Filomat 37:24 (2023), 8067–8077
https://doi.org/10.2298/FIL2324067J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On holomorphic mappings with relatively p-compact range

A. Jiménez-Vargas[a]

[a] *Departamento de Matemáticas, Universidad de Almería, 04120, Almería, Spain*

Abstract. Related to the concept of p-compact operators with $p \in [1, \infty]$ introduced by Sinha and Karn [20], this paper deals with the space $\mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$ of all Banach-valued holomorphic mappings on an open subset U of a complex Banach space E whose ranges are relatively p-compact subsets of F. We characterize such holomorphic mappings as those whose Mujica's linearisations on the canonical predual of $\mathcal{H}^\infty(U)$ are p-compact operators. This fact allows us to make a complete study of them. We show that $\mathcal{H}^\infty_{\mathcal{K}_p}$ is a surjective Banach ideal of bounded holomorphic mappings which is generated by composition with the ideal of p-compact operators and contains the Banach ideal of all right p-nuclear holomorphic mappings. We also characterize holomorphic mappings with relatively p-compact ranges as those bounded holomorphic mappings which factorize through a quotient space of ℓ_{p^*} or as those whose transposes are quasi p-nuclear operators (respectively, factor through a closed subspace of ℓ_p).

1. Introduction

Inspired by classical Grothendieck's characterization of a relatively compact subset of a Banach space as a subset of the convex hull of a norm null sequence of vectors [12], Sinha and Karn [20] introduced and studied p-compact sets and p-compact operators with $p \in [1, \infty]$.

Let E be a complex Banach space with closed unit ball B_E. Let $p \in (1, \infty)$ and let p^* denote the *conjugate index of* p given by $p^* = p/(p-1)$. Given $p \in [1, \infty)$, $\ell_p(E)$ denotes the Banach space of all absolutely p-summable sequences (x_n) in E, equipped with the norm

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p},$$

and $c_0(E)$ stands for the Banach space of all norm null sequences (x_n) in E, endowed with the norm

$$\|(x_n)\|_\infty = \sup \{\|x_n\| : n \in \mathbb{N}\}.$$

In the case $E = \mathbb{C}$, we will simply write ℓ_p and c_0.

2020 *Mathematics Subject Classification*. Primary 47B07; Secondary 47B10, 46E15, 46E40.
Keywords. Vector-valued holomorphic mapping, p-compact set, p-compact operator, locally p-compact holomorphic mapping.
Received: 10 February 2023; Accepted: 06 April 2023
Communicated by Dragan S. Djordjević
This research was partially supported by project UAL-FEDER grant UAL2020-FQM-B1858, by Junta de Andalucía grants P20_00255 and FQM194, and by grant PID2021-122126NB-C31 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of making Europe".
Email address: ajimenez@ual.es (A. Jiménez-Vargas)

For $p \in (1,\infty)$, the *p-convex hull* of a sequence $(x_n) \in \ell_p(E)$ is defined by

$$p\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n \colon (a_n) \in B_{\ell_{p^*}} \right\}.$$

Similarly, we set

$$1\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n \colon (a_n) \in B_{c_0} \right\}, \qquad (x_n) \in \ell_1(E),$$

$$\infty\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n \colon (a_n) \in B_{\ell_1} \right\}, \qquad (x_n) \in c_0(E).$$

Note that ∞-conv(x_n) coincides with $\overline{\text{abco}}(\{x_n \colon n \in \mathbb{N}\})$, the norm-closed absolutely convex hull of the set $\{x_n \colon n \in \mathbb{N}\}$ in E.

Following [20], given $p \in [1,\infty]$, a set $K \subseteq E$ is said to be *relatively p-compact* if there is a sequence $(x_n) \in \ell_p(E)$ ($(x_n) \in c_0(E)$ if $p = \infty$) such that $K \subseteq p$-conv(x_n). Such a sequence is not unique but a *measure of the size of the p-compact set K* is introduced in [10, p. 297] (see also [14]) defining

$$m_p(K,E) = \begin{cases} \inf\{\|(x_n)\|_p : (x_n) \in \ell_p(E),\ K \subseteq p\text{-conv}(x_n)\} & \text{if } 1 \leq p < \infty, \\ \inf\{\|(x_n)\|_p : (x_n) \in c_0(E),\ K \subseteq p\text{-conv}(x_n)\} & \text{if } p = \infty, \end{cases}$$

and $m_p(K,E) = \infty$ if K is not p-compact. We frequently will write $m_p(K)$ instead of $m_p(K,E)$.

A linear operator between Banach spaces $T \colon E \to F$ is said to be *p-compact* if $T(B_E)$ is a relatively p-compact subset of F. The space of all p-compact linear operators from E to F is denoted by $\mathcal{K}_p(E,F)$, and $\mathcal{K}_p$ is a Banach operator ideal endowed with the norm $k_p(T) = m_p(T(B_E))$ (see [20, Theorem 4.2] and [10, Proposition 3.15]).

From the cited Grothendieck's result [12], a set $K \subseteq E$ is relatively compact if and only if for every $\varepsilon > 0$, there is a sequence $(x_n) \in c_0(E)$ with $\|(x_n)\|_\infty \leq \sup_{x \in K} \|x\| + \varepsilon$ such that $K \subseteq \infty$-conv(x_n). Hence we can consider compact sets as ∞-compact sets and, in this way, compact operators can be viewed as ∞-compact operators with k_∞ being the usual operator norm.

The work of Sinha and Karn [20] motivated many papers on p-compactness in operator spaces (see [3, 8–11, 14, 19], among others) and also in Lipschitz spaces [1, 2].

The extension of this theory to the polynomial and holomorphic settings was addressed in [5, 6]. In these environments, the property of p-compactness was studied from the following local point of view: a mapping $f \colon U \to F$ is said to be *locally p-compact* if every point $x \in U$ has a neighborhood $U_x \subseteq U$ such that $f(U_x)$ is relatively p-compact in F.

The aim of this note is to study a subclass of locally p-compact holomorphic mappings, namely, *holomorphic mappings with relatively p-compact range*. Notice that every such mapping is locally p-compact but the converse is not true. For instance, if $\mathbb{D}$ denotes the open complex unit disc, the holomorphic mapping $f \colon \mathbb{D} \to c_0$ defined by $f(z) = \sum_{n=1}^{\infty} z^n e_n$, where (e_n) is the canonical basis of ℓ_1, is locally 1-compact but it has not relatively compact range (see [15, Example 3.2]) and, consequently, neither relatively p-compact range for any $p \geq 1$.

Our motivation to deal with this class of mappings also arises from the study (initiated in [15] and continued in [7, 13]) on the Banach space $\mathcal{H}^\infty_{\mathcal{K}}(U,F)$ formed by all holomorphic mappings from U to F with relatively compact range, equipped with the supremum norm.

We now briefly describe the content of this paper. Let E and F be complex Banach spaces, U an open subset of E and $p \in [1,\infty]$. Let $\mathcal{H}^\infty(U,F)$ denote the Banach space of all bounded holomorphic mappings from U into F, endowed with the supremum norm. In particular, $\mathcal{H}^\infty(U)$ stands for $\mathcal{H}^\infty(U,\mathbb{C})$.

In [15], Mujica provided a linearisation method of the members of $\mathcal{H}^\infty(U,F)$, which will be an essential tool in our analysis of the subject. If $\mathcal{G}^\infty(U)$ is the canonical predual of $\mathcal{H}^\infty(U)$ obtained by Mujica [15] via an identification denoted g_U, we will establish that a bounded holomorphic mapping $f\colon U \to F$ has relatively p-compact range if and only if Mujica's linearisation $T_f\colon \mathcal{G}^\infty(U) \to F$ is a p-compact operator. This fact has some interesting applications.

If $\mathcal{H}^\infty_{\mathcal{K}_p}(U,F)$ denotes the space of all holomorphic mappings with relatively p-compact range $f\colon U \to F$ with the natural norm $k_p^{\mathcal{H}^\infty}(f) = m_p(f(U))$, we will prove that $\mathcal{H}^\infty_{\mathcal{K}_p}$ is a surjective Banach ideal of bounded holomorphic mappings which is generated by composition with the ideal $\mathcal{K}_p$ of p-compact operators. This means that each mapping $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U,F)$ admits a factorization $f = T \circ g$, where G is a complex Banach space, $g \in \mathcal{H}^\infty(U,G)$ and $T \in \mathcal{K}_p(G,F)$. Moreover, $k_p^{\mathcal{H}^\infty}(f)$ coincides with $\inf\{k_p(T)\left\|g\right\|_\infty\}$, where the infimum is extended over all such factorizations of f and, curiously, this infimum is attained at the factorization $f = T_f \circ g_U$ due to Mujica [15].

In parallelism with the linear case, we introduce the notion of right p-nuclear holomorphic mapping from U to F, study its linearisation on $\mathcal{G}^\infty(U)$, analyse its ideal property and show that every right p-nuclear holomorphic mapping has relatively p-compact range.

Moreover, we characterize the members of $\mathcal{H}^\infty_{\mathcal{K}_p}(U,F)$ as those bounded holomorphic mappings from U to F which factorize through a quotient space of ℓ_{p^*}, and also as those whose transposes are quasi p-nuclear operators (respectively, factor through a closed subspace of ℓ_p).

2. The results

From now on, unless otherwise stated, E and F will denote complex Banach spaces, U will be an open subset of E and $p \in [1,\infty]$.

As usual, $\mathcal{L}(E,F)$ denotes the Banach space of all bounded linear operators from E to F endowed with the operator canonical norm and E^* stands for the dual space of E. The subspaces of $\mathcal{L}(E,F)$ consisting of all compact operators and finite-rank bounded operators from E to F will be denoted by $\mathcal{K}(E,F)$ and $\mathcal{F}(E,F)$, respectively.

In this section, we will study holomorphic mappings $f\colon U \to F$ so that $f(U)$ is a relatively p-compact subset of F. We denote by $\mathcal{H}^\infty_{\mathcal{K}_p}(U,F)$ the set formed by such mappings and, for each $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U,F)$, we define $k_p^{\mathcal{H}^\infty}(f) = m_p(f(U))$.

The space of all holomorphic mappings with relatively compact range from U to F, denoted $\mathcal{H}^\infty_{\mathcal{K}}(U,F)$, is a Banach space equipped with the supremum norm (see [13, Corollary 2.11]). On account of the following result, we will only study in this paper the case $1 \leq p < \infty$.

Proposition 2.1. *$\mathcal{H}^\infty_{\mathcal{K}_\infty}(U,F) = \mathcal{H}^\infty_{\mathcal{K}}(U,F)$ and $k_\infty^{\mathcal{H}^\infty}(f) = \left\|f\right\|_\infty$ for all $\mathcal{H}^\infty_{\mathcal{K}_\infty}(U,F)$.*

Proof. Let $f \in \mathcal{H}^\infty_{\mathcal{K}_\infty}(U,F)$ and let $(y_n) \in c_0(F)$ be such that $f(U) \subseteq \infty\text{-conv}(y_n)$. Since $\infty\text{-conv}(y_n)$ is relatively compact in F, it follows that $f \in \mathcal{H}^\infty_{\mathcal{K}}(U,F)$ with $\left\|f\right\|_\infty \leq \left\|(y_n)\right\|_\infty$, and taking infimum over all such sequences (y_n), we have $\left\|f\right\|_\infty \leq k_\infty^{\mathcal{H}^\infty}(f)$.

Conversely, let $f \in \mathcal{H}^\infty_{\mathcal{K}}(U,F)$. By classical Grothendieck's result, for every $\varepsilon > 0$, there exists $(y_n) \in c_0(F)$ with $\left\|(y_n)\right\|_\infty \leq \left\|f\right\|_\infty + \varepsilon$ such that $f(U) \subseteq \infty\text{-conv}(y_n)$. Hence $f \in \mathcal{H}^\infty_{\mathcal{K}_\infty}(U,F)$ and $k_\infty^{\mathcal{H}^\infty}(f) \leq \left\|f\right\|_\infty$. □

2.1. Linearisation

Our first aim is to characterize holomorphic mappings with relatively p-compact range in terms of the p-compactness of their linearisations on the canonical predual of $\mathcal{H}^\infty(U)$.

Towards this end, we first recall the following result due to Mujica [15] concerning linearisation of holomorphic mappings on Banach spaces.

Theorem 2.2. *[15] Let E be a complex Banach space and U be an open set in E. Let $\mathcal{G}^\infty(U)$ denote the norm-closed linear subspace of $\mathcal{H}^\infty(U)^*$ generated by the functionals $\delta(x) \in \mathcal{H}^\infty(U)^*$ with $x \in U$, defined by $\delta(x)(f) = f(x)$ for all $f \in \mathcal{H}^\infty(U)$.*

(i). *The mapping* $g_U : U \to \mathcal{G}^\infty(U)$ *defined by* $g_U(x) = \delta(x)$ *is holomorphic with* $\|\delta(x)\| = 1$ *for all* $x \in U$.

(ii). *For every complex Banach space* F *and every mapping* $f \in \mathcal{H}^\infty(U, F)$, *there exists a unique operator* $T_f \in \mathcal{L}(\mathcal{G}^\infty(U), F)$ *such that* $T_f \circ g_U = f$. *Furthermore,* $\left\|T_f\right\| = \left\|f\right\|_\infty$.

(iii). *For every complex Banach space* F, *the mapping* $f \mapsto T_f$ *is an isometric isomorphism from* $\mathcal{H}^\infty(U, F)$ *onto* $\mathcal{L}(\mathcal{G}^\infty(U), F)$.

(iv). $\mathcal{H}^\infty(U)$ *is isometrically isomorphic to* $\mathcal{G}^\infty(U)^*$, *via the mapping* $J_U : \mathcal{H}^\infty(U) \to \mathcal{G}^\infty(U)^*$ *given by* $J_U(f)(g_U(x)) = f(x)$ *for all* $f \in \mathcal{H}^\infty(U)$ *and* $x \in U$.

(v). $B_{\mathcal{G}^\infty(U)}$ *coincides with* $\overline{\mathrm{abco}}(g_U(U))$. □

We are now ready to state the aforementioned characterization.

Theorem 2.3. *Let* $p \in [1, \infty)$ *and* $f \in \mathcal{H}^\infty(U, F)$. *The following conditions are equivalent:*

(i). f *has relatively* p*-compact range.*

(ii). $T_f : \mathcal{G}^\infty(U) \to F$ *is a* p*-compact linear operator.*

In this case, $k_p^{\mathcal{H}^\infty}(f) = k_p(T_f)$. *Furthermore,* $f \mapsto T_f$ *is an isometric isomorphism from* $(\mathcal{H}^\infty_{\mathcal{K}_p}(U, F), k_p^{\mathcal{H}^\infty})$ *onto* $(\mathcal{K}_p(\mathcal{G}^\infty(U), F), k_p)$.

Proof. Using Theorem 2.2, we have the following inclusions:

$$
\begin{aligned}
f(U) = T_f \circ g_U(U) &\subseteq T_f(\overline{\mathrm{abco}}(g_U(U))) = T_f(B_{\mathcal{G}^\infty(U)}) \\
&\subseteq \overline{\mathrm{abco}}(T_f \circ g_U(U)) = \overline{\mathrm{abco}}(f(U)).
\end{aligned}
$$

$(ii) \Rightarrow (i)$: If $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$, then $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$ with

$$k_p^{\mathcal{H}^\infty}(f) = m_p(f(U)) \leq m_p(T_f(B_{\mathcal{G}^\infty(U)})) = k_p(T_f),$$

by the first inclusion above.

$(i) \Rightarrow (ii)$: If $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$, then $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ with

$$k_p(T_f) = m_p(T_f(B_{\mathcal{G}^\infty(U)})) \leq m_p(\overline{\mathrm{abco}}(f(U))) = m_p(f(U)) = k_p^{\mathcal{H}^\infty}(f),$$

by the second inclusion above and the known fact that a set $K \subseteq F$ is p-compact if and only if $\overline{\mathrm{abco}}(K)$ is p-compact, in whose case $m_p(K) = m_p(\overline{\mathrm{abco}}(K))$.

The last assertion of the statement follows immediately from part (iii) of Theorem 2.2 and from the above proof. □

2.2. Banach ideal property

Our next goal is to study the Banach ideal structure of $(\mathcal{H}^\infty_{\mathcal{K}_p}, k_p^{\mathcal{H}^\infty})$. Inspired by the notion of Banach operator ideal [18], the following type of ideals was considered in [7].

An *ideal of bounded holomorphic mappings* (or simply, a *bounded-holomorphic ideal*) is a subclass $\mathcal{I}^{\mathcal{H}^\infty}$ of the class of bounded holomorphic mappings $\mathcal{H}^\infty$ such that for each complex Banach space E, each open subset U of E and each complex Banach space F, the components

$$\mathcal{I}^{\mathcal{H}^\infty}(U, F) := \mathcal{I}^{\mathcal{H}^\infty} \cap \mathcal{H}^\infty(U, F)$$

satisfy the following three conditions:

(I1) $\mathcal{I}^{\mathcal{H}^\infty}(U, F)$ is a linear subspace of $\mathcal{H}^\infty(U, F)$,

(I2) For any $g \in \mathcal{H}^\infty(U)$ and $y \in F$, the mapping $g \cdot y : x \mapsto g(x)y$ from U to F is in $\mathcal{I}^{\mathcal{H}^\infty}(U, F)$, (I3) *The ideal property*: if H, G are complex Banach spaces, V is an open subset of H, $h \in \mathcal{H}(V, U)$, $f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F)$ and $S \in \mathcal{L}(F, G)$, then $S \circ f \circ h \in \mathcal{I}^{\mathcal{H}^\infty}(V, G)$.

Suppose that a function $\|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty}} : \mathcal{I}^{\mathcal{H}^\infty} \to \mathbb{R}_0^+$ satisfies the following three properties:

(N1) $(\mathcal{I}^{\mathcal{H}^\infty}(U, F), \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty}})$ is a normed (Banach) space with $\|f\|_\infty \leq \|f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$ for all $f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F)$,

(N2) $\|g \cdot y\|_{\mathcal{I}^{\mathcal{H}^\infty}} = \|g\|_\infty \|y\|$ for all $g \in \mathcal{H}^\infty(U)$ and $y \in F$,

(N3) If H, G are complex Banach spaces, V is an open subset of H, $h \in \mathcal{H}(V, U)$, $f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F)$ and $S \in \mathcal{L}(F, G)$, then $\|S \circ f \circ h\|_{\mathcal{I}^{\mathcal{H}^\infty}} \leq \|S\| \|f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$.

Then $(\mathcal{I}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty}})$ is called a *normed (Banach) bounded-holomorphic ideal*.

A normed bounded-holomorphic ideal $\mathcal{I}^{\mathcal{H}^\infty}$ is said to be:

(R) *regular* if for any $f \in \mathcal{H}^\infty(U, F)$, we have that $f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F)$ with $\|f\|_{\mathcal{I}^{\mathcal{H}^\infty}} = \|\kappa_F \circ f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$ whenever $\kappa_F \circ f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F^{**})$, where κ_F denotes the isometric linear embedding from F into F^{**}.

(S) *surjective* if for any mapping $f \in \mathcal{H}^\infty(U, F)$, any open subset V of a complex Banach space G and any surjective mapping $\pi \in \mathcal{H}(V, U)$, we have that $f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F)$ with $\|f\|_{\mathcal{I}^{\mathcal{H}^\infty}} = \|f \circ \pi\|_{\mathcal{I}^{\mathcal{H}^\infty}}$ whenever $f \circ \pi \in \mathcal{I}^{\mathcal{H}^\infty}(V, F)$.

Bearing in mind Theorem 2.3, Theorem 3.2 in [4] (see also Theorem 2.4 in [7]) shows that $\mathcal{H}^\infty_{\mathcal{K}_p}$ is generated by composition with the operator ideal $\mathcal{K}_p$ (see [7, Definition 2.3]).

Corollary 2.4. *Let $p \in [1, \infty)$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent:*

(i). *$f : U \to F$ has relatively p-compact range.*

(ii). *$f = T \circ g$ for some complex Banach space G, $g \in \mathcal{H}^\infty(U, G)$ and $T \in \mathcal{K}_p(G, F)$.*

In this case, we have

$$k_p^{\mathcal{H}^\infty}(f) = \|f\|_{\mathcal{K}_p \circ \mathcal{H}^\infty} := \inf\{k_p(T) \|g\|_\infty\},$$

where the infimum runs over all factorizations of f as in (ii), and this infimum is attained at $T_f \circ g_U$ (Mujica's factorization of f [15]).

Furthermore, $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{K}_p \circ \mathcal{H}^\infty(U, F), \|\cdot\|_{\mathcal{K}_p \circ \mathcal{H}^\infty})$ onto $(\mathcal{K}_p(\mathcal{G}^\infty(U), F), k_p)$. □

The following result gathers some Banach ideal properties of $\mathcal{H}^\infty_{\mathcal{K}_p}$.

Theorem 2.5. *For each $p \in [1, \infty)$, $(\mathcal{H}^\infty_{\mathcal{K}_p}, k_p^{\mathcal{H}^\infty})$ is a surjective Banach bounded-holomorphic ideal. Furthermore, the ideal $(\mathcal{H}^\infty_{\mathcal{K}_p}(U, F), k_p^{\mathcal{H}^\infty})$ is regular whenever F is reflexive.*

Proof. In view of Corollary 2.4, Corollary 2.5 in [7] yields that $(\mathcal{H}^\infty_{\mathcal{K}_p}, k_p^{\mathcal{H}^\infty})$ is a Banach bounded-holomorphic ideal. Then we only have to study its surjectivity and its regularity.

(S) Let $f \in \mathcal{H}^\infty(U, F)$ and assume that $f \circ \pi \in \mathcal{H}^\infty_{\mathcal{K}_p}(V, F)$, where V is an open subset of a complex Banach space G and $\pi \in \mathcal{H}(V, U)$ is surjective. Since $f(U) = (f \circ \pi)(V)$, it is immediate that $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$ with $k_p^{\mathcal{H}^\infty}(f) = k_p^{\mathcal{H}^\infty}(f \circ \pi)$. Hence $(\mathcal{H}^\infty_{\mathcal{K}_p}, k_p^{\mathcal{H}^\infty})$ is surjective.

(R) Suppose now that F is reflexive. Let $f \in \mathcal{H}^\infty(U, F)$ and assume that $\kappa_F \circ f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F^{**})$. We can take a sequence (y_n) in $\ell_p(F)$ (in $c_0(F)$ if $p = 1$) such that $(\kappa_F \circ f)(U) \subseteq p\text{-conv}(\kappa_F(y_n))$, that is, $\kappa_F(f(U)) \subseteq \kappa_F(p\text{-conv}(y_n))$ which yields $f(U) \subseteq p\text{-conv}(y_n)$ by the injectivity of κ_F. Hence $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$ with $k_p^{\mathcal{H}^\infty}(f) \leq \|(y_n)\|_p = \|(\kappa_F(y_n))\|_p$ and so $k_p^{\mathcal{H}^\infty}(f) \leq k_p^{\mathcal{H}^\infty}(\kappa_F \circ f)$ by extending this infimum over all such sequences $(\kappa_F(y_n))$. The converse inequality follows from the condition (N3) satisfied by $(\mathcal{H}^\infty_{\mathcal{K}_p}, k_p^{\mathcal{H}^\infty})$ and this completes the proof. □

2.3. Factorization

We now present a factorization result for holomorphic mappings with relatively p-compact range which should be compared with [11, Proposition 2.9].

Corollary 2.6. *Let $p \in [1, \infty)$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent:*

(i). *$f : U \to F$ has relatively p-compact range.*

(ii). *There exist a closed subspace M in ℓ_{p^*} (c_0 instead of ℓ_{p^*} if $p = 1$), a separable Banach space G, an operator T in $\mathcal{K}_p(\ell_{p^*}/M, G)$, a mapping g in $\mathcal{H}^\infty_{\mathcal{K}}(U, \ell_{p^*}/M)$ and an operator S in $\mathcal{K}(G, F)$ such that $f = S \circ T \circ g$.*

In this case, $k_p^{\mathcal{H}^\infty}(f) = \inf\{\|S\| k_p(T) \|g\|_\infty\}$, where the infimum is extended over all factorizations of f as in (ii).

Proof. We will only prove it for $p \in (1, \infty)$. The case $p = 1$ is similarly obtained.

$(i) \Rightarrow (ii)$: Suppose that $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$. By Theorem 2.3, $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ with $k_p(T_f) = k_p^{\mathcal{H}^\infty}(f)$. Applying [11, Proposition 2.9], for each $\varepsilon > 0$, there exist a closed subspace $M \subseteq \ell_{p^*}$, a separable Banach space G, an operator $T \in \mathcal{K}_p(\ell_{p^*}/M, G)$, an operator $S \in \mathcal{K}(G, F)$ and an operator $R \in \mathcal{K}(\mathcal{G}^\infty(U), \ell_{p^*}/M)$ such that $T_f = S \circ T \circ R$ with $\|S\| k_p(T) \|R\| \leq k_p(T_f) + \varepsilon$. Moreover, $R = T_g$ with $\|g\|_\infty = \|R\|$ for some $g \in \mathcal{H}^\infty_{\mathcal{K}}(U, \ell_{p^*}/M)$ by [13, Corollary 2.11]. Thus we obtain

$$f = T_f \circ g_U = S \circ T \circ R \circ g_U = S \circ T \circ T_g \circ g_U = S \circ T \circ g,$$

with

$$\|S\| k_p(T) \|g\|_\infty = \|S\| k_p(T) \|R\| \leq k_p(T_f) + \varepsilon = k_p^{\mathcal{H}^\infty}(f) + \varepsilon.$$

Since ε was arbitrary, we deduce that $\|S\| k_p(T) \|g\|_\infty \leq k_p^{\mathcal{H}^\infty}(f)$.

$(ii) \Rightarrow (i)$: Assume that $f = S \circ T \circ g$ is a factorization as in (ii). Since $S \circ T \in \mathcal{K}_p(\ell_{p^*}/M, F)$ by the ideal property of $\mathcal{K}_p$, Corollary 2.4 yields that $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$ with

$$k_p^{\mathcal{H}^\infty}(f) \leq k_p(S \circ T) \|g\|_\infty \leq \|S\| k_p(T) \|g\|_\infty,$$

and taking infimum over all such factorizations of f, we have $k_p^{\mathcal{H}^\infty}(f) \leq \inf\{\|S\| k_p(T) \|g\|_\infty\}$. □

2.4. Transposition

Let us recall that the *transpose* of a mapping $f \in \mathcal{H}^\infty(U, F)$ is the bounded linear operator $f^t : F^* \to \mathcal{H}^\infty(U)$ defined by

$$f^t(y^*) = y^* \circ f \qquad (y^* \in F^*).$$

Moreover, $\|f^t\| = \|f\|_\infty$ and $f^t = J_U^{-1} \circ (T_f)^*$, where $J_U : \mathcal{H}^\infty(U) \to \mathcal{G}^\infty(U)^*$ is the isometric isomorphism defined in Theorem 2.2.

The Banach ideal $\mathcal{K}_p$ is associated by duality with the ideal of quasi-p-nuclear operators. According to [17], for every $p \in [1, \infty)$, an operator $T \in \mathcal{L}(E, F)$ is said to be *quasi p-nuclear* if there is a sequence $(x_n^*) \in \ell_p(E^*)$ such that

$$\|T(x)\| \leq \left(\sum_{n=1}^{\infty} |x_n^*(x)|^p \right)^{1/p} \qquad (x \in E).$$

If $\mathcal{QN}_p(E, F)$ denotes the set formed by such operators, then $\mathcal{QN}_p$ is a Banach operator ideal equipped with the norm

$$\nu_p^{\mathcal{Q}}(T) = \inf \left\{ \|(x_n^*)\|_p : \ \|T(x)\| \leq \left(\sum_{n=1}^{\infty} |x_n^*(x)|^p \right)^{1/p}, \ \forall x \in E \right\}.$$

By [10, Proposition 3.8], an operator $T \in \mathcal{K}_p(E, F)$ if and only if its adjoint $T^* \in \mathcal{QN}_p(F^*, E^*)$. Moreover, $k_p(T) = \nu_p^{\mathcal{Q}}(T^*)$ by [11, Corollary 2.7]. A holomorphic version of this result can be stated as follows.

Theorem 2.7. *Let $p \in [1,\infty)$ and $f \in \mathcal{H}^\infty(U,F)$. The following conditions are equivalent:*

(i). *$f\colon U \to F$ has relatively p-compact range.*
(ii). *$f^t\colon F^* \to \mathcal{H}^\infty(U)$ is a quasi p-nuclear operator.*

In this case, $k_p^{\mathcal{H}^\infty}(f) = \nu_p^{\mathcal{Q}}(f^t)$.

Proof. Applying Theorem 2.3, [11, Corollary 2.7] and [17, p. 32], respectively, we have

$$\begin{aligned} f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U,F) &\Leftrightarrow T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U),F)\\ &\Leftrightarrow (T_f)^* \in \mathcal{QN}_p(F^*,\mathcal{G}^\infty(U)^*)\\ &\Leftrightarrow f^t = J_U^{-1}\circ (T_f)^* \in \mathcal{QN}_p(F^*,\mathcal{H}^\infty(U)).\end{aligned}$$

In this case, $k_p^{\mathcal{H}^\infty}(f) = k_p(T_f) = \nu_p^{\mathcal{Q}}((T_f)^*) = \nu_p^{\mathcal{Q}}(f^t)$. □

Given $p \in [1,\infty)$, let us recall (see [18]) that an operator $T \in \mathcal{L}(E,F)$ is *p-summing* if there exists a constant $C \geq 0$ such that, regardless of the natural number n and regardless of the choice of vectors $x_1,\ldots,x_n$ in E, we have

$$\left(\sum_{i=1}^n \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{x^*\in B_{E^*}} \left(\sum_{i=1}^n |x^*(x_i)|^p\right)^{\frac{1}{p}}.$$

The infimum of such constants C is denoted by $\pi_p(T)$ and the linear space of all p-summing operators from E into F by $\Pi_p(E,F)$.

The following result could be compared with [10, Proposition 3.13] stated in the linear setting.

Proposition 2.8. *Let $f \in \mathcal{H}^\infty(U,F)$ and $g \in \mathcal{H}^\infty_{\mathcal{K}}(U,F^*)$. Assume that $T_f \in \Pi_p(\mathcal{G}^\infty(U),F)$ with $p \in [1,\infty)$. Then $f^t \circ g \in \mathcal{H}^\infty_{\mathcal{K}_p}(U,\mathcal{H}^\infty(U))$ with $k_p^{\mathcal{H}^\infty}(f^t\circ g) \leq \pi_p(T_f)\left\|g\right\|_\infty$.*

Proof. By Theorem 2.3 and Proposition 2.1, $T_g \in \mathcal{K}(\mathcal{G}^\infty(U),F^*)$ with $\|T_g\| = \left\|g\right\|_\infty$. Consequently, by [10, Proposition 3.13], the linear operator $(T_f)^*\circ T_g \in \mathcal{K}_p(\mathcal{G}^\infty(U),\mathcal{G}^\infty(U)^*)$ with $k_p((T_f)^*\circ T_g) \leq \pi_p(T_f)\|T_g\|$. From the equality $f^t\circ T_g = J_U^{-1}\circ (T_f)^*\circ T_g$, we infer that $f^t \circ T_g \in \mathcal{K}_p(\mathcal{G}^\infty(U),\mathcal{H}^\infty(U))$ with $k_p(f^t\circ T_g) = k_p((T_f)^*\circ T_g)$ by the ideal property of $\mathcal{K}_p$. Applying Theorem 2.3, there exists $h \in \mathcal{H}^\infty_{\mathcal{K}_p}(U,\mathcal{H}^\infty(U))$ with $k_p^{\mathcal{H}^\infty}(h) = k_p(T_h)$ such that $f^t\circ T_g = T_h$. Hence $f^t\circ g = h$ and thus $f^t\circ g \in \mathcal{H}^\infty_{\mathcal{K}_p}(U,\mathcal{H}^\infty(U))$ with $k_p^{\mathcal{H}^\infty}(f^t\circ g) = k_p(T_h) = k_p((T_f)^*\circ T_g) \leq \pi_p(T_f)\left\|T_g\right\| = \pi_p(T_f)\left\|g\right\|_\infty$. □

p-Compact operators were characterized as those operators whose adjoints factor through a subspace of ℓ_p [20, Theorem 3.2]. We now obtain a similar factorization for the transpose of a holomorphic mapping with relatively p-compact range (compare also to [10, Proposition 3.10]).

Corollary 2.9. *Let $p \in [1,\infty)$ and $f \in \mathcal{H}^\infty(U,F)$. The following conditions are equivalent:*

(i). *$f\colon U \to F$ has relatively p-compact range.*
(ii). *There exist a closed subspace $M \subseteq \ell_p$ and $R \in \mathcal{QN}_p(F^*,M)$, $S \in \mathcal{L}(M,\mathcal{H}^\infty(U))$ such that $f^t = S\circ R$.*

Proof. $(i) \Rightarrow (ii)$: If $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U,F)$, then $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U),F)$ by Theorem 2.3. By [10, Proposition 3.10], there exist a closed subspace $M \subseteq \ell_p$ and operators $R \in \mathcal{QN}_p(F^*,M)$ and $S_0 \in \mathcal{L}(M,\mathcal{G}^\infty(U)^*)$ such that $(T_f)^* = S_0\circ R$. Taking $S = J_U^{-1}\circ S_0 \in \mathcal{L}(M,\mathcal{H}^\infty(U))$, we have $f^t = S\circ R$.

$(ii) \Rightarrow (i)$: Assume that $f^t = S\circ R$ being S and R as in the statement. It follows that $(T_f)^* = J_U\circ f^t = J_U\circ S\circ R$, and thus $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U),F)$ by [10, Proposition 3.10]. Hence $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U,F)$ by Theorem 2.3 □

2.5. Inclusions

We will study the inclusion relations of holomorphic mappings with relatively p-compact range between them and with other classes of bounded holomorphic mappings.

Our first result follows immediately by applying Theorem 2.3 and the fact stated in [20, Proposition 4.3] that $\mathcal{K}_p \subseteq \mathcal{K}_q$ whenever $1 \leq p \leq q \leq \infty$.

Corollary 2.10. *If $1 \leq p \leq q < \infty$ and $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$, then $f \in \mathcal{H}^\infty_{\mathcal{K}_q}(U, F)$ and $k_q^{\mathcal{H}^\infty}(f) \leq k_p^{\mathcal{H}^\infty}(f)$.* □

Let us recall that a mapping $f \in \mathcal{H}^\infty(U, F)$ has *finite dimensional rank* if the linear hull of its range is a finite dimensional subspace of F. We denote by $\mathcal{H}^\infty_{\mathcal{F}}(U, F)$ the set of all finite-rank bounded holomorphic mappings from U to F. In the light of Theorem 2.5, it is clear that $\mathcal{H}^\infty_{\mathcal{F}}(U, F)$ is a linear subspace of $\mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$.

In similarity with the linear case, it seems natural to introduce the following class of holomorphic mappings.

Definition 2.11. *Let $p \in [1, \infty)$. A mapping $f \in \mathcal{H}^\infty(U, F)$ is said to be p-approximable if there exists a sequence (f_n) in $\mathcal{H}^\infty_{\mathcal{F}}(U, F)$ such that $k_p^{\mathcal{H}^\infty}(f_n - f) \to 0$ as $n \to \infty$. We denote by $\mathcal{H}^\infty_{\overline{\mathcal{F}}_p}(U, F)$ the space of all p-approximable holomorphic mappings from U to F.*

Proposition 2.12. *For $p \in [1, \infty)$, every p-approximable holomorphic mapping from U to F has relatively p-compact range.*

Proof. Let $f \in \mathcal{H}^\infty_{\overline{\mathcal{F}}_p}(U, F)$. Hence there is a sequence (f_n) in $\mathcal{H}^\infty_{\mathcal{F}}(U, F)$ such that $k_p^{\mathcal{H}^\infty}(f_n - f) \to 0$ as $n \to \infty$. Since $T_{f_n} \in \mathcal{F}(\mathcal{G}^\infty(U), F)$ by [15, Proposition 3.1], $\mathcal{F}(\mathcal{G}^\infty(U), F) \subseteq \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ by [20, Theorem 4.2] and $k_p(T_{f_n} - T_f) = k_p(T_{f_n - f}) = k_p^{\mathcal{H}^\infty}(f_n - f)$ for all $n \in \mathbb{N}$ by Theorems 2.2 and 2.3, we deduce that $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ by [20, Theorem 4.2], and so $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$ by Theorem 2.3. □

Given $p \in [1, \infty)$, $\ell_p^{\mathrm{weak}}(E)$ denotes the Banach space of all weakly p-summable sequences (x_n) in E, endowed with the norm

$$\|(x_n)\|_p^{\mathrm{weak}} = \sup\left\{\left(\sum_{n=1}^{\infty} \left|f(x_n)\right|^p\right)^{1/p} : f \in B_{E^*}\right\}.$$

Let us recall (see [16]) that an operator $T \in \mathcal{L}(E, F)$ is said to be *right p-nuclear* if there are sequences $(x_n^*) \in \ell_{p^*}^{\mathrm{weak}}(E^*)$ and $(y_n) \in \ell_p(F)$ such that

$$T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n \qquad (x \in E),$$

and the series converges in $\mathcal{L}(E, F)$. The set of such operators, denoted $\mathcal{N}^p(E, F)$, is a Banach space with the norm

$$\nu^p(T) = \inf\left\{\left\|(x_n^*)\right\|_{p^*}^{\mathrm{weak}} \left\|(y_n)\right\|_p\right\},$$

where the infimum is taken over all representations of T as above.

A holomorphic variant of this class of operators can be introduced as follows.

Definition 2.13. *Given $p \in [1, \infty)$, a holomorphic mapping $f \colon U \to F$ is said to be right p-nuclear if there exist sequences (g_n) in $\ell_{p^*}^{\mathrm{weak}}(\mathcal{H}^\infty(U))$ and (y_n) in $\ell_p(F)$ such that $f = \sum_{n=1}^{\infty} g_n \cdot y_n$ in $(\mathcal{H}^\infty(U, F), \|\cdot\|_\infty)$. We set*

$$\nu^{p\mathcal{H}^\infty}(f) = \inf\left\{\left\|(g_n)\right\|_{p^*}^{\mathrm{weak}} \left\|(y_n)\right\|_p\right\},$$

with the infimum taken over all right p-nuclear holomorphic representations of f as above. Let $\mathcal{H}^\infty_{\mathcal{N}^p}(U, F)$ denote the set of all right p-nuclear holomorphic mappings from U into F.

We now establish the relationships of a right p-nuclear holomorphic mapping $f\colon U \to F$ with its linearisation $T_f\colon \mathcal{G}^\infty(U) \to F$.

Theorem 2.14. *Let $p \in [1,\infty)$ and $f \in \mathcal{H}^\infty(U,F)$. The following conditions are equivalent:*

(i). *$f\colon U \to F$ is right p-nuclear.*
(ii). *$T_f\colon \mathcal{G}^\infty(U) \to F$ is a right p-nuclear operator.*

In this case, $\nu^{p\mathcal{H}^\infty}(f) = \nu^p(T_f)$. Furthermore, $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{H}^\infty_{\mathcal{N}^p}(U,F), \nu^{p\mathcal{H}^\infty})$ onto $(\mathcal{N}^p(\mathcal{G}^\infty(U),F), \nu^p)$.

Proof. $(i) \Rightarrow (ii)$: Assume that $f \in \mathcal{H}^\infty_{\mathcal{N}^p}(U,F)$ and let $\sum_{n\geq 1} g_n \cdot y_n$ be a right p-nuclear holomorphic representation of f. By Theorem 2.2, there is a unique operator $T_f \in \mathcal{L}(\mathcal{G}^\infty(U,F)$ such that $T_f \circ g_U = f$. Similarly, for each $n \in \mathbb{N}$, there is a functional $T_{g_n} \in \mathcal{G}^\infty(U)^*$ with $\|T_{g_n}\| = \|g_n\|_\infty$ such that $T_{g_n} \circ g_U = g_n$. Notice that $\sum_{n=1}^{+\infty} T_{g_n} \cdot y_n \in \mathcal{L}(\mathcal{G}^\infty(U),F)$ since

$$\sum_{k=1}^{m} \left\|T_{g_k} \cdot y_k\right\| = \sum_{k=1}^{m} \left\|T_{g_k}\right\| \left\|y_k\right\| = \sum_{k=1}^{m} \left\|g_k\right\|_\infty \left\|y_k\right\| \leq \left\|(g_n)\right\|_{p^*}^{\text{weak}} \left\|(y_n)\right\|_p,$$

for all $m \in \mathbb{N}$. We can write

$$f = \sum_{n=1}^{\infty} g_n \cdot y_n = \sum_{n=1}^{\infty} (T_{g_n} \circ g_U) \cdot y_n = \left(\sum_{n=1}^{\infty} T_{g_n} \cdot y_n\right) \circ g_U,$$

in $(\mathcal{H}^\infty(U,F), \|\cdot\|_\infty)$. Hence $T_f = \sum_{n=1}^{\infty} T_{g_n} \cdot y_n$ by Theorem 2.2, where $(T_{g_n}) \in \ell_{p^*}^{\text{weak}}(\mathcal{G}^\infty(U)^*)$ with $\left\|(T_{g_n})\right\|_{p^*}^{\text{weak}} \leq \left\|(g_n)\right\|_{p^*}^{\text{weak}}$. Therefore $T_f \in \mathcal{N}^p(\mathcal{G}^\infty(U),F)$ with $\nu^p(T_f) \leq \left\|(g_n)\right\|_{p^*}^{\text{weak}} \left\|(y_n)\right\|_p$. Taking infimum over all right p-nuclear holomorphic representation of f, we deduce that $\nu^p(T_f) \leq \nu^{p\mathcal{H}^\infty}(f)$.

$(ii) \Rightarrow (i)$: Suppose that $T_f \in \mathcal{N}^p(\mathcal{G}^\infty(U),F)$ and let $\sum_{n\geq 1} \phi_n \cdot y_n$ be a right p-nuclear representation of T_f. By Theorem 2.2, for each $n \in \mathbb{N}$, there is a $g_n \in \mathcal{H}^\infty(U)$ such that $J_U(g_n) = \phi_n$ with $\left\|g_n\right\|_\infty = \|\phi_n\|$. We have

$$\begin{aligned}\left\|\left(f - \sum_{k=1}^{n} g_k \cdot y_k\right)(x)\right\| &= \left\|f(x) - \sum_{k=1}^{n} g_k(x) y_k\right\| = \left\|T_f(g_U(x)) - \sum_{k=1}^{n} J_U(g_k)(g_U(x)) y_k\right\| \\ &= \left\|\left(T_f - \sum_{k=1}^{n} \phi_k \cdot y_k\right)(g_U(x))\right\| \leq \left\|T_f - \sum_{k=1}^{n} \phi_k \cdot y_k\right\| \left\|g_U(x)\right\| \\ &= \left\|T_f - \sum_{k=1}^{n} \phi_k \cdot y_k\right\|,\end{aligned}$$

for all $x \in U$ and $n \in \mathbb{N}$. Taking supremum over all $x \in U$, we obtain

$$\left\|f - \sum_{k=1}^{n} g_k \cdot y_k\right\|_\infty \leq \left\|T_f - \sum_{k=1}^{n} \phi_k \cdot y_k\right\|,$$

for all $n \in \mathbb{N}$. Hence $f = \sum_{n=1}^{\infty} g_n \cdot y_n$ in $(\mathcal{H}^\infty(U,F), \|\cdot\|_\infty)$, where $(g_n) \in \ell_{p^*}^{\text{weak}}(\mathcal{H}^\infty(U))$ with $\left\|(g_n)\right\|_{p^*}^{\text{weak}} \leq \left\|(\phi_n)\right\|_{p^*}^{\text{weak}}$. So $f \in \mathcal{H}^\infty_{\mathcal{N}^p}(U,F)$ with $\nu^{p\mathcal{H}^\infty}(f) \leq \left\|(\phi_n)\right\|_{p^*}^{\text{weak}} \left\|(y_n)\right\|_p$, and this implies that $\nu^{p\mathcal{H}^\infty}(f) \leq \nu^p(T_f)$.

The last assertion in the statement follows easily from what was proved above and from Theorem 2.2. □

Combining Theorem 2.14, firstly with [4, Theorem 3.2] and secondly with [7, Corollary 2.5], we derive the following two results.

Corollary 2.15. *Let $p \in [1, \infty)$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent:*

(i). *$f : U \to F$ is right p-nuclear.*
(ii). *$f = T \circ g$ for some complex Banach space G, $g \in \mathcal{H}^\infty(U, G)$ and $T \in \mathcal{N}^p(G, F)$.*

In this case, we have

$$\nu^{p\mathcal{H}^\infty}(f) = \left\|f\right\|_{\mathcal{N}^p \circ \mathcal{H}^\infty} := \inf\{\nu^p(T) \left\|g\right\|_\infty\},$$

where the infimum is taken over all factorizations of f as in (ii) and this infimum is attained at $T_f \circ g_U$.

As a consequence, the correspondence $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{N}^p \circ \mathcal{H}^\infty(U, F), \|\cdot\|_{\mathcal{N}^p \circ \mathcal{H}^\infty})$ onto $(\mathcal{N}^p(\mathcal{G}^\infty(U), F), \nu^p)$. □

Corollary 2.16. *For each $p \in [1, \infty)$, $(\mathcal{H}^\infty_{\mathcal{N}^p}, \nu^{p\mathcal{H}^\infty})$ is a Banach bounded-holomorphic ideal.* □

The following relation is readily obtained.

Corollary 2.17. *Let $p \in [1, \infty)$ and $f \in \mathcal{H}^\infty_{\mathcal{N}^p}(U, F)$. Then $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$ with $k_p^{\mathcal{H}^\infty}(f) \leq \nu^{p\mathcal{H}^\infty}(f)$.*

Proof. By Proposition 2.14, we have $T_f \in \mathcal{N}^p(\mathcal{G}^\infty(U), F)$ with $\nu^p(T_f) = \nu^{p\mathcal{H}^\infty}(f)$. It follows that $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ with $k_p(T_f) \leq \nu^p(T_f)$ (see [10, p. 295]). Hence $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$ with $k_p^{\mathcal{H}^\infty}(f) \leq \nu^{p\mathcal{H}^\infty}(f)$ by Theorem 2.3. □

Given Banach spaces E, F, G, let us recall that a normed operator ideal $\mathcal{I}$ is *surjective* if for every surjection $Q \in \mathcal{L}(G, E)$ and every $T \in \mathcal{L}(E, F)$, it follows from $T \circ Q \in \mathcal{I}(G, F)$ that $T \in \mathcal{I}(E, F)$ with $\|T\|_\mathcal{I} = \|T \circ Q\|_\mathcal{I}$. The smallest surjective ideal which contains $\mathcal{I}$, denoted by $\mathcal{I}^{\mathrm{sur}}$, is called the *surjective hull of $\mathcal{I}$*.

We now introduce the analogue concept in the holomorphic setting.

Definition 2.18. *The surjective hull of a bounded-holomorphic ideal $\mathcal{I}^{\mathcal{H}^\infty}$ is the smallest surjective ideal which contains $\mathcal{I}^{\mathcal{H}^\infty}$ and it is denoted by $(\mathcal{I}^{\mathcal{H}^\infty})^{\mathrm{sur}}$.*

We have seen that $\mathcal{H}^\infty_{\mathcal{K}_p}$ is a surjective bounded-holomorphic ideal which contains $\mathcal{H}^\infty_{\mathcal{N}^p}$, and therefore $(\mathcal{H}^\infty_{\mathcal{N}^p})^{\mathrm{sur}} \subseteq \mathcal{H}^\infty_{\mathcal{K}_p}$, but we do not know if both sets are equal as it happens in the linear case (see [10, Proposition 3.11]) .

References

[1] D. Achour, E. Dahia and P. Turco, *Lipschitz p-compact mappings*, Monatsh. Math. **189** (2019), no. 4, 595–609.
[2] D. Achour, E. Dahia, P. Turco, *The Lipschitz injective hull of Lipschitz operator ideals and applications*, Banach J. Math. Anal. **14** (2020), no. 3, 1241–1257.
[3] K. Ain, R. Lillemets, E. Oja, *Compact operators which are defined by ℓ_p-spaces*, Quaest. Math. **35** (2012), no. 2, 145–159.
[4] R. Aron, G. Botelho, D. Pellegrino, P. Rueda, *Holomorphic mappings associated to composition ideals of polynomials*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **21** (2010), no. 3, 261–274.
[5] R. Aron, E. Çalişkan, D. García, M. Maestre, *Behavior of holomorphic mappings on p-compact sets in a Banach space*, Trans. Amer. Math. Soc. **368** (2016), no. 7, 4855–4871.
[6] R. Aron, M. Maestre, P. Rueda, *p-Compact holomorphic mappings*, RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. **104** (2) (2010), 353–364.
[7] M. G. Cabrera-Padilla, A. Jiménez-Vargas, D. Ruiz-Casternado, *On composition ideals and dual ideals of bounded holomorphic mappings*, Results Math. (2023), 78:103 https://doi.org/10.1007/s00025-023-01868-9
[8] J. A. Chávez-Domínguez, V. Dimant, D. Galicer, *Operator p-compact mappings*, J. Funct. Anal. **277** (2019), no. 8, 2865–2891.
[9] Y. S. Choi, J. M. Kim, *The dual space of $(\mathcal{L}(X, Y), \tau_p)$ and the p-approximation property*, J. Funct. Anal. **259** (2010), no. 9, 2437–2454.
[10] J. M. Delgado, C. Piñeiro, E. Serrano, *Operators whose adjoints are quasi p-nuclear*, Stud. Math. **197** (2010), no. 3, 291–304.
[11] D. Galicer, S. Lassalle, P. Turco, *The ideal of p-compact operators: a tensor product approach*, Stud. Math. **211** (2012), no. 3, 269–286.
[12] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **1955** (1955), no. 16, 140 pp. (French).
[13] A. Jiménez-Vargas, D. Ruiz-Casternado, J. M. Sepulcre, *On holomorphic mappings with compact type range*, Bull. Malays. Math. Sci. Soc. **46** (2023), no. 1, Paper No. 20, 16 pp.
[14] S. Lassalle, P. Turco, *On p-compact mappings and the p-approximation properties*, J. Math. Anal. Appl. **389** (2012), no. 2, 1204–1221.
[15] J. Mujica, *Linearization of bounded holomorphic mappings on Banach spaces*, Trans. Amer. Math. Soc. **324** (1991), no. 2, 867–887.
[16] A. Persson, *On some properties of p-nuclear and p-integral operators*, Studia Math. **33** (1969), 213–222.

[17] A. Persson, A. Pietsch, *p-nuklear und p-integrale Abbildungen in Banachräumen*, Stud. Math. **33** (1969), 19–62.
[18] A. Pietsch, *Operator ideals*, North-Holland Mathematical Library, vol. 20, North-Holland Publishing Co., Amsterdam-New York, 1980. Translated from German by the author.
[19] A. Pietsch, *The ideal of p-compact operators and its maximal hull*, Proc. Am. Math. Soc. **142** (2), (2014), 519–530.
[20] D. P. Sinha, A. K. Karn, *Compact operators whose adjoints factor through subspaces of ℓ_p*, Studia Math. **150** (1) (2002), 17–33.