b^\infty(0,T;\mathbf{H}^s)\right)^2$, where b and s are suitably chosen. Now we estimate the term $\|\mathbf{T}(u_1,v_1)-\mathbf{T}(u_2,v_2)\|_{b,s}$ where $(u_1,v_1),(u_2,v_2)$ are solutions of (9). We use $|e^{-y}-e^{-z}| \le |y-z|$, for any $y,z\in\mathbb{R}, y>0, z>0$ to obtain

$$\begin{aligned}
&\left|\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u_1)(r), l(v_1)(r)\right) dr\right) - \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u_2)(r), l(v_2)(r)\right) dr\right)\right| \\
&\quad\le \frac{\lambda_j}{1+k\lambda_j}\int_0^t \left|\mathcal{B}\left(l(u_1)(r), l(v_1)(r)\right) - \mathcal{B}\left(l(u_2)(r), l(v_2)(r)\right)\right| dr \\
&\quad\le \frac{\lambda_j}{1+k\lambda_j} L\|f\|_{L^2(\Omega)}\int_0^t \Big(\|u_1(.,r)-u_2(.,r)\|_{L^2(\Omega)} + \|v_1(.,r)-v_2(.,r)\|_{L^2(\Omega)}\Big)dr.
\end{aligned} \tag{17}$$

This implies that

$$\begin{aligned}
&e^{-2bt}\left\|\mathcal{T}_1(u_1,v_1)(t) - \mathcal{T}_1(u_2,v_2)(t)\right\|^2_{\mathbf{H}^s(\Omega)} \\
&= e^{-2bt}\sum_j \lambda_j^{2s} \\
&\quad \left|\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u_1)(r), l(v_1)(r)\right) dr\right)\right. \\
&\qquad \left. - \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u_2)(r), l(v_2)(r)\right) dr\right)\right|^2 |u_{0,j}|^2 \\
&\qquad \le L^2\|f\|^2_{L^2(\Omega)} k^{-2}\left(\int_0^t e^{-b(t-r)}e^{-br}\|u_1(.,r)-u_2(.,r)\|_{L^2(\Omega)}dr\right)^2\left(\sum_j \lambda_j^{2s}|u_{0,j}|^2\right) \\
&\qquad + L^2\|f\|^2_{L^2(\Omega)} k^{-2}\left(\int_0^t e^{-b(t-r)}e^{-br}\|v_1(.,r)-v_2(.,r)\|_{L^2(\Omega)}dr\right)^2\left(\sum_j \lambda_j^{2s}|u_{0,j}|^2\right)
\end{aligned} \tag{18}$$

where we note that $\frac{\lambda_j}{1+k\lambda_j} \le k^{-1}$. Using the Hölder inequality, we bound the integral term on the right hand side of (18) as follows

$$\begin{aligned}
\int_0^t e^{-b(t-r)}e^{-br}\|u_1(.,r)-u_2(.,r)\|_{\mathbf{H}^s(\Omega)}dr &\le \left(\int_0^t e^{-b(t-r)}dr\right)\|u_1-u_2\|_{L_b^\infty(0,T;\mathbf{H}^s(\Omega))} \\
&\le \frac{1}{b}\|u_1-u_2\|_{L_b^\infty(0,T;\mathbf{H}^s(\Omega))}.
\end{aligned} \tag{19}$$

In a similar way, we get

$$\begin{aligned}
\int_0^t e^{-b(t-r)}e^{-br}\|v_1(.,r)-v_2(.,r)\|_{\mathbf{H}^s(\Omega)}dr &\le \left(\int_0^t e^{-b(t-r)}dr\right)\|v_1-v_2\|_{L_b^\infty(0,T;\mathbf{H}^s(\Omega))} \\
&\le \frac{1}{b}\|v_1-v_2\|_{L_b^\infty(0,T;\mathbf{H}^s(\Omega))}.
\end{aligned} \tag{20}$$

Combining (18) and (19), we obtain that

$$e^{-2bt}\Big\|\mathcal{T}_1(u_1,v_1)(t)-\mathcal{T}_1(u_2,v_2)(t)\Big\|^2_{\mathbf{H}^s(\Omega)} \le L^2\|f\|^2_{L^2}k^{-2}b^{-2}\|u_1-u_2\|^2_{L^\infty_b(0,T;\mathbf{H}^s(\Omega))}\|u_0\|^2_{\mathbf{H}^s(\Omega)} + L^2\|f\|^2_{L^2}k^{-2}b^{-2}\|v_1-v_2\|^2_{L^\infty_b(0,T;\mathbf{H}^s(\Omega))}\|u_0\|^2_{\mathbf{H}^s(\Omega)}. \tag{21}$$

By a similar explanation, we find that

$$e^{-2bt}\Big\|\mathcal{T}_2(u_1,v_1)(t)-\mathcal{T}_2(u_2,v_2)(t)\Big\|^2_{\mathbf{H}^s(\Omega)} \le L^2\|f\|^2_{L^2}k^{-2}b^{-2}\|u_1-u_2\|^2_{L^\infty_b(0,T;\mathbf{H}^s(\Omega))}\|v_0\|^2_{\mathbf{H}^s(\Omega)} + L^2\|f\|^2_{L^2}k^{-2}b^{-2}\|v_1-v_2\|^2_{L^\infty_b(0,T;\mathbf{H}^s(\Omega))}\|v_0\|^2_{\mathbf{H}^s(\Omega)}. \tag{22}$$

Combining (21) and (22), we arrive at the following bound

$$\begin{aligned}&\|\mathbf{T}(u_1,v_1)(t)-\mathbf{T}(u_2,v_2)(t)\Big\|_{\left(L^\infty_b(0,T;\mathbf{H}^s(\Omega))\right)^2}\\ &\le L\|f\|_{L^2}k^{-1}b^{-1}\|(u_1,v_1)-(u_2,v_2)\|_{\left(L^\infty_b(0,T;\mathbf{H}^s(\Omega))\right)^2}\|(u_0,v_0)\|_{(\mathbf{H}^s(\Omega))^2}.\end{aligned} \tag{23}$$

We can now choose b such that

$$b > L\|f\|_{L^2}k^{-1}\|(u_0,v_0)\|_{(\mathbf{H}^s(\Omega))^2}.$$

From the estimate (23), we deduce that $\mathbf{T}$ is a contraction on the space $\left(L^\infty_b(0,T;\mathbf{H}^s(\Omega))\right)^2$. From (15), we see that if $(u_3,v_3)=(0,0)$ then

$$\mathbf{T}(u_3,v_3)=(u_0,v_0)\in\left(L^\infty_b(0,T;\mathbf{H}^s(\Omega))\right)^2.$$

Therefore, we can conclude that the nonlinear equation (9) has a unique solution (u,v) in the space $\left(L^\infty_b(0,T;\mathbf{H}^s(\Omega))\right)^2$. □

3.2. *Asymptotic behaviour of the mild solution to Problem* (8) *when* $k\to 0$

In this section, we focus on the asymptotic behaviour of the mild solution to Problem (8) when $k\to 0$. Let the mild solution of Problem (8) be $\left(u^{(k)}(x,t),v^{(k)}(x,t)\right)$. Let (u^+,v^+) be the mild solution of the following problem

$$\begin{cases}u_t-\mathcal{B}\left(l(u),l(v)\right)\Delta u=0, & \text{in } \Omega\times(0,T),\\ v_t-\mathcal{B}\left(l(u),l(v)\right)\Delta v=0, & \text{in } \Omega\times(0,T),\\ u(x,t)=0, & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & \text{in } \Omega.\end{cases} \tag{24}$$

First, we show the existence of Problem (24).

Theorem 3.2. *Let the initial datum* $(u_0,v_0)\in\mathbf{H}^{s+3/2}(\Omega)\times\mathbf{H}^{s+3/2}(\Omega)$*. Then Problem* (24) *has a unique solution* u *which satisfies the nonlinear equation*

$$\begin{cases}u^+(x,t)=\sum_j\exp\left(-\lambda_j\int_0^t\mathcal{B}\left(l(u^+)(r),l(v^+)(r)\right)dr\right)u_{0,j}\varphi_j,\\ v^+(x,t)=\sum_j\exp\left(-\lambda_j\int_0^t\mathcal{B}\left(l(u^+)(r),l(v^+)(r)\right)dr\right)v_{0,j}\varphi_j.\end{cases} \tag{25}$$

Proof. The proof is similar to the proof of Theorem 3.1, so we omit it here. □

In the following theorem, we will show that $\left(u^{(k)}(x,t),v^{(k)}(x,t)\right)$ tends to (u^+,v^+) in the appropriate space when $k\to 0^+$.

Theorem 3.3. *Let $(u_0, v_0) \in \mathbf{H}^{s+3/2}(\Omega) \times \mathbf{H}^{s+3/2}(\Omega)$. Then we get the following estimate for any $0 \le t \le T$,*

$$\begin{aligned} &\|u^+(.,t) - u^{(k)}(.,t)\|^2_{\mathbf{H}^s(\Omega)} + \|v^+(.,t) - v^{(k)}(.,t)\|^2_{\mathbf{H}^s(\Omega)} \\ &\qquad \le \frac{k}{2} L^2 \|f\|^2_{L^2(\Omega)} T \overline{\mathcal{D}}(T, f, u_0, v_0) \exp(\mathcal{D}(T, u_0, v_0)(T-t)), \end{aligned} \tag{26}$$

where

$$\mathcal{D}(T, f, u_0, v_0) = 4L^2 T \|f\|^2_{L^2(\Omega)} \Big(\|u_0\|^2_{\mathbf{H}^{1+s}(\Omega)} + \|v_0\|^2_{\mathbf{H}^{1+s}(\Omega)} \Big)$$

and

$$\overline{D}(T, u_0, v_0) = \left(\|u_0\|^2_{\mathbf{H}^{s+3/2}(\Omega)} + \|v_0\|^2_{\mathbf{H}^{s+3/2}(\Omega)} \right) \left(\|v_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)} \right).$$

Remark 3.4. *The result* (26) *holds for any $t \in [0, T]$, so it is clear that*

$$\lim_{k \to 0} \left\| \left(u^{(k)}, v^{(k)} \right) - (u^+, v^+) \right\|_{(L^\infty(0,T;\mathbf{H}^s(\Omega)))^2} = 0. \tag{27}$$

Proof. We have in view of Theorem 3.1,

$$\begin{cases} u^{(k)}(x,t) = \sum_j \exp\left(-\frac{\lambda_j}{1+k\lambda_j} \int_0^t \mathcal{B}\left(l(u^{(k)})(r), l(v^{(k)})(r) \right) dr \right) u_{0,j} \varphi_j \\ v^{(k)}(x,t) = \sum_j \exp\left(-\frac{\lambda_j}{1+k\lambda_j} \int_0^t \mathcal{B}\left(l(u^{(k)})(r), l(v^{(k)})(r) \right) dr \right) v_{0,j} \varphi_j \end{cases} \tag{28}$$

and Theorem 3.2,

$$\begin{cases} u^+(x,t) = \sum_j \exp\left(-\lambda_j \int_0^t \mathcal{B}\left(l(u^+)(r), l(v^+)(r) \right) dr \right) u_{0,j} \varphi_j, \\ v^+(x,t) = \sum_j \exp\left(-\lambda_j \int_0^t \mathcal{B}\left(l(u^+)(r), l(v^+)(r) \right) dr \right) v_{0,j} \varphi_j. \end{cases} \tag{29}$$

Let $\mathbf{P}_k$ and $\mathbf{Q}_k$ be

$$\mathbf{P}_k u^+(x,t) = \sum_j \exp\left(-\frac{\lambda_j}{1+k\lambda_j} \int_0^t \mathcal{B}\left(l(u^+)(r), l(v^+)(r) \right) dr \right) u_{0,j} \varphi_j. \tag{30}$$

and

$$\mathbf{Q}_k v^+(x,t) = \sum_j \exp\left(-\frac{\lambda_j}{1+k\lambda_j} \int_0^t \mathcal{B}\left(l(u^+)(r), l(v^+)(r) \right) dr \right) v_{0,j} \varphi_j. \tag{31}$$

Step 1. Estimate the quantity $\|u^{(k)}(.,t) - \mathbf{P}_k u^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} + \|v^{(k)}(.,t) - \mathbf{Q}_k v^+(.,t)\|^2_{\mathbf{H}^s(\Omega)}$.
Applying $|e^{-y} - e^{-z}| \le |y - z|$, for any $y, z \in \mathbb{R}, y > 0, z > 0$ and Remark 2.2, we obtain the following inequality

$$\begin{aligned} &\left| \exp\left(-\frac{\lambda_j}{1+k\lambda_j} \int_0^t \mathcal{B}\left(l(u^{(k)})(r), l(v^{(k)})(r) \right) dr \right) - \exp\left(-\frac{\lambda_j}{1+k\lambda_j} \int_0^t \mathcal{B}\left(l(u^+)(r), l(v^+)(r) \right) dr \right) \right| \\ &\le \frac{\lambda_j}{1+k\lambda_j} \int_0^t \left| \mathcal{B}\left(l(u^{(k)})(r), l(v^{(k)})(r) \right) - \mathcal{B}\left(l(u^+)(r), l(v^+)(r) \right) \right| dr \\ &\le \lambda_j L \|f\|_{L^2(\Omega)} \int_0^t \|u^{(k)}(.,r) - u^+(.,r)\|_{L^2(\Omega)} dr + \lambda_j L \|f\|_{L^2(\Omega)} \int_0^t \|v^{(k)}(.,r) - v^+(.,r)\|_{L^2(\Omega)} dr. \end{aligned} \tag{32}$$

Using Parseval's equality and Hölder's inequality, it follows from (32), (28) and (30) that

$$
\begin{aligned}
&\|u^{(k)}(.,t) - \mathbf{P}_k u^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} \\
&\le 2L^2\|f\|^2_{L^2(\Omega)}\left(\sum_j \lambda_j^{2+2s}u_{0,j}^2\right)\left(\int_0^t \|u^{(k)}(.,r) - u^+(.,r)\|_{L^2(\Omega)}dr\right)^2 \\
&\qquad + 2L^2\|f\|^2_{L^2(\Omega)}\left(\sum_j \lambda_j^{2+2s}u_{0,j}^2\right)\left(\int_0^t \|v^{(k)}(.,r) - v^+(.,r)\|_{L^2(\Omega)}dr\right)^2 \\
&\le 2L^2T\|f\|^2_{L^2(\Omega)}\|u_0\|^2_{\mathbf{H}^{1+s}(\Omega)}\left(\int_0^t \|u^{(k)}(.,r) - u^+(.,r)\|^2_{L^2(\Omega)}dr + \int_0^t \|v^{(k)}(.,r) - v^+(.,r)\|^2_{L^2(\Omega)}dr\right).
\end{aligned} \tag{33}
$$

From Parseval's equality and Hölder's inequality, it follows from (32), (28) and (31) that

$$
\begin{aligned}
&\|v^{(k)}(.,t) - \mathbf{Q}_k v^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} \\
&\le 2L^2\|f\|^2_{L^2(\Omega)}\left(\sum_j \lambda_j^{2+2s}v_{0,j}^2\right)\left(\int_0^t \|u^{(k)}(.,r) - u^+(.,r)\|_{L^2(\Omega)}dr\right)^2 \\
&\qquad + 2L^2\|f\|^2_{L^2(\Omega)}\left(\sum_j \lambda_j^{2+2s}v_{0,j}^2\right)\left(\int_0^t \|v^{(k)}(.,r) - v^+(.,r)\|_{L^2(\Omega)}dr\right)^2 \\
&\le 2L^2T\|f\|^2_{L^2(\Omega)}\|v_0\|^2_{\mathbf{H}^{1+s}(\Omega)}\left(\int_0^t \|u^{(k)}(.,r) - u^+(.,r)\|^2_{L^2(\Omega)}dr + \int_0^t \|v^{(k)}(.,r) - v^+(.,r)\|^2_{L^2(\Omega)}dr\right).
\end{aligned} \tag{34}
$$

The inequality $(c+d)^2 \le 2c^2 + 2d^2$ and the two estimates (33), (34) allow us to obtain

$$
\begin{aligned}
&\|u^{(k)}(.,t) - \mathbf{P}_k u^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} + \|v^{(k)}(.,t) - \mathbf{Q}_k v^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} \\
&\qquad \le 2\|u^{(k)}(.,t) - \mathbf{P}_k u^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} + \|v^{(k)}(.,t) - \mathbf{Q}_k v^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} \\
&\qquad \le 4L^2T\|f\|^2_{L^2(\Omega)}\left(\|u_0\|^2_{\mathbf{H}^{1+s}(\Omega)} + \|v_0\|^2_{\mathbf{H}^{1+s}(\Omega)}\right) \\
&\qquad\qquad \left(\int_0^t \|u^{(k)}(.,r) - u^+(.,r)\|^2_{L^2(\Omega)}dr + \int_0^t \|v^{(k)}(.,r) - v^+(.,r)\|^2_{L^2(\Omega)}dr\right).
\end{aligned} \tag{35}
$$

Step 2. Estimate the term $\|u^+(.,t) - \mathbf{P}_k u^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} + \|v^+(.,t) - \mathbf{Q}_k v^+(.,t)\|^2_{\mathbf{H}^s(\Omega)}$.
Using the inequality $|e^{-y} - e^{-z}| \le |y - z|$, for any $y, z \in \mathbb{R}, y > 0, z > 0$, we find that

$$
\begin{aligned}
&\|u^+(.,t) - \mathbf{P}_k u^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} \\
&= \sum_j \lambda_j^{2s} \\
&\left|\exp\left(-\lambda_j \int_0^t \mathcal{B}(l(u^+)(r), l(v^+)(r))\,dr\right) - \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}(l(u^+)(r), l(v^+)(r))\,dr\right)\right|^2 u_{0,j}^2 \\
&\le \left(\sum_j \lambda_j^{2s}\left(\lambda_j - \frac{\lambda_j}{1+k\lambda_j}\right)^2 u_{0,j}^2\right)\left(\int_0^t \mathcal{B}(l(u^+)(r), l(v^+)(r))\,dr\right)^2.
\end{aligned} \tag{36}
$$

Also note

$$\sum_j \lambda_j^{2s}\left(\lambda_j - \frac{\lambda_j}{1+k\lambda_j}\right)^2 u_{0,j}^2 = k^2 \sum_j \frac{\lambda_j^{2s+4}}{(1+k\lambda_j)^2} u_{0,j}^2$$
$$\leq \frac{k}{2}\sum_j \lambda_j^{2s+3} u_{0,j}^2 = \frac{k}{2}\|u_0\|^2_{\mathbf{H}^{s+3/2}(\Omega)}, \tag{37}$$

where we have used the fact that $(1+k\lambda_j)^2 \geq 2k\lambda_j$. Next, we need to control the integral term on the right hand side of (36). Applying Remark 2.2, we get

$$\Big|\mathcal{B}\left(l(u^+)(r), l(v^+)(r)\right)\Big| = \Big|\mathcal{B}\left(l(u^+)(r), l(v^+)(r)\right) - \mathcal{B}\left(l(0), l(0)\right)\Big|$$
$$\leq L\|f\|_{L^2(\Omega)}\|u^+(.,r)\|_{L^2(\Omega)} + L\|f\|_{L^2(\Omega)}\|v^+(.,r)\|_{L^2(\Omega)}. \tag{38}$$

Now use Hölder's inequality to obtain

$$\Big[\int_0^t \mathcal{B}\left(l(u^+)(r), l(v^+)(r)\right) dr\Big]^2 \leq T\int_0^t |\mathcal{B}\left(l(u^+)(r), l(v^+)(r)\right)|^2 dr$$
$$\leq L^2\|f\|^2_{L^2(\Omega)} T\int_0^t \|u^+(.,r)\|^2_{L^2(\Omega)} dr + L^2\|f\|^2_{L^2(\Omega)} T\int_0^t \|v^+(.,r)\|^2_{L^2(\Omega)} dr$$
$$\leq L^2\|f\|^2_{L^2(\Omega)} T\Big(\|u^+\|_{L^\infty(0,T;L^2(\Omega))} + \|v^+\|_{L^\infty(0,T;L^2(\Omega))}\Big). \tag{39}$$

From (29), we have

$$\|u^+(.,t)\|_{L^2(\Omega)} = \sum_j \exp\left(-2\lambda_j\int_0^t \mathcal{B}\left(l(u^+)(r)\right) dr\right) u_{0,j}^2 \leq \sum_j u_{0,j}^2 = \|u_0\|^2_{L^2(\Omega)} \tag{40}$$

which allows us to deduce that

$$\|u^+\|_{L^\infty(0,T;L^2(\Omega))} \leq \|u_0\|^2_{L^2(\Omega)}. \tag{41}$$

By a similar argument, we get that

$$\|v^+\|_{L^\infty(0,T;L^2(\Omega))} \leq \|v_0\|^2_{L^2(\Omega)}. \tag{42}$$

Combining (36), (37), (39), (41) and (42), we deduce that

$$\|u^+(.,t) - \mathbf{P}_k u^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} \leq \frac{k}{2}L^2\|f\|^2_{L^2(\Omega)} T\|u_0\|^2_{\mathbf{H}^{s+3/2}(\Omega)}\left(\|v_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}\right). \tag{43}$$

By a similar argument, we also find that

$$\|v^+(.,t) - \mathbf{Q}_k v^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} \leq \frac{k}{2}L^2\|f\|^2_{L^2(\Omega)} T\|v_0\|^2_{\mathbf{H}^{s+3/2}(\Omega)}\left(\|v_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}\right). \tag{44}$$

From the two above estimates, we obtain

$$\|u^+(.,t) - \mathbf{P}_k u^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} + \|v^+(.,t) - \mathbf{Q}_k v^+(.,t)\|^2_{\mathbf{H}^s(\Omega)}$$
$$\leq \frac{k}{2}L^2\|f\|^2_{L^2(\Omega)} T\Big(\|u_0\|^2_{\mathbf{H}^{s+3/2}(\Omega)} + \|v_0\|^2_{\mathbf{H}^{s+3/2}(\Omega)}\Big)\left(\|v_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}\right). \tag{45}$$

Also from (33) , (35), (45) and the inequality $(c+d)^2 \leq 2c^2 + 2d^2$ it follows that

$$\begin{aligned}
&\|u^+(.,t) - u^{(k)}(.,t)\|^2_{\mathbf{H}^s(\Omega)} + \|v^+(.,t) - v^{(k)}(.,t)\|^2_{\mathbf{H}^s(\Omega)} \\
&\quad \leq 2\|u^{(k)}(.,t) - \mathbf{P}_k u^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} + 2\|v^{(k)}(.,t) - \mathbf{Q}_k v^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} \\
&\quad + 2\|u^+(.,t) - \mathbf{P}_k u^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} + 2\|v^+(.,t) - \mathbf{Q}_k v^+(.,t)\|^2_{\mathbf{H}^s(\Omega)} \\
&\quad \leq \frac{k}{2} L^2 \|f\|^2_{L^2(\Omega)} T\Big(\|u_0\|^2_{\mathbf{H}^{s+3/2}(\Omega)} + \|v_0\|^2_{\mathbf{H}^{s+3/2}(\Omega)}\Big)\Big(\|v_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}\Big) \\
&\quad + 4L^2 T \|f\|^2_{L^2(\Omega)}\Big(\|u_0\|^2_{\mathbf{H}^{1+s}(\Omega)} + \|v_0\|^2_{\mathbf{H}^{1+s}(\Omega)}\Big) \\
&\qquad \left(\int_0^t \|u^{(k)}(.,r) - u^+(.,r)\|^2_{L^2(\Omega)} dr + \int_0^t \|v^{(k)}(.,r) - v^+(.,r)\|^2_{L^2(\Omega)} dr\right). \qquad (46)
\end{aligned}$$

Applying Grönwall's inequality, we get the desired result (26). The proof is completed. □

4. The nonlocal value problem

In this section, we are interested in considering the following problem

$$\begin{cases}
u_t^{\epsilon,\beta} - k\Delta u_t^{\epsilon,\beta} - \mathcal{B}\left(l(u^{\epsilon,\beta}), l(v^{\epsilon,\beta})\right)\Delta u^{\epsilon,\beta} = 0, & \text{in } \Omega \times (0,T), \\
v_t^{\epsilon,\beta} - k\Delta v_t^{\epsilon,\beta} - \mathcal{B}\left(l(u^{\epsilon,\beta}), l(v^{\epsilon,\beta})\right)\Delta u^{\epsilon,\beta} = 0, & \text{in } \Omega \times (0,T), \\
u^{\epsilon,\beta}(x,t) = v^{\epsilon,\beta}(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\
\epsilon u^{\epsilon,\beta}(x,0) + \beta u^{\epsilon,\beta}(x,T) = \theta(x), & \text{in } \Omega, \\
\epsilon v^{\epsilon,\beta}(x,0) + \beta v^{\epsilon,\beta}(x,T) = \psi(x), & \text{in } \Omega,
\end{cases} \qquad (47)$$

where ϵ, β are non-negative constants. Note that if $\epsilon = 0$ and $\beta = 1$ then the above problem is called the backward in time problem which is well-known to be ill-posed in the sense of Hadamard.

Now, under a small enough input data θ, we will show the existence and uniqueness of solution of the nonlocal problem. Let

$$C^\gamma([0,T];\mathbf{H}^s)) = \left\{v \in C([0,T];\mathbf{H}^s)) : \sup_{0\leq t<t'\leq T} \frac{\|v(.,t) - v(.,t')\|_{\mathbf{H}^s}}{|t-t'|^\gamma} < \infty\right\} \text{ for } \gamma > 0. \qquad (48)$$

Theorem 4.1. *Let the input data $\theta \in \mathbf{H}^s(\Omega)$ for $s \geq 0$ and suppose we can choose L such that*

$$L\|f\|_{L^2} k^{-1} T\left(\left(\epsilon + \beta\exp(k^{-1}T\mu_1)\right)^{-1} + 2\beta\left(\epsilon + \beta\exp(k^{-1}T\mu_1)\right)^{-2} C_s\right)\|\theta\|_{\mathbf{H}^s(\Omega)} \leq p, \qquad (49)$$

where $p \in (0,1)$; here C_s is an embedding constant (noted in the proof). Then Problem (47) has a unique solution in $L^\infty(0,T;\mathbf{H}^s(\Omega)) \cap C^\gamma([0,T];\mathbf{H}^s(\Omega))$.

Proof. Note

$$\left[\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right)dr\right)\right] u_j^{\epsilon,\beta}(0) = \theta_j, \qquad (50)$$

$$\left[\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right)dr\right)\right] v_j^{\epsilon,\beta}(0) = \psi_j, \qquad (51)$$

where $\theta_j = (\theta, \varphi_j)$, $\psi_j = (\psi, \varphi_j)$. Hence, we obtain

$$u_j^{\epsilon,\beta}(0) = \left[\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right)dr\right)\right]^{-1} \theta_j \qquad (52)$$

and

$$v_j^{\epsilon,\beta}(0) = \left[\epsilon + \beta \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right) dr\right)\right]^{-1} \psi_j. \tag{53}$$

This yields immediately that

$$u^{\epsilon,\beta}(x,t) = \sum_j \frac{\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right) dr\right)}{\epsilon + \beta \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right) dr\right)} \theta_j \varphi_j(x) \tag{54}$$

and

$$v^{\epsilon,\beta}(x,t) = \sum_j \frac{\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right) dr\right)}{\epsilon + \beta \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right) dr\right)} \psi_j \varphi_j(x). \tag{55}$$

Let us define the operator

$$\mathbf{S}(u^{\epsilon,\beta}, v^{\epsilon,\beta})(t) = \left(\mathcal{Q}_{\epsilon,\beta}(u^{\epsilon,\beta}, v^{\epsilon,\beta})(t), \mathcal{P}_{\epsilon,\beta}(u^{\epsilon,\beta}, v^{\epsilon,\beta})(t)\right), \tag{56}$$

where $\mathcal{Q}_{\epsilon,\beta}$ and $\mathcal{P}_{\epsilon,\beta}$ are defined by

$$\begin{aligned} \mathcal{Q}_{\epsilon,\beta}(u^{\epsilon,\beta}, v^{\epsilon,\beta})(t) &= \sum_j \frac{\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right) dr\right)}{\epsilon + \beta \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right) dr\right)} \theta_j \varphi_j(x), \\ \mathcal{P}_{\epsilon,\beta}(u^{\epsilon,\beta}, v^{\epsilon,\beta})(t) &= \sum_j \frac{\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right) dr\right)}{\epsilon + \beta \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r), l(v^{\epsilon,\beta})(r)\right) dr\right)} \theta_j \psi_j(x). \end{aligned} \tag{57}$$

Let

$$\mathscr{D}(u,v)(t) = \int_0^t \mathcal{B}\left(l(u)(r), l(v)(r)\right) dr. \tag{58}$$

It is easy to verify that

$$\mathcal{Q}_{\epsilon,\beta}\left(u_1^{\epsilon,\beta}, v_1^{\epsilon,\beta}\right)(x,t) - \mathcal{Q}_{\epsilon,\beta}\left(u_2^{\epsilon,\beta}, v_2^{\epsilon,\beta}\right)(x,t) = \text{Error}_1(x,t) + \text{Error}_2(x,t), \tag{59}$$

where the first quantity $\text{Error}_1(x,t)$ is given by

$$\begin{aligned} &\text{Error}_1(x,t) \\ &= \frac{\epsilon\left(\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_1^{\epsilon,\beta}, v_1^{\epsilon,\beta}\right)(t)\right) - \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_2^{\epsilon,\beta}, v_2^{\epsilon,\beta}\right)(t)\right)\right)}{\left(\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_1^{\epsilon,\beta}, v_1^{\epsilon,\beta}\right)(T)\right)\right)\left(\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_2^{\epsilon,\beta}, v_2^{\epsilon,\beta}\right)(T)\right)\right)} \theta_j \varphi_j \end{aligned} \tag{60}$$

and the second term $\text{Error}_2(x,t)$ is equal to

$$\begin{aligned}
&\text{Error}_2(x,t) \\
&= \frac{\beta \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_1^{\epsilon,\beta}, v_1^{\epsilon,\beta}\right)(t)\right)\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_2^{\epsilon,\beta}, v_2^{\epsilon,\beta}\right)(T)\right)}{\left(\epsilon+\beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_1^{\epsilon,\beta}, v_1^{\epsilon,\beta}\right)(T)\right)\right)\left(\epsilon+\beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_2^{\epsilon,\beta}, v_2^{\epsilon,\beta}\right)(T)\right)\right)}\theta_j\varphi_j \\
&- \frac{\beta \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_2^{\epsilon,\beta}, v_2^{\epsilon,\beta}\right)(t)\right)\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_1^{\epsilon,\beta}, v_1^{\epsilon,\beta}\right)(T)\right)}{\left(\epsilon+\beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_1^{\epsilon,\beta}, v_1^{\epsilon,\beta}\right)(T)\right)\right)\left(\epsilon+\beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\mathscr{D}\left(u_2^{\epsilon,\beta}, v_2^{\epsilon,\beta}\right)(T)\right)\right)}\theta_j\varphi_j. \qquad (61)
\end{aligned}$$

Step 1. Estimate the term $\|Q_{1,\epsilon,\beta}(u^{\epsilon,\beta}) - Q_{1,\epsilon,\beta}(v^{\epsilon,\beta})\|_{b,s}$.

First, using the assumption $(H2)$ and $\frac{\lambda_j}{1+k\lambda_j} \le k^{-1}$, we note that

$$\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right) \ge \epsilon + \beta\exp(k^{-1}T\mu_1) \qquad (62)$$

and

$$\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right) \ge \epsilon + \beta\exp(k^{-1}T\mu_1). \qquad (63)$$

Since the Sobolev embedding $\mathbf{H}^s(\Omega) \hookrightarrow L^2(\Omega)$ holds and C_s is an embedding constant, we see that

$$\begin{aligned}
&\left\|Q_{1,\epsilon,\beta}(u^{\epsilon,\beta})(t) - Q_{1,\epsilon,\beta}(v^{\epsilon,\beta})(t)\right\|^2_{\mathbf{H}^s(\Omega)} \\
&= \sum_j \frac{\epsilon^2\lambda_j^{2s}\left(\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right) - \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right)\right)^2}{\left(\epsilon+\beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)\right)^2\left(\epsilon+\beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right)\right)^2}\theta_j^2 \\
&\le C_s^2L^2\|f\|^2_{L^2}k^{-2}\left(\epsilon+\beta\exp(k^{-1}T\mu_1)\right)^{-2} \\
&\qquad \times\left(\int_0^t \|u^{\epsilon,\beta}(.,r) - v^{\epsilon,\beta}(.,r)\|_{\mathbf{H}^s(\Omega)}dr\right)^2\left(\sum_j \lambda_j^{2s}\theta_j^2\right) \\
&\le C_s^2L^2\|f\|^2_{L^2}k^{-2}T^2\left(\epsilon+\beta\exp(k^{-1}T\mu_1)\right)^{-2}\|u^{\epsilon,\beta} - v^{\epsilon,\beta}\|^2_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\|\theta\|^2_{\mathbf{H}^s(\Omega)}. \qquad (64)
\end{aligned}$$

The right hand side of (64) is independent of t, so we deduce

$$\begin{aligned}
&\|Q_{1,\epsilon,\beta}(u^{\epsilon,\beta}) - Q_{1,\epsilon,\beta}(v^{\epsilon,\beta})\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))} \\
&\le C_sL\|f\|_{L^2}k^{-1}T\left(\epsilon+\beta\exp(k^{-1}T\mu_1)\right)^{-1}\|u^{\epsilon,\beta} - v^{\epsilon,\beta}\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\|\theta\|_{\mathbf{H}^s(\Omega)}. \qquad (65)
\end{aligned}$$

Step 2. Estimate the term $\|Q_{2,\epsilon,\beta}(u^{\epsilon,\beta}) - Q_{2,\epsilon,\beta}(v^{\epsilon,\beta})\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}$.

Note

$$Q_{2,\epsilon,\beta}(u^{\epsilon,\beta})(t) - Q_{2,\epsilon,\beta}(v^{\epsilon,\beta})(t) = (I)(x,t) + (II)(x,t), \qquad (66)$$

where

$$(I)(x,t) = \beta \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right) \frac{\left[\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right) - \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u)(r)\right)dr\right)\right]}{\left(\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)\right)\left(\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right)\right)}\theta_j\varphi_j \tag{67}$$

and

$$(II)(x,t) = \beta \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right) \frac{\left[\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right) - \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(v)(r)\right)dr\right)\right]}{\left(\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)\right)\left(\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t T\mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right)\right)}\theta_j\varphi_j. \tag{68}$$

Let us treat the term $\|(I)(.,t)\|_{\mathbf{H}^s(\Omega)}$. We see that

$$\left\|(I)(.,t)\right\|^2_{\mathbf{H}^s(\Omega)} = \beta^2\lambda_j^{2s}\exp\left(-2\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right) \frac{\left[\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right) - \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u)(r)\right)dr\right)\right]^2}{\left(\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)\right)^2\left(\epsilon + \beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right)\right)^2}\theta_j^2. \tag{69}$$

Now

$$\left[\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right) - \exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)\right]^2 \leq C_s^2L^2\|f\|^2_{L^2}k^{-2}\left(\int_0^T \|u^{\epsilon,\beta}(.,r) - v^{\epsilon,\beta}(.,r)\|_{\mathbf{H}^s(\Omega)}dr\right)^2. \tag{70}$$

Noting that $\exp\left(-2\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)$ is less than 1 and using Hölder's inequality, it follows from (69) that

$$\begin{aligned}
&\left\|(I)(.,t)\right\|^2_{\mathbf{H}^s(\Omega)} \\
&\leq \beta^2\left(\epsilon + \beta\exp(k^{-1}T\mu_1)\right)^{-4}C_s^2L^2\|f\|^2_{L^2}k^{-2} \left(\int_0^T \|u^{\epsilon,\beta}(.,r) - v^{\epsilon,\beta}(.,r)\|_{\mathbf{H}^s(\Omega)}dr\right)^2\left(\sum_j \lambda_j^{2s}\theta_j^2\right) \\
&\leq \beta^2\left(\epsilon + \beta\exp(k^{-1}T\mu_1)\right)^{-4}C_s^2L^2\|f\|^2_{L^2}k^{-2}T^2\|u^{\epsilon,\beta} - v^{\epsilon,\beta}\|^2_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\|\theta\|^2_{\mathbf{H}^s(\Omega)}.
\end{aligned} \tag{71}$$

This implies immediately that

$$\left\|(I)(.,t)\right\|_{\mathbf{H}^s(\Omega)} \leq \beta\left(\epsilon + \beta\exp(k^{-1}T\mu_1)\right)^{-2}C_sL\|f\|_{L^2}k^{-1}T\|u - v\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\|\theta\|_{\mathbf{H}^s(\Omega)}. \tag{72}$$

Next, we look at the term $\left\|(II)\right\|_{\mathbf{H}^s(\Omega)}$. Now

$$\begin{aligned}
&\left\|(II)(.,t)\right\|^2_{\mathbf{H}^s(\Omega)}\\
&= \beta^2\lambda_j^{2s}\exp\left(-2\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(u)(r)\right)dr\right)\\
&\frac{\left[\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)-\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)\right]^2}{\left(\epsilon+\beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right)\right)^2\left(\epsilon+\beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^T \mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right)\right)^2}\theta_j^2. \qquad (73)
\end{aligned}$$

Also note

$$\begin{aligned}
&\left[\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(v^{\epsilon,\beta})(r)\right)dr\right)-\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^t \mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)\right]^2\\
&\qquad\le C_s^2L^2\|f\|^2_{L^2}k^{-2}\left(\int_0^t \|u^{\epsilon,\beta}(.,r)-v^{\epsilon,\beta}(.,r)\|_{\mathbf{H}^s(\Omega)}dr\right)^2\\
&\qquad\le C_s^2L^2\|f\|^2_{L^2}k^{-2}T^2\|u^{\epsilon,\beta}-v^{\epsilon,\beta}\|^2_{L^\infty(0,T;\mathbf{H}^s(\Omega))}. \qquad (74)
\end{aligned}$$

Then, we obtain

$$\left\|(II)(.,t)\right\|_{\mathbf{H}^s(\Omega)} \le \beta\Big(\epsilon+\beta\exp(k^{-1}T\mu_1)\Big)^{-2}C_sL\|f\|_{L^2}k^{-1}T\|u-v\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\|\theta\|_{\mathbf{H}^s(\Omega)}. \qquad (75)$$

From the observations (66), (73) and (74), we obtain

$$\begin{aligned}
\left\|Q_{2,\epsilon,\beta}(u^{\epsilon,\beta})(t)-Q_{2,\epsilon,\beta}(v^{\epsilon,\beta})(t)\right\|_{\mathbf{H}^s(\Omega)} &\le \left\|(I)(.,t)\right\|_{\mathbf{H}^s(\Omega)}+\left\|(II)(.,t)\right\|_{\mathbf{H}^s(\Omega)}\\
&\le 2\beta\epsilon^{-2}C_sL\|f\|_{L^2}k^{-1}T\|u^{\epsilon,\beta}-v^{\epsilon,\beta}\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\|\theta\|_{\mathbf{H}^s(\Omega)}. \qquad (76)
\end{aligned}$$

The right hand side of (76) is independent of t, so we deduce

$$\begin{aligned}
&\|Q_{2,\epsilon,\beta}(u^{\epsilon,\beta})-Q_{2,\epsilon,\beta}(v^{\epsilon,\beta})\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\\
&\le 2\beta\Big(\epsilon+\beta\exp(k^{-1}T\mu_1)\Big)^{-2}C_sL\|f\|_{L^2}k^{-1}T\|u^{\epsilon,\beta}-v^{\epsilon,\beta}\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\|\theta\|_{\mathbf{H}^s(\Omega)}. \qquad (77)
\end{aligned}$$

Combining (59), (65), (77), we find that

$$\begin{aligned}
&\|Q_{\epsilon,\beta}(u^{\epsilon,\beta})-Q_{\epsilon,\beta}(v^{\epsilon,\beta})\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\\
&\le \|Q_{1,\epsilon,\beta}(u^{\epsilon,\beta})-Q_{1,\epsilon,\beta}(v^{\epsilon,\beta})\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}+\|Q_{2,\epsilon,\beta}(u^{\epsilon,\beta})-Q_{2,\epsilon,\beta}(v^{\epsilon,\beta})\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\\
&\le L\|f\|_{L^2}k^{-1}T\left(\Big(\epsilon+\beta\exp(k^{-1}T\mu_1)\Big)^{-1}+2\beta\Big(\epsilon+\beta\exp(k^{-1}T\mu_1)\Big)^{-2}C_s\right)\\
&\qquad\times\|u^{\epsilon,\beta}-v^{\epsilon,\beta}\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\|\theta\|_{\mathbf{H}^s(\Omega)}. \qquad (78)
\end{aligned}$$

From condition (49), we have that $Q_{\epsilon,\beta}$ is a contraction in $L^\infty(0,T;\mathbf{H}^s(\Omega))$. Thus, there exists a function $u_{\epsilon,\beta}$ which is a solution of Problem (47). If $w^{\epsilon,\beta}=0$ then we get the following equality $Q_{\epsilon,\beta}(w^{\epsilon,\beta})=\sum_j \frac{1}{\epsilon}\theta_j\varphi_j(x)$. It follows from (78) that

$$\begin{aligned}
\|u^{\epsilon,\beta}\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))} &\le \|Q_{\epsilon,\beta}(u^{\epsilon,\beta})-Q_{\epsilon,\beta}(w^{\epsilon,\beta})\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}+\|Q_{\epsilon,\beta}(w^{\epsilon,\beta})\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\\
&\le L\|f\|_{L^2}k^{-1}T\left(\Big(\epsilon+\beta\exp(k^{-1}T\mu_1)\Big)^{-1}+2\beta\Big(\epsilon+\beta\exp(k^{-1}T\mu_1)\Big)^{-2}C_s\right)\\
&\quad\times\|u^{\epsilon,\beta}\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\|\theta\|_{\mathbf{H}^s(\Omega)}+\frac{1}{\epsilon}\|\theta\|_{\mathbf{H}^s(\Omega)}. \qquad (79)
\end{aligned}$$

Hence, we deduce that the regularity of the mild solution $u^{\epsilon,\beta}$ is as follows

$$\begin{aligned}
&\|u^{\epsilon,\beta}\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}\\
&\leq \frac{\frac{1}{\epsilon}\|\theta\|_{\mathbf{H}^s(\Omega)}}{1-L\|f\|_{L^2}k^{-1}T\left(\left(\epsilon+\beta\exp(k^{-1}T\mu_1)\right)^{-1}+2\beta\left(\epsilon+\beta\exp(k^{-1}T\mu_1)\right)^{-2}C_s\right)\|\theta\|_{\mathbf{H}^s(\Omega)}}\\
&\leq \frac{1}{\epsilon(1-p)}\|\theta\|_{\mathbf{H}^s(\Omega)} \quad \text{where assumption (49) is used to evaluate the denominator.}
\end{aligned} \tag{80}$$

Let $0 \leq t \leq t' \leq T$. It is easy to see that

$$\begin{aligned}
&u^{\epsilon,\beta}(x,t') - u^{\epsilon,\beta}(x,t)\\
&= \sum_j \frac{\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^{t'}\mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)-\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^{t}\mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)}{\epsilon+\beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^{T}\mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)}\theta_j\varphi_j(x).
\end{aligned} \tag{81}$$

Therefore, using Parseval's equality and using $|e^{-m}-e^{-n}| \leq C_\gamma |m-n|^\gamma$, for any $\gamma > 0$, C_γ is a constant depending on γ, we obtain at

$$\begin{aligned}
&\left\|u^{\epsilon,\beta}(.,t') - u^{\epsilon,\beta}(.,t)\right\|^2_{\mathbf{H}^s(\Omega)}\\
&= \sum_j \lambda_j^{2s}\left[\frac{\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^{t'}\mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)-\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^{t}\mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)}{\epsilon+\beta\exp\left(-\frac{\lambda_j}{1+k\lambda_j}\int_0^{T}\mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)}\right]^2\theta_j^2\\
&\leq \frac{C_\gamma^2}{\epsilon^2}\sum_j \lambda_j^{2s}\left(\frac{\lambda_j}{1+k\lambda_j}\right)^{2\gamma}\left(\int_t^{t'}\mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)^{2\gamma}\theta_j^2\\
&\leq \frac{C_\gamma^2}{\epsilon^2 k^{2\gamma}}\left(\int_t^{t'}\mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr\right)^{2\gamma}\|\theta\|^2_{\mathbf{H}^s(\Omega)}.
\end{aligned} \tag{82}$$

From Remark 2.2 we have

$$\begin{aligned}
\left|\mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)\right| = \left|\mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)-\mathcal{B}\left(l(0)\right)\right| &\leq L\|f\|_{L^2(\Omega)}\|u^{\epsilon,\beta}(.,r)\|_{L^2(\Omega)}\\
&\leq CL\|f\|_{L^2(\Omega)}\|u^{\epsilon,\beta}\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}.
\end{aligned} \tag{83}$$

Hence, we get immediately that

$$\int_t^{t'}\mathcal{B}\left(l(u^{\epsilon,\beta})(r)\right)dr \leq CL\|f\|_{L^2(\Omega)}(t-t')\|u^{\epsilon,\beta}\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}. \tag{84}$$

Combining (82) and (84), we deduce that $u^{\epsilon,\beta} \in C^\gamma([0,T];\mathbf{H}^s(\Omega))$. □

5. Numerical tests

In this section, we present some examples to compare the evaluations between the theoretical and numerical estimates. For convenience in calculations, we begin with a domain $\Omega = (0,L) = (0,\pi)$ and $T = 1$. In addition, we consider the Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = v(0,t) = v(\pi,t) = 0, \ t \in (0,1). \tag{85}$$

Then, we have the eigenvalues of the problem (1)

$$\lambda_j = \left(\frac{j\pi}{L}\right)^2 = j^2, \quad j \geq 1$$

and the corresponding eigenfunctions are given by

$$\varphi_j(x) = \sin\left(\frac{j\pi x}{L}\right) = \sin(jx), \quad j \geq 1.$$

First, we define the inner product in the $L^2(0,\pi)$ space between two functions f and g as follows

$$\langle f, g\rangle_{L^2(0,\pi)} = \int_0^\pi f(\tau)g(\tau)\mathrm{d}\tau$$

and the norm respectively

$$\|f\|_{L^2(0,\pi)} = \sqrt{\langle f, f\rangle} = \left(\int_0^\pi |f|^2 \,\mathrm{d}x\right)^{1/2}.$$

Second, by applying the middle Riemann sum method, we can approximate an integral as follows. Let $f(x)$ be defined on the closed interval $[a,b]$ and let $P = \{x_1, x_2, \ldots, x_{n+1}\}$ be a partition of $[a,b]$, with

$$a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b.$$

Let Δx_i denote the length of the i^{th} subinterval $[x_i, x_{i+1}]$ and let c_i denote any value in the i^{th} subinterval. The sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

is a Riemann sum of $f(x)$ on $[a,b]$. Then

i) When the n subintervals have equal length, $\Delta x_i = \Delta x = \dfrac{b-a}{n}$.

ii) The i^{th} term of the partition is $x_i = a + (i-1)\Delta x$. (This makes $x_{n+1} = b$.)

iii) The Midpoint Rule summation is: $\displaystyle\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x$.

Approximating f at the midpoint of intervals gives $f(a + \Delta x/2)$ for the first interval, for the next one $f(a + 3\Delta x/2)$, and so on until $f(b - \Delta x/2)$. Summing up gives

$$A_{\text{mid}} = \Delta x\left[f\left(a + \frac{\Delta x}{2}\right) + f\left(a + \frac{3\Delta x}{2}\right) + \cdots + f\left(b - \frac{\Delta x}{2}\right)\right].$$

The error of this formula will be

$$\left|\int_a^b f(x)dx - A_{\text{mid}}\right| \leq \frac{M_2(b-a)^3}{24n^2}$$

where M_2 is the maximum value of the absolute value of $f''(x)$ on the interval. This error is half of that of the Trapezoid rule and as such the Midpoint rule is the most accurate approach to the Riemann sum.

Next, the time and spatial domains are partitioned as follows

$$x_i = \frac{(i-1)\pi}{N_x}, \quad t_j = \frac{j-1}{N_t}, \text{ for } i = 1, \ldots, N_x + 1 \text{ and } j = 1, \ldots, N_t + 1,$$

where two given integer numbers are N_x, $N_t > 0$. By applying the finite difference method with the following partitions of temporal and spatial variables, for $x \in [0, \pi]$ and $t \in [0, 1]$, we consider the following partitions $\mathcal{D}_x \times \mathcal{D}_t$

$$\mathcal{D}_x := \left\{x_1 = 0,\ x_2 = \frac{1}{N_x},\ x_3 = \frac{2}{N_x},\ \dots, x_k = \frac{i-1}{N_x},\ \dots, x_{N_{x+1}} = 1,\ \text{for } i = 1, 2, \dots, N_x + 1\right\},$$
$$\mathcal{D}_t := \left\{t_1 = 0,\ t_2 = \frac{1}{N_t},\ t_3 = \frac{2}{N_t},\ \dots, t_j = \frac{j-1}{N_t},\ \dots, t_{N_{t+1}} = 1,\ \text{for } j = 1, 2, \dots, N_t + 1\right\}.$$

In test examples, the illustration is supported by the Python software (version 3.10) run on Windows 10-64bit, 16GB RAM, GPU NVIDIA 3050Ti, i7-11370H. Here, CPU times (seconds) are estimated by the Python Calculate Runtime in the library `import time` by the command code `start = time.time()`, `end = time.time()`.

Test 1. We focus on the estimate in Remark 2.2

In this test, we compare the estimate between theory and numerical estimates of the operation $\mathcal{B}$ which is a non-local term in Remark 2.2. In this example, we choose the functions

$$\mathcal{B}(z_1, z_2) = \sin(z_1) + \sin(z_2) + 3 \tag{86}$$

and we choose the following test functions

$$\begin{cases} u(x,t) = (t+1)\sin(3x) \\ v(x,t) = (t+2)\sin(4x). \end{cases} \tag{87}$$

Then, we obtain $\mu_0 = 1$ and $\mu_1 = 5$.

$\{t, t' = t + \varepsilon\}$	Truncated series $\mathcal{N}_k = 20, N_x = 100, N_t = 100$			
	ε	Theory errors	Calculation errors	CPU times(s) by Python
$\{0.1, 0.11\}$	0.01	0.15216252426680665	0.15542878878449115	8.624
$\{0.1, 0.101\}$	0.001	0.11875858857869495	0.09103388168847913	7.314
$\{0.1, 0.1001\}$	0.0001	0.06634852287631532	0.06713696707319854	8.160
$\{0.5, 0.51\}$	0.01	0.10253211218646219	0.10988490556048607	5.332
$\{0.5, 0.501\}$	0.001	0.08177708379028531	0.02701317361533908	4.670
$\{0.5, 0.5001\}$	0.0001	0.03926458697054691	0.04495235041540915	6.299
$\{0.7, 0.71\}$	0.01	0.09119491386297059	0.08903635480883027	7.305
$\{0.7, 0.701\}$	0.001	0.08232492332941625	0.07585845147500128	5.442
$\{0.7, 0.7001\}$	0.0001	0.04568552758279035	0.04164531720712175	7.500

Table 1: The error estimate at $t \in \{0.1, 0.5, 0.7\}$ and $t' + \varepsilon$, respectively

Test 2. In this test, we compare the estimate between theory and numerical estimation of $\|Q_{2,\epsilon,\beta}(u^{\epsilon,\beta}) - Q_{2,\epsilon,\beta}(v^{\epsilon,\beta})\|_{L^\infty(0,T;\mathbf{H}^s(\Omega))}$ which is given by 66. In this example, we choose

$$\begin{cases} \epsilon = \frac{1}{2},\ \beta = \frac{1}{3}, \\ \theta(x) = \frac{7}{6}\sin(3x), \\ \psi(x) = 2\sin(4x). \end{cases} \tag{88}$$

$\{t, t' = t + \varepsilon\}$	Truncated series $N_k = 20, N_x = 100, N_t = 100$			
	ε	Theory errors	Calculation errors	CPU times(s) by Python
$\{0.1, 0.11\}$	0.01	0.12447115325905414	0.11455702385634205	08.593
$\{0.1, 0.101\}$	0.001	0.07696337246573869	0.07315372279363776	15.348
$\{0.1, 0.1001\}$	0.0001	0.03472844795540667	0.04378750215898009	12.793
$\{0.5, 0.51\}$	0.01	0.09859619072167503	0.09999250647428548	10.044
$\{0.5, 0.501\}$	0.001	0.07904475978421273	0.07463925359386366	09.512
$\{0.5, 0.5001\}$	0.0001	0.03978042763065304	0.04069826837800516	14.365
$\{0.7, 0.71\}$	0.01	0.08023423716025158	0.07703612966733741	11.978
$\{0.7, 0.701\}$	0.001	0.05483434132912536	0.03921421759010715	19.356
$\{0.7, 0.7001\}$	0.0001	0.02857589727522703	0.02857589727527073	15.177

Table 2: The error estimate at $t \in \{0.1, 0.5, 0.7\}$ and $t' + \varepsilon$, respectively

The results obtained from Tables 1 and 2 show that the errors are tolerable. In other words, when the values os the time are close together, we have the smaller error between the solutions.

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