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Bounded version of approximate module character amenability of Banach algebras

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Abstract. The bounded version of approximate module character amenability of Banach algebras is introduced and studied. This new concept is characterized by several different concepts such as bounded approximate module character means. Moreover, this new concept is investigated for second dual, unitization, tensor product and l^p-direct sums of Banach algebras.

1. Introduction and preliminaries

Througout this paper, A and $\mathfrak{A}$ are Banach algebras. For a Banach A-bimodule X, a *derivation* is a bounded linear map $D : A \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A).$$

For each $x \in X$, the derivation $D_x : A \to X$ given by $D_x(a) = a \cdot x - x \cdot a$ is called an *inner derivation*. A derivation $D : A \to X$ is called *approximately inner*, if there exists a net $(x_i) \subset X$ such that

$$D(a) = \lim_i D_{x_i}(a) \qquad (a \in A),$$

if also there is $L > 0$ such that

$$\sup \left\| D_{x_i}(a) \right\| \le L\|a\| \qquad (a \in A),$$

then D is called *boundedly approximately inner*.

Let $\phi \in \sigma(A)$ be a character on A, and let $\mathcal{M}_\phi^A$ [resp. ${}_\phi\mathcal{M}^A$] denotes the class of Banach A-bimodules X such that $x \cdot a = \phi(a)x$ [resp. $a \cdot x = \phi(a)x$] for all $a \in A$ and $x \in X$, [10]. Obviously, $X \in {}_\phi\mathcal{M}^A$ iff $X^* \in \mathcal{M}_\phi^A$, where X^* denotes the dual space of X.

Definition 1.1. *Let A be a Banach algebra and $\phi \in \sigma(A)$. Then*

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(i) *A is called* (*approximately*) (*boundedly approximately*) *amenable if for each A-bimodule X, every derivation* $D : A \to X^*$ *is (approximately) (boundedly approximately) inner.*

(ii) *A is called right* [*left*] (*approximately*) (*boundedly approximately*) ϕ*-amenable if for each* $X \in {}_\phi\mathcal{M}^A$ *[resp.* $\mathcal{M}_\phi^A$*], every derivation* $D : A \to X^*$ *is (approximately) (boundedly approximately) inner.*

(iii) *A is called right* [*left*] (*approximately*) (*boundedly approximately*) *character amenable if it is right [left] (approximately) (boundedly approximately)* ϕ*-amenable for each* $\phi \in \sigma(A)$*.*

(iv) *A is called* (*approximately*) (*boundedly approximately*) *character amenable if it is both left and right (approximately) (boundedly approximately) character amenable.*

Throughout this paper, A is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b \quad , \quad (ab) \cdot \alpha = a(b \cdot \alpha) \qquad (a, b \in A \ , \ \alpha \in \mathfrak{A}).$$

Let X be a Banach A-bimodule and Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x \quad , \quad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x \quad , \quad (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \qquad (a \in A \ , \ \alpha \in \mathfrak{A} \ , \ x \in X),$$

and similarly for the right and two-sided actions, in this case we say that X is a Banach A-$\mathfrak{A}$*-module*. If moreover, $\alpha \cdot x = x \cdot \alpha$ for all $\alpha \in \mathfrak{A}$ and $x \in X$, then X is called a *commutative* A-$\mathfrak{A}$*-module*.

A bounded map $D : A \to X$ is called an $\mathfrak{A}$*-module derivation* if it is $\mathfrak{A}$-bimodule homomorphism and

$$D(a \pm b) = D(a) \pm D(b) \quad , \quad D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in A).$$

The boundedness of D means that there is $L > 0$ such that $\|D(a)\| \leq L\|a\|$, for all $a \in A$. When X is a commutative A-$\mathfrak{A}$-module, then for each $x \in X$ the map $D_x : A \to X$ given by $D_x(a) = a \cdot x - x \cdot a$ is called *inner* $\mathfrak{A}$*-module derivation* [1].

Definition 1.2. *The Banach algebra A is called* (*approximately*) $\mathfrak{A}$*-module* (*or module*) *amenable if for any commutative Banach* A-$\mathfrak{A}$*-module X, each* $\mathfrak{A}$*-module derivation* $D : A \to X^*$ *is (approximately) inner [1, 14].*

Ghahramani and Loy generalized the theory of classical amenable Banach algebras in [6, 7], introduced by Johnson in 1972 [11], to approximate amenability. The concepts of ϕ-amenable and character amenable Banach algebras were introduced by Kaniuth, Lau and Pym in [12] and by Monfared et.al. in [10, 13]. Pourmahmood, Shi and Wu introduced the concept of approximate character amenability and characterized this notion in several ways [16]. On the other hand, Amini [1] introduced the notion of module amenability for a class of Banach algebras which could be considered as a generalization of the Johnson's amenability. He showed that for an inverse semigroup S with the set of idempotents E, the semigroup algebra $l^1(S)$ is module amenable, as a Banach module over $l^1(E)$, if and only if S is amenable. After that, Bodaghi and Amini [2, 4] introduced the concept of module (ϕ, φ)-amenability for Banach algebras and investigated a module character amenable Banach algebra. They showed that such Banach algebras posses module character virtual (approximate) diagonals. On the other hand, Bodaghi in [3] studied the module amenability of the projective module tensor product. In [14], Pourmahmood and Bodaghi introduced the concept of module approximate amenability (and contractibility) for Banach algebras. Finally, the concept of (approximate) module character amenability was introduced by Bodaghi et.al. in [5]. The bounded versions of above concepts were introduced by several authors. For instance, Ghahramani and Read introduced the class of boundedly approximately amenable Banach algebras [8]. In addition, the bounded versions of approximate character amenability and approximate module amenability were studied by authors in [9, 15].

In this paper, we introduce the bounded version of approximate module character amenability and some of its characterizations and heredity properties. In addition, we have some results for second dual, unitization, tensor products and l^p-direct sums of Banach algebras. The bounded version of approximate module amenability can be one of the consequences of this paper.

2. Bounded approximate module character amenability

Throughout this paper A and $\mathfrak{A}$ are Banach algebras and A is Banach $\mathfrak{A}$-bimodule with compatible actions and $\varphi \in \sigma(\mathfrak{A})$ is a character on $\mathfrak{A}$. Consider the multiplicative linear map $\phi : A \to \mathfrak{A}$ such that

$$\phi(a \cdot \alpha) = \phi(\alpha \cdot a) = \varphi(\alpha)\phi(a) \quad (a \in A \ , \ \alpha \in \mathfrak{A}),$$

we denote the set of all such maps by Ω_A or $\Omega_{\mathfrak{A}}(A)$.

Definition 2.1. *[5] Let $\varphi \in \sigma(\mathfrak{A})$ and $\phi \in \Omega_A$. We say that the Banach space X is a $\big((\phi,\varphi), A\text{-}\mathfrak{A}\big)$-module or $X \in {}_{(\phi,\varphi)}\mathcal{M}^{A,\mathfrak{A}}$, if left module action of A on X is given by*

$$a \cdot x = \phi(a) \cdot x \quad (a \in A \ , \ x \in X),$$

and the actions of $\mathfrak{A}$ on X is given by

$$\alpha \cdot x = x \cdot \alpha = \varphi(\alpha)x \quad (\alpha \in \mathfrak{A} \ , \ x \in X).$$

Note that in this case we can write $a \cdot x = \phi(a) \cdot x = \varphi \circ \phi(a)x$, for all $a \in A$ and $x \in X$. Similarly, we say that X is $\big(A\text{-}\mathfrak{A}, (\phi,\varphi)\big)$-module or $X \in \mathcal{M}^{A,\mathfrak{A}}_{(\phi,\varphi)}$, if right module action of A on X is given by

$$x \cdot a = \phi(a) \cdot x \quad (a \in A \ , \ x \in X),$$

and the actions of $\mathfrak{A}$ on X is given by

$$\alpha \cdot x = x \cdot \alpha = \varphi(\alpha)x \quad (\alpha \in \mathfrak{A} \ , \ x \in X).$$

The authors in [5] defined the concept (approximate) module character amenability for A, and now we introduce the bounded version of this concept. Then we present some characterizations for this concept.

Definition 2.2. *Let A be a Banach $\mathfrak{A}$-bimodule, $\phi \in \Omega_A$ and $\varphi \in \sigma(\mathfrak{A})$. Then*

(i) A is called right (boundedly) approximately module (ϕ,φ)-amenable, if every $\mathfrak{A}$-module derivation $D : A \to X^*$ is (boundedly) approximately inner, for all $X \in {}_{(\phi,\varphi)}\mathcal{M}^{A,\mathfrak{A}}$. There is a similar definition for left (boundedly) approximately module (ϕ,φ)-amenable Banach $\mathfrak{A}$-bimodule.

(ii) A is called (boundedly) approximately module (ϕ,φ)-amenable, if it is left and right (boundedly) approximately module (ϕ,φ)-amenable.

(iii) A is called (boundedly) approximately module character amenable, if it is (boundedly) approximately module (ϕ,φ)-amenable for all $\phi \in \Omega_A$ and all $\varphi \in \sigma(\mathfrak{A})$.

Notation. We will use the abbreviated symbol ($b \cdot app \cdot m \cdot (\phi,\varphi)$-am.) for bounded approximate module (ϕ,φ)-amenability, and ($b \cdot app \cdot m \cdot char \cdot am$) for bounded approximate module character amenability.

We remind that, if $\mathfrak{A} = \mathbb{C}$ and φ is the identity map, then all of the above definitions coincide with their classical case.

Proposition 2.3. *If A has a right [left] multiplier bounded approximate identity, then A is right [left] $b \cdot app \cdot m \cdot (0,\varphi)$-am. for all $\varphi \in \sigma(\mathfrak{A})$.*

Proof. If (e_i) is a right multiplier bounded approximate identity for A, then there exists a $K > 0$ such that for all $a \in A$, $ae_i \to a$ and $\|ae_i\| \leq K\|a\|$. Let $X \in {}_{(0,\varphi)}\mathcal{M}^{A,\mathfrak{A}}$ and $D : A \to X^*$ be an $\mathfrak{A}$-module derivation, so for $a, b \in A$, $\alpha \in \mathfrak{A}$ and $f \in X^*$

$$a \cdot x = 0 \ \to \ f \cdot a = 0 \ , \ \alpha \cdot x = x \cdot \alpha = \varphi(\alpha)x.$$

Therefore $D(ab) = a \cdot D(b)$ and we have

$$\begin{aligned} D(a) = D(\lim_i a \cdot e_i) &= \lim_i a \cdot D(e_i) \\ &= \lim_i \left[a \cdot D(e_i) - D(e_i) \cdot a\right] \\ &= \lim_i D_{D(e_i)}(a), \end{aligned}$$

also

$$\|D_{D(e_i)}(a)\| = \|D(a \cdot e_i)\| \leq \|D\|\|a \cdot e_i\| \leq \|D\|K\|a\|.$$

This shows that D is boundedly approximately inner, thus A is right $b \cdot app \cdot m \cdot (0, \varphi)$-am. □

Definition 2.4. *A net $(m_i) \subset A^{**}$ is called a right bounded approximate module (ϕ, φ)-mean. ($b \cdot app \cdot m \cdot (\phi, \varphi)$-mean) if $m_i(\varphi \circ \phi) = 1$ and*

$$\begin{aligned} a \cdot m_i - \phi(a) \cdot m_i &\to 0 \quad (a \in A), \\ \alpha \cdot m_i - \varphi(\alpha) m_i &\to 0 \quad (\alpha \in \mathfrak{A}), \end{aligned}$$

and also there exist $L, L^{'} > 0$ such that

$$\begin{aligned} \|a \cdot m_i - \phi(a) \cdot m_i\| &\leq L\|a\| \quad (a \in A), \\ \|\alpha \cdot m_i - \varphi(\alpha) m_i\| &\leq L^{'}\|\alpha\| \quad (\alpha \in \mathfrak{A}). \end{aligned}$$

We have a similar definition for left $b \cdot app \cdot m \cdot (\phi, \varphi)$-mean.

Proposition 2.5. *The following statements are equivalent:*

(i) *A is right $b \cdot app \cdot m \cdot (\phi, \varphi)$-am;*

(ii) *Every $\mathfrak{A}$-module derivation $D : A \to K^{**}$ is boundedly approximately inner, in which $K = \ker(\varphi \circ \phi)$, and the right A-module action on $K^{**} = \ker(\varphi \circ \phi)^{**}$ is given by $m \cdot a = \phi(a) \cdot m$, for $a \in A$ and $m \in K^{**}$, and with the natural left A-module action on K^{**}. Also, with $\mathfrak{A}$-module actions on K^{**} given by $\alpha \cdot m = m \cdot \alpha = \varphi(\alpha) m$, for $\alpha \in \mathfrak{A}$ and $m \in K^{**}$;*

(iii) *There exists a right $b \cdot app \cdot m \cdot (\phi, \varphi)$-mean;*

(iv) *There exists a net $(m_i) \subset A^{**}$ such that $m_i(\varphi \circ \phi) \to 1$, and*

$$\begin{aligned} a \cdot m_i - \phi(a) \cdot m_i &\to 0 \quad (a \in A), \\ \alpha \cdot m_i - \varphi(\alpha) m_i &\to 0 \quad (\alpha \in \mathfrak{A}), \end{aligned}$$

and there are $L, L^{'} > 0$ such that

$$\begin{aligned} \|a \cdot m_i - \phi(a) \cdot m_i\| &\leq L\|a\| \quad (a \in A), \\ \|\alpha \cdot m_i - \varphi(\alpha) m_i\| &\leq L^{'}\|\alpha\| \quad (\alpha \in \mathfrak{A}). \end{aligned}$$

Proof. (i) ⇒ (ii) and (iii) ⇒ (iv) are obvious.

(ii) ⇒ (iii) Take $b \in A$ with $\varphi \circ \phi(b) = 1$, and define the $\mathfrak{A}$-module derivation $D : A \to K^{**}$ by $D(a) = a\hat{b} - \hat{b}a$, where $\hat{b}$ is the canonical image of b in A^{**}, and with following actions on K^{**}

$$m \cdot a = \phi(a) \cdot m \ , \ \alpha \cdot m = m \cdot \alpha = \varphi(\alpha) m \quad (a \in A \ , \ \alpha \in \mathfrak{A} \ , \ m \in K^{**}).$$

So we have $D(a) = a\hat{b} - \hat{b}a = a\hat{b} - \phi(a) \cdot \hat{b}$. By hypothesis, there is a net $(n_i) \subset K^{**}$ and $L > 0$ such that for all $a \in A$

$$\begin{aligned} a\hat{b} - \phi(a) \cdot \hat{b} &= \lim_i \left(a \cdot n_i - \phi(a) \cdot n_i\right), \\ \|a \cdot n_i &- \phi(a) \cdot n_i\| \leq L\|a\|. \end{aligned}$$

Put $m_i = \hat{b} - n_i$. Then $m_i(\varphi \circ \phi) = 1$ and for all $a \in A$

$$\|a \cdot m_i - \phi(a) \cdot m_i\| = \|a\hat{b} - a \cdot n_i - \phi(a) \cdot \hat{b} + \phi(a) \cdot n_i\| \to 0$$

and

$$\begin{aligned} \|a \cdot m_i - \phi(a) \cdot m_i\| &\leq \|D(a)\| + \|D_{n_i}(a)\| \\ &\leq 2\|b\|\|a\| + L\|a\| \\ &= (2\|b\| + L)\|a\|. \end{aligned}$$

Also for all $\alpha \in \mathfrak{A}$, $\alpha \cdot m_i = m_i \cdot \alpha = \varphi(\alpha) m_i$. This shows that (m_i) is a right $b \cdot app \cdot m \cdot (\phi, \varphi)$-mean.

(iv) ⇒ (i) Consider the net $(m_i) \subset A^{**}$ satisfies in (iv). Therefore, for all $a \in A$ we have

$$\begin{aligned} \|a \cdot m_i - \varphi \circ \phi(a) m_i\| &= \|a \cdot m_i - \phi(a) \cdot m_i\| + \|\phi(a) \cdot m_i - \varphi \circ \phi(a) m_i\| \\ &\leq L\|a\| + L'\|\phi(a)\| \\ &\leq (L + L'\|\phi\|)\|a\|. \end{aligned}$$

In addition, we conclude that $\|a \cdot m_i - \varphi \circ \phi(a) m_i\| \to 0$. Now, suppose that $X \in {}_{(\phi,\varphi)}\mathcal{M}^{A,\mathfrak{A}}$, $D : A \to X^*$ is an $\mathfrak{A}$-module derivation and there is $M > 0$ such that for all $a \in A$, $\|D(a)\| \leq M\|a\|$. We can write

$$a \cdot x = \phi(a) \cdot x = \varphi \circ \phi(a) x \qquad (x \in X \ , \ a \in A).$$

Since each $f \in X^*$ is a linear map, then for all $a \in A$ we have $f \cdot a = \varphi \circ \phi(a) f$. Put $D' = D^*|_X : X \to A^*$ and $g_i =: (D')^*(m_i)$, so $g_i \in X^*$ and for all $a, b, x \in A$

$$\begin{aligned} \langle D'(xa), b\rangle &= \langle D^*(\widehat{xa}), b\rangle \\ &= \langle D(b), xa\rangle \\ &= \langle a \cdot D(b), x\rangle \\ &= \langle D(ab) - D(a) \cdot b, x\rangle \\ &= \langle D(ab) - \varphi \circ \phi(b) D(a), x\rangle \\ &= \langle D(ab), x\rangle - \varphi \circ \phi(b)\langle D(a), x\rangle \\ &= \langle D^*(\hat{x}), ab\rangle - \varphi \circ \phi(b)\langle D(a), x\rangle \\ &= \langle D'(x) \cdot a, b\rangle - \varphi \circ \phi(b)\langle D(a), x\rangle, \end{aligned}$$

we conclude that

$$D'(xa) = D'(x) \cdot a - \langle D(a), x\rangle \varphi \circ \phi.$$

Hence

$$\begin{aligned} \langle a \cdot g_i, x\rangle &= \langle g_i, x \cdot a\rangle \\ &= \langle (D')^*(m_i), x \cdot a\rangle \\ &= \langle m_i, D'(xa)\rangle \\ &= \langle m_i, D'(x) \cdot a\rangle - \langle D(a), x\rangle\langle m_i, \varphi \circ \phi\rangle \\ &= \langle a \cdot m_i, D'(x)\rangle - \langle D(a), x\rangle\langle m_i, \varphi \circ \phi\rangle, \end{aligned}$$

and

$$\begin{aligned} & |\langle a \cdot g_i, x\rangle - \varphi \circ \phi(a)\langle g_i, x\rangle + \langle D(a), x\rangle| \\ = \ & |\langle a \cdot m_i, D'(x)\rangle - \langle D(a), x\rangle\langle m_i, \varphi \circ \phi\rangle - \varphi \circ \phi(a)\langle g_i, x\rangle + \langle D(a), x\rangle| \\ \leq \ & |\langle a \cdot m_i, D'(x)\rangle - \varphi \circ \phi(a)\langle g_i, x\rangle| + |\langle D(a), x\rangle||\langle m_i, \varphi \circ \phi\rangle - 1| \\ = \ & |\langle a \cdot m_i, D'(x)\rangle - \varphi \circ \phi(a)\langle m_i, D'(x)\rangle| + |\langle D(a), x\rangle||\langle m_i, \varphi \circ \phi\rangle - 1| \\ = \ & |\langle a \cdot m_i - \varphi \circ \phi(a) m_i, D'(x)\rangle| + |\langle D(a), x\rangle||\langle m_i, \varphi \circ \phi\rangle - 1| \\ \leq \ & \|a \cdot m_i - \varphi \circ \phi(a) m_i\|\|D'(x)\| + \|D(a)\|\|x\|\|m_i(\varphi \circ \phi) - 1\|, \end{aligned}$$

we conclude that $\|a \cdot g_i - \varphi \circ \phi(a)g_i + D(a)\| \to 0$ or $D(a) = \lim_i D_{-g_i}(a)$. This shows that D is approximately inner. On the other hand, since $m_i(\varphi \circ \phi) \to 1$, then we can assume that $|m_i(\varphi \circ \phi)| \leq 1$ and we have

$$\begin{aligned}
|\langle D_{-g_i}(a), x\rangle| &= |\langle a \cdot g_i, x\rangle - \varphi \circ \phi(a)\langle g_i, x\rangle| \\
&= |\langle a \cdot m_i, D^{'}(x)\rangle - \langle D(a), x\rangle\langle m_i, \varphi \circ \phi\rangle - \varphi \circ \phi\langle m_i, D^{'}(x)\rangle| \\
&\leq |\langle a \cdot m_i - \varphi \circ \phi(a)m_i, D^{'}(x)\rangle| - |\langle D(a), x\rangle||\langle m_i, \varphi \circ \phi\rangle| \\
&\leq \|a \cdot m_i - \varphi \circ \phi(a)m_i\|\|D^{'}(x)\| + \|D(a)\|\|x\| \\
&\leq (L + L^{'}\|\phi\|)\|a\|M\|x\| + M\|a\|\|x\| \\
&= \Big[(L + L^{'}\|\phi\|)M + M\Big]\|a\|\|x\|,
\end{aligned}$$

we conclude that

$$\|D_{-g_i}(a)\| \leq \Big[(L + L^{'}\|\phi\|)M + M\Big]\|a\|,$$

and this shows that D is boundedly approximately inner. Hence, A is right $b \cdot app \cdot m \cdot (\phi, \varphi)$-am. □

Proposition 2.6. *Let A and B be $\mathfrak{A}$-bimodules, and $\theta : A \to B$ be [norm-preserving] continuous $\mathfrak{A}$-module epimorphism. Then right [left] [bounded] $app \cdot m \cdot (\phi \circ \theta, \varphi)$-am. of A implies right [left] [bounded] $app \cdot m \cdot (\phi, \varphi)$-am. of B.*

Proof. Let A be right $app \cdot m \cdot (\phi \circ \theta, \varphi)$-am, then by Proposition 2.5 there exist a net $(m_i) \subset A^{**}$ and $L, L^{'} > 0$ such that $m_i(\varphi \circ \phi \circ \theta) = 1$, and for all $a \in A$ and $\alpha \in \mathfrak{A}$

$$\begin{aligned}
\|a \cdot m_i - \phi \circ \theta(a) \cdot m_i\| \to 0 \quad &, \quad \|a \cdot m_i - \phi \circ \theta(a) \cdot m_i\| \leq L\|a\|, \\
\|\alpha \cdot m_i - \varphi(\alpha)m_i\| \to 0 \quad &, \quad \|\alpha \cdot m_i - \varphi(\alpha)m_i\| \leq L^{'}\|\alpha\|.
\end{aligned}$$

Set $n_i = \theta^{**}(m_i) \in B^{**}$, we have $\langle n_i, g\rangle = \langle m_i, g \circ \theta\rangle$ for $g \in B^*$, so $\langle n_i, \varphi \circ \phi\rangle = \langle m_i, \varphi \circ \phi \circ \theta\rangle = 1$. Since θ is surjective, then for each $b \in B$ there is $a \in A$ such that $\theta(a) = b$, and for $g \in B^*$ we have

$$\begin{aligned}
\langle b \cdot n_i - \phi(b) \cdot n_i, g\rangle &= \langle n_i, g \cdot \theta(a) - \phi\big(\theta(a)\big) \cdot g\rangle \\
&= \langle m_i, \big(g \cdot \theta(a)\big) \circ \theta - \phi \circ \theta(a) \cdot g \circ \theta\rangle \\
&= \langle m_i, (g \circ \theta) \cdot a - \phi \circ \theta(a) \cdot g \circ \theta\rangle \\
&= \langle a \cdot m_i - \phi \circ \theta(a) \cdot m_i, g \circ \theta\rangle \\
&\to 0,
\end{aligned}$$

and for all $\alpha \in \mathfrak{A}$ and $g \in B^*$

$$\begin{aligned}
\langle \alpha \cdot n_j - \varphi(\alpha)n_j, g\rangle &= \langle n_j, g \cdot \alpha - \varphi(\alpha)g\rangle \\
&= \langle m_j, (g \cdot \alpha) \circ \theta - \varphi(\alpha)g \circ \theta\rangle \\
&= \langle m_i, (g \circ \theta) \cdot \alpha - \varphi(\alpha)g \circ \theta\rangle \\
&= \langle \alpha \cdot m_i - \varphi(\alpha)m_i, g \circ \theta\rangle \\
&\to 0.
\end{aligned}$$

Hence the net $(n_i) \subset B^{**}$ is a right $app \cdot m \cdot (\phi, \varphi)$-mean for B and B is right $app \cdot m \cdot (\phi, \varphi)$-am. by Proposition 2.5. Now assume that θ is norm preserving and (m_i) is a right $b \cdot app \cdot m \cdot (\phi \circ \theta, \varphi)$-mean for A, then $\|b\| = \|a\|$ and there exist $L, L^{'} > 0$ such that for all $\alpha \in \mathfrak{A}$ and $a \in A$

$$\begin{aligned}
\|a \cdot m_i - \phi \circ \theta(a) \cdot m_i\| &\leq L\|a\|, \\
\|\alpha \cdot m_i - \varphi(\alpha)m_i\| &\leq L^{'}\|\alpha\|.
\end{aligned}$$

Therefore, for all $\alpha \in \mathfrak{A}$, $b = \theta(a) \in B$ and $g \in B^*$

$$\begin{aligned} |\langle b \cdot n_i - \phi(b) \cdot n_i, g\rangle| &= |\langle a \cdot m_i - \phi \circ \theta(a) \cdot m_i, g \circ \theta\rangle| \\ &\leq \|a \cdot m_i - \phi \circ \theta(a) \cdot m_i\| \|g \circ \theta\| \\ &\leq L\|a\|\|g\|\|\theta\| \\ &= L\|b\|\|g\|\|\theta\|, \end{aligned}$$

and we concluded that

$$\|b \cdot n_i - \phi(b) n_i\| \leq (L\|\theta\|)\|b\|,$$

similarly, we conclude that

$$\begin{aligned} \|\alpha \cdot n_i - \varphi(\alpha) n_i\| &\leq \|\alpha \cdot m_i - \varphi(\alpha) m_i\| \|\theta\| \\ &\leq L^{'}\|\alpha\|\|\theta\| \\ &= (L^{'}\|\theta\|)\|\alpha\|, \end{aligned}$$

this shows that (n_i) is a right $b \cdot app \cdot m \cdot (\phi, \varphi)$-mean for B. □

3. The second dual of Banach algebras

In this section, we assume that A^{**}, the second dual of A is equipped with the first Arens product, and we denote it by □. The canonical image of $a \in A$ in A^{**} is denoted by $\hat{a}$, and $\hat{A} = \{\hat{a} \ : \ a \in A\}$. Let $F = w^* - \lim_i \hat{a}_i$ and $G = w^* - \lim_j \hat{b}_j$ are members of A^{**} and $\Lambda = w^* - \lim_k \hat{\alpha}_k \in \mathfrak{A}^{**}$, where (a_i) and (b_j) are nets in A and (α_k) is a net in $\mathfrak{A}$. We consider the module $\mathfrak{A}^{**}$ actions on A^{**} by

$$\Lambda \cdot F = w^* - \lim_k w^* - \lim_i (\alpha_k \cdot a_i)\hat{} \quad , \quad F \cdot \Lambda = w^* - \lim_i w^* - \lim_k (a_i \cdot \alpha_k)\hat{},$$

and also for the first Arens product □ on A^{**} we have

$$F \square G = w^* - \lim_i w^* - \lim_j (a_i b_j)\hat{}.$$

Let $\varphi \in \sigma(\mathfrak{A})$ and $\phi \in \Omega_A$. If φ^{**} and ϕ^{**} are the double conjugates of φ and ϕ, respectively, then $\varphi^{**} \in \sigma(\mathfrak{A}^{**})$ and $\phi^{**} \in \Omega_{A^{**}}$.

Proposition 3.1. *Let A^{**} be right [left] $b \cdot app \cdot m \cdot (\phi^{**}, \varphi^{**})$-am., as an $\mathfrak{A}^{**}$-bimodule, then A is right [left] $b \cdot app \cdot m \cdot (\phi, \varphi)$-am. as an $\mathfrak{A}$-bimodule.*

Proof. By Proposition 2.5, there is a right $b \cdot app \cdot m \cdot (\phi^{**}, \varphi^{**})$-mean $(m_i) \subset A^{****}$, satisfying $\varphi^{**} \circ \phi^{**}(m_i) = 1$, $\|F \cdot m_i - \phi^{**}(F) \cdot m_i\| \to 0$, $\|\mathcal{F} \cdot m_i - \varphi^{**}(\mathcal{F}) m_i\| \to 0$, for all $F \in A^{**}$ and $\mathcal{F} \in \mathfrak{A}^{**}$, and also for $L, L^{'} > 0$ we have

$$\|F \cdot m_i - \phi^{**}(F) \cdot m_i\| < L\|F\| \quad , \quad \|\mathcal{F} \cdot m_i - \varphi^{**}(\mathcal{F}) m_i\| < L^{'}\|\mathcal{F}\|.$$

Now, we define $m_i^{'} \in A^{**}$ by $m_i^{'}(f) =: m_i|_{A^*}(\hat{f})$, and we have

$$m_i^{'}(\varphi \circ \phi) = m_i|_{A^*}(\varphi \circ \phi)^\wedge = m_i(\varphi^{**} \circ \phi^{**}) = 1.$$

Moreover, for $F = \hat{a} \in A^{**}$, $\mathcal{F} = \hat{\alpha} \in \mathfrak{A}^{**}$ and $f \in A^*$

$$\begin{aligned} |\langle a \cdot m_i^{'} - \phi(a) \cdot m_i^{'}, f\rangle| &= |\langle \hat{a} \cdot m_i - \phi^{**}(\hat{a}) \cdot m_i, \hat{f}\rangle| \\ &\leq \|\hat{a} \cdot m_i - \phi^{**}(\hat{a}) \cdot m_i\| \|\hat{f}\| \\ &\to 0, \end{aligned}$$

Therefore

$$\|a \cdot m_i^{'} - \phi(a) \cdot m_i^{'}\| \to 0,$$
$$\|a \cdot m_i^{'} - \phi(a) \cdot m_i^{'}\| \leq L\|\hat{a}\| \leq L\|a\|,$$

and also

$$\begin{aligned} |\langle \alpha \cdot m_i^{'} - \varphi(\alpha)m_i^{'}, f\rangle| &= |\langle \hat{\alpha} \cdot m_i - \varphi^{**}(\hat{\alpha})m_i, \hat{f}\rangle| \\ &\leq \|\hat{\alpha} \cdot m_i - \varphi^{**}(\hat{\alpha})m_i\|\|\hat{f}\| \\ &\to 0, \end{aligned}$$

hence

$$\|\alpha \cdot m_i^{'} - \varphi(\alpha)m_i^{'}\| \to 0,$$
$$\|\alpha \cdot m_i^{'} - \varphi(\alpha)m_i^{'}\| \leq L^{'}\|\hat{\alpha}\| \leq L^{'}\|\alpha\|.$$

The above relations show that $(m_i^{'}) \subset A^{**}$ is a right $b\cdot app\cdot m\cdot(\phi,\varphi)$-mean. Hence A is right $b\cdot app\cdot m\cdot(\phi,\varphi)$-am. by Proposition 2.5. □

4. Unitizations of Banach algebras

In this section, $A^{\#} = A \oplus \mathbb{C}$ and $\mathfrak{A}^{\#} = \mathfrak{A} \oplus \mathbb{C}$ are unitizations of A and $\mathfrak{A}$, respectively. According to notations in [5], let $B = A \oplus \mathfrak{A}^{\#}$ with following multiplication

$$(a,u)(b,v) =: (ab + a \cdot v + u \cdot b, uv) \quad (a,b \in A \ , \ u,v \in \mathfrak{A}^{\#}),$$

in which $\mathfrak{A}^{\#}$-module actions on A defined by

$$a \cdot (\alpha,\lambda) =: a \cdot \alpha + \lambda a \ , \ (\alpha,\lambda) \cdot a =: \alpha \cdot a + \lambda a \quad (a \in A \ , \ (\alpha,\lambda) \in \mathfrak{A}^{\#}).$$

Moreover, we can define $\mathfrak{A}^{\#}$-module actions on B by

$$u \cdot (a,v) =: (u \cdot a, uv) \ , \ (a,v) \cdot u =: (a \cdot u, vu) \quad (a \in A \ ; \ u,v \in \mathfrak{A}^{\#}).$$

Then, B is a unital Banach algebra and Banach $\mathfrak{A}^{\#}$-bimodule with compatible actions and with identity $e_B = (0, e_{\mathfrak{A}^{\#}})$, where $e_{\mathfrak{A}^{\#}} = (0,1)$ is the identity of $\mathfrak{A}^{\#}$.

Now, suppose that $\phi \in \Omega_{\mathfrak{A}}(A)$ and $\varphi \in \sigma(\mathfrak{A})$. We can define the extensions of ϕ and φ by

$$\begin{aligned} &\tilde{\varphi} : \mathfrak{A}^{\#} \to \mathbb{C} \quad , &&\tilde{\varphi}(\alpha,\lambda) =: \varphi(\alpha) + \lambda. \\ &\phi_e : A \to \mathfrak{A}^{\#} \quad , &&\phi_e(a) =: \big(\phi(a), 0\big). \\ &\tilde{\phi} : B = A \oplus \mathfrak{A}^{\#} \to \mathfrak{A}^{\#} \quad , &&\tilde{\phi}(a,u) = \big(\phi(a), \tilde{\varphi}(u)\big). \end{aligned}$$

It is easy to check that $\phi_e \in \Omega_{\mathfrak{A}^{\#}}(A)$, $\tilde{\varphi} \in \sigma(\mathfrak{A}^{\#})$, $\tilde{\phi} \in \Omega_{\mathfrak{A}^{\#}}(B)$ and for $\alpha \in \mathfrak{A}$, $a \in A$ and $u \in \mathfrak{A}^{\#}$ we have

$$\begin{aligned} \tilde{\varphi}(\alpha, 0) &= \varphi(\alpha), \\ \tilde{\phi}(a, 0) &= \phi_e(a), \\ \tilde{\varphi} \circ \phi_e &= \varphi \circ \phi, \\ \tilde{\varphi} \circ \tilde{\phi}(a,u) &= \tilde{\varphi} \circ \phi_e(a) + \tilde{\varphi}(u) = \varphi \circ \phi(a) + \tilde{\varphi}(u). \end{aligned}$$

In addition, we can identify the dul space B^* with $A^* \oplus \mathbb{C}h_0$, where $h_0 \in B^*$ and $h_0|_A = 0$. Also, $B^{**} \cong A^{**} \oplus \mathbb{C}m_0$, in which $m_0 \in B^{**}$ and $m_0|_{A^*} = 0$. Moreover, we can extend the $\mathfrak{A}^{\#}$-bimodule actions on A and B to $\mathfrak{A}^{\#}$-bimodule actions on A^*, A^{**}, B^* and B^{**}.

Proposition 4.1. *The following statements are equivalent*

(i) A is right [left] $b \cdot app \cdot m \cdot (\phi, \varphi)$-am. as an $\mathfrak{A}$-bimodule,
(ii) A is right [left] $b \cdot app \cdot m \cdot (\phi_e, \tilde{\varphi})$-am. as an $\mathfrak{A}^\#$-bimodule,
(iii) B is right [left] $b \cdot app \cdot m \cdot (\tilde{\phi}, \tilde{\varphi})$-am. as an $\mathfrak{A}^\#$-bimodule.

Proof. (i) ⇒ (ii) By Proposition 2.5, there exist a net $(m_i) \subset A^{**}$ and $L, L' > 0$ such that $m_i(\varphi \circ \phi) = 1$ and for all $a \in A$ and $\alpha \in \mathfrak{A}$

$$a \cdot m_i - \phi(a) \cdot m_i \to 0 \quad , \quad \|a \cdot m_i - \phi(a) \cdot m_i\| \leq L\|a\|,$$
$$\alpha \cdot m_i - \varphi(\alpha) m_i \to 0 \quad , \quad \|\alpha \cdot m_i - \varphi(\alpha) m_i\| \leq L'\|\alpha\|.$$

Therefore, $m_i(\tilde{\varphi} \circ \phi_e) = m_i(\varphi \circ \varphi) = 1$, and for all $a \in A$ and $u = (\alpha, \lambda) \in \mathfrak{A}^\#$ we have

$$\begin{aligned} \|a \cdot m_i - \phi_e(a) \cdot m_i\| &= \|a \cdot m_i - \phi(a) \cdot m_i\| \to 0, \\ \|a \cdot m_i - \phi_e(a) \cdot m_i\| &= \|a \cdot m_i - \phi(a) \cdot m_i\| \leq L\|a\|, \end{aligned}$$

and

$$\begin{aligned} \|u \cdot m_i - \tilde{\varphi}(u) m_i\| &= \|\alpha \cdot m_i + \lambda m_i - \varphi(\alpha) m_i - \lambda m_i\| \\ &= \|\alpha \cdot m_i - \varphi(\alpha) m_i\| \to 0, \end{aligned}$$

$$\|u \cdot m_i - \tilde{\varphi}(u) m_i\| \leq L'\|\alpha\| \leq L'(\|\alpha\| + \|\lambda\|) = L'\|u\|.$$

Thus, $(m_i) \subset A^{**}$ is a right $b \cdot app \cdot m \cdot (\phi_e, \tilde{\varphi})$-mean and A is right $b \cdot app \cdot m \cdot (\phi_e, \tilde{\varphi})$-am. by Proposition 2.5.

(ii) ⇒ (i) If $(m_i) \subset A^{**}$ is a right $b \cdot app \cdot m \cdot (\phi_e, \tilde{\varphi})$-mean, then it is easy to check that it is a right $b \cdot app \cdot m \cdot (\phi, \varphi)$-mean. Thus the assertion is hold by Proposition 2.5.

(ii) ⇒ (iii) By Proposition 2.5, there exist a net $(m_i) \subset A^{**}$ and $L, L' > 0$ such that $m_i(\tilde{\varphi} \circ \phi_e) = 1$ and for all $a \in A$ and $u = (\alpha, \lambda) \in \mathfrak{A}^\#$

$$a \cdot m_i - \phi_e(a) \cdot m_i \to 0 \quad , \quad \|a \cdot m_i - \phi_e(a) \cdot m_i\| \leq L\|a\|,$$
$$u \cdot m_i - \tilde{\varphi}(u) m_i \to 0 \quad , \quad \|u \cdot m_i - \tilde{\varphi}(u) m_i\| \leq L'\|u\|.$$

We define $n_i \in B^{**}$ by $n_i(h + \mu h_0) = m_i(h)$, where $h + \mu h_0 \in B^* \cong A^* \oplus \mathbb{C}h_0$. Since $\tilde{\varphi} \circ \tilde{\phi}(a, u) = \tilde{\varphi} \circ \phi_e(a) + \tilde{\varphi}(u)$, then $n_i(\tilde{\varphi} \circ \tilde{\phi}) = m_i(\tilde{\varphi} \circ \phi_e) = 1$. On the other hand, for all $(a, u), (b, v) \in B$ and $h + \mu h_0 \in B^*$ we have

$$\Big[(h + \mu h_0) \cdot (a, u)\Big](b, v) = (h \cdot a + h \cdot u)(b) + h(av) + \mu h_0(0, uv).$$

Therefore, $n_i\Big[(h + \mu h_0) \cdot (a, u)\Big] = m_i(h \cdot a + h \cdot u)$, and

$$\begin{aligned} |\langle (a, u) \cdot n_i - \tilde{\phi}(a, u) \cdot n_i, h + \mu h_0 \rangle| &= |\langle n_i, (h + \mu h_0) \cdot (a, u) \rangle - \tilde{\phi}(a, u) \langle n_i, h + \mu h_0 \rangle| \\ &= |\langle m_i, h \cdot a + h \cdot u \rangle - \Big(\phi(a), \tilde{\varphi}(u)\Big) \langle m_i, h \rangle| \\ &= |\langle a \cdot m_i + u \cdot m_i, h \rangle - \langle \phi_e(a) \cdot m_i + \tilde{\varphi}(u) m_i, h \rangle| \\ &\leq \Big[\|a \cdot m_i - \phi_e(a) \cdot m_i\| + \|u \cdot m_i - \tilde{\varphi}(u) m_i\|\Big]\|h\|, \end{aligned}$$

so $\|(a, u) \cdot n_i - \tilde{\phi}(a, u) \cdot n_i\| \to 0$, and also

$$\begin{aligned} \|(a, u) \cdot n_i - \tilde{\phi}(a, u) \cdot n_i\| &\leq L\|a\| + L'\|u\| \\ &\leq L\Big(\|a\| + \|u\|\Big) + L'\Big(\|a\| + \|u\|\Big) \\ &= (L + L')\|(a, u)\|. \end{aligned}$$

In addition, by using $\mathfrak{A}^\#$-bimodule actions on B^*, for $u = (\alpha, \lambda) \in \mathfrak{A}^\#$, $(b, v) \in B$ and $h + \mu h_0 \in B^*$ we have

$$\Big[(h + \mu h_0) \cdot u\Big](b, v) = h \cdot u(b) + \mu h_0(0, vu).$$

Therefore, $n_i\Big[(h+\mu h_0)\cdot u\Big] = m_i(h\cdot u)$ and

$$\begin{aligned} |\langle u\cdot n_i - \tilde{\varphi}(u)n_i, h+\mu h_0\rangle| &= |\langle n_i,(h+\mu h_0)\cdot u\rangle - \tilde{\varphi}(u)\langle n_i, h+\mu h_0\rangle| \\ &= |\langle m_i, h\cdot u\rangle - \tilde{\varphi}(u)\langle m_i, h\rangle| \\ &= |\langle u\cdot m_i - \tilde{\varphi}(u)m_i, h\rangle| \\ &\leq \|u\cdot m_i - \tilde{\varphi}(u)m_i\|\|h\|, \end{aligned}$$

so $\|u\cdot n_i - \tilde{\varphi}(u)n_i\| \to 0$, and also

$$\|u\cdot n_i - \tilde{\varphi}(u)n_i\| \leq L^{'}\|u\|.$$

These show that $(n_i) \subset B^{**}$ is a right $b\cdot app\cdot m\cdot(\tilde{\phi},\tilde{\varphi})$-mean and so B is right $b\cdot app\cdot m\cdot(\tilde{\phi},\tilde{\varphi})$-am. by Proposition 2.5.

(iii)⇒ (ii) By Proposition 2.5, there exist a net $(m_i) \subset B^{**}$ and $L, L^{'} > 0$ such that $m_i(\tilde{\varphi}\circ\tilde{\phi}) = 1$ and for all $b = (a,u) \in B$ and $v = (\alpha,\lambda) \in \mathfrak{A}^{\#}$

$$\begin{aligned} b\cdot m_i - \tilde{\phi}(b)\cdot m_i \to 0 \quad &, \quad \|b\cdot m_i - \tilde{\phi}(b)\cdot m_i\| \leq L\|a\|, \\ v\cdot m_i - \tilde{\varphi}(v)m_i \to 0 \quad &, \quad \|v\cdot m_i - \tilde{\varphi}(v)m_i\| \leq L^{'}\|v\|. \end{aligned}$$

We know that $m_i = n_i - \mu n_0 \in B^{**} \cong A^{**} + \mathbb{C}n_0$, where $n_i, n_0 \in A^{**}$ and $n_0|_{A^*} = 0$. Since $\tilde{\varphi}\circ\tilde{\phi}(a,u) = \tilde{\varphi}\circ\phi_e(a) + \tilde{\varphi}(u)$, then $n_i(\tilde{\varphi}\circ\phi_e) = m_i(\tilde{\varphi}\circ\tilde{\phi}) = 1$. Now for $b = (a,0) \in B$ and $h + \mu h_0 \in B^* \cong A^* \oplus \mathbb{C}h_0$ we have

$$\begin{aligned} (b\cdot m_i)(h+\mu h_0) &= m_i\Big[(h+\mu h_0)\cdot(a,0)\Big] \\ &= m_i(h\cdot a) \\ &= n_i(h\cdot a) \\ &= a\cdot n_i(h), \end{aligned}$$

so for all $a \in A$, $h \in A^*$

$$\begin{aligned} |\langle a\cdot n_i - \phi_e(a)\cdot n_i, h\rangle| &= |\langle (a,0)\cdot m_i - \tilde{\phi}(a,0)\cdot m_i, h+0h_0\rangle| \\ &\leq \|(a,0)\cdot m_i - \tilde{\phi}(a,0)\cdot m_i\|\|h\|, \end{aligned}$$

so $\|a\cdot n_i - \phi_e(a)\cdot n_i\| \to 0$, and also

$$\|a\cdot n_i - \phi_e(a)\cdot n_i\| \leq L\|(a,0)\| = L\|a\|.$$

In addition, for $v = (\alpha,\lambda) \in \mathfrak{A}^{\#}$ and $h \in B^*$ we have

$$\langle m_i, (h+0h_0)\cdot(\alpha,\lambda)\rangle = \langle n_i, h\cdot(\alpha,\lambda)\rangle,$$

and then

$$\begin{aligned} |\langle (\alpha,\lambda)\cdot n_i - \tilde{\varphi}(\alpha,\lambda)n_i, h\rangle| &= |\langle n_i, h\cdot(\alpha,\lambda) - \tilde{\varphi}(\alpha,\lambda)h\rangle| \\ &= |\langle m_i, (h+0h_0)\cdot(\alpha,\lambda) - \tilde{\varphi}(\alpha,\lambda)(h+0h_0)\rangle| \\ &\leq \|(\alpha,\lambda)\cdot m_i - \tilde{\varphi}(\alpha,\lambda)m_i\|\|h+0h_0\|, \end{aligned}$$

so $\|(\alpha,\lambda)\cdot n_i - \tilde{\varphi}(\alpha,\lambda)n_i\| \to 0$, and also

$$\|(\alpha,\lambda)\cdot n_i - \tilde{\varphi}(\alpha,\lambda)n_i\| \leq L^{'}\|(\alpha,\lambda)\|.$$

These show that $(n_i) \subset A^{**}$ is a right $b\cdot app\cdot m\cdot(\phi_e,\tilde{\varphi})$-mean, thus A is right $b\cdot app\cdot m\cdot(\phi_e,\tilde{\varphi})$-am. by Proposition 2.5. □

5. Ideals and quotients of Banach algebras

Throughout this section, I is a closed ideal and $\mathfrak{A}$-submodule of A such that $I \subseteq \ker\phi$ ($\phi \in \Omega_A$), so we can define $\phi_I : A/I \to \mathfrak{A}$ by $\phi_I(a+I) = \phi(a)$. Furthermore, we consider the closed ideal $J = J_A$ of A generated by $\left\{(a\cdot\alpha)b - a(\alpha\cdot b) \ : \ a, b \in A \ , \ \alpha \in \mathfrak{A}\right\}$, then J and A/J are $\mathfrak{A}$-bimodules. Since $J \subseteq \ker\phi$, then ϕ lifts to $\phi_J : A/J \to \mathfrak{A}$, and clearly $\varphi \circ \phi_J \in \sigma(A/J)$ in which $\varphi \in \sigma(\mathfrak{A})$.

Proposition 5.1. *The Banach $\mathfrak{A}$-bimodule A is right [left] $app\cdot m\cdot(\phi,\varphi)$-am. if and only if A/J is right [left] $app\cdot m\cdot(\phi_J,\varphi)$-am. Moreover, if A/J is right [left] $b\cdot app\cdot m\cdot(\phi_J,\varphi)$-am. then A is right [left] $b\cdot app\cdot m\cdot(\phi,\varphi)$-am.*

Proof. Suppose that A is right $app\cdot m\cdot(\phi,\varphi)$-am. then A/J is right $app\cdot m\cdot(\phi_J,\varphi)$-am. by Proposition 2.6, in which $\theta : A \to A/J$ is the canonical mapping.

For the converse, let A/J be right $b\cdot app\cdot m\cdot(\phi_J,\varphi)$-am. and let $X \in {}_{(\phi,\varphi)}\mathcal{M}^{A,\mathfrak{A}}$ and $D : A \to X^*$ be an $\mathfrak{A}$-module derivation. Since $XJ = JX = 0$, then X is an A/J-bimodule by well defined actions

$$\begin{aligned}(a+J)\cdot x &= a\cdot x = \phi(a)\cdot x = \phi_J(a+J)\cdot x,\\ x\cdot(a+J) &= x\cdot a.\end{aligned}$$

In addition, we can extend D to an $\mathfrak{A}$-module derivation $\tilde{D} : A/J \to X^*$ defined by $\tilde{D}(a+J) = D(a)$. By hypothesis, there is a net $(f_i) \subset X^*$ and $L > 0$ such that for all $(a+J) \in A/J$

$$\begin{aligned}&\tilde{D}(a+J) = \lim_i \left[(a+J)\cdot f_i - f_i\cdot(a+J)\right],\\ &\|(a+J)\cdot f_i - f_i\cdot(a+J)\| \leq L\|a+J\|.\end{aligned}$$

Thus for all $a \in A$ we have

$$\begin{aligned}&D(a) = \lim_i (a\cdot f_i - f_i\cdot a),\\ &\|a\cdot x_i - x_i\cdot a\| \leq L\|a+J\| \leq L\|a\|.\end{aligned}$$

This shows that D is boundedly approximately inner, so A is right $b\cdot app\cdot m\cdot(\phi,\varphi)$-am. □

Proposition 5.2. *If A is right [left] $app\cdot m\cdot(\phi,\varphi)$-am. then A/I is right [left] $app\cdot m\cdot(\phi_I,\varphi)$-am. The boundedness holds only if $I = \{0\}$.*

Proof. This is a consequence of Proposition 2.6. □

Proposition 5.3. *Let A/I is right [left] $b\cdot app\cdot m\cdot(\phi_I,\varphi)$-am. and I is module $(\phi|_I,\varphi)$-amenable, then A is right [left] $b\cdot app\cdot m\cdot(\phi,\varphi)$-am.*

Proof. Let $X \in {}_{(\phi,\varphi)}\mathcal{M}^{A,\mathfrak{A}}$ and let $D : A \to X^*$ be an $\mathfrak{A}$-module derivation. Then $D|_I : I \to X^*$ is an $\mathfrak{A}$-module derivation and by hypothesis, there is $f \in X^*$ such that $D|_I = D_f$. Now, we can extend $\tilde{D} = D - D_f$ to a well defined $\mathfrak{A}$-module derivation $\tilde{\tilde{D}} : A/I \to X^*$ by $\tilde{\tilde{D}}(a+I) = \tilde{D}(a)$. By hypothesis, there is a net $(f_i) \subset X^*$ and $L > 0$ such that $\tilde{\tilde{D}} = \lim_i D_{f_i}$ and $\|D_{f_i}(a+I)\| \leq L\|a+I\|$, for all $a+I \in A/I$. Therefore, $\tilde{D} = \lim_i D_{f_i}$, $D = \lim_i D_{f_i+f}$ and for all $a \in A$ we have

$$\begin{aligned}\|D_{f_i+f}(a)\| &\leq \|D_{f_i}(a)\| + \|D_f(a)\|\\ &\leq \left(L + 2\|f\|\right)\|a\|.\end{aligned}$$

This shows that D is boundedly approximately inner, hence A is right $b\cdot app\cdot m\cdot(\phi,\varphi)$-am. □

Proposition 5.4. *Let I be a closed weakly complemented ideal, and $\mathfrak{A}$-submodule of A such that $I \subset \ker\phi$. If A is right [left] $b \cdot app \cdot m \cdot (\phi, \varphi)$-am., then I is right [left] $b \cdot app \cdot m \cdot (\phi|_I, \varphi)$-am.*

Proof. By the hypothesis and Proposition 2.5, there exist a net $(m_i) \subset A^{**}$ and $L, L' > 0$ such that $m_i(\varphi \circ \phi) = 1$ and for all $a \in A$ and $\alpha \in \mathfrak{A}$

$$a \cdot m_i - \phi(a) \cdot m_i \to 0 \quad , \quad \|a \cdot m_i - \phi(a) \cdot m_i\| \le L\|a\|,$$
$$\alpha \cdot m_i - \varphi(\alpha) m_i \to 0 \quad , \quad \|\alpha \cdot m_i - \varphi(\alpha) m_i\| \le L'\|\alpha\|.$$

Since I is weakly complemented in A, there exists a closed subspace X of A^* such that $A^* = I^{\perp} \oplus X$, and there exists $K > 0$ such that for any $F \in A^*$, $F = g_F + h_F$, where $g_F \in I^{\perp}$ and $h_F \in X$, and $\|g_F\| \le K\|F\|$, $\|h_F\| \le K\|F\|$. Now for all $a \in I$, $g_F \cdot a = 0$, thus $|\big(\varphi \circ \phi\big)(a) m_i(g_F)| \to 0$. Choose $a \in I$ with $\varphi \circ \phi(a) = 1$, then $|m_i(g_F)| \to 0$. We set $n_i : I^* \to \mathbb{C}$ defined by $n_i(f) = m_i(h_F)$, where $f \in I^*$ and F is any Hahn-Banach extension of f. Since $\varphi \circ \phi = g_{\varphi\circ\phi} + h_{\varphi\circ\phi} = g_{\varphi\circ\phi} + (\varphi \circ \phi)|_I = g_{\varphi\circ\phi} + \varphi \circ \phi|_I$, we have

$$n_i(\varphi \circ \phi|_I) = m_i(\varphi \circ \phi|_I) = m_i(\varphi \circ \phi) - m_i(g_{\varphi\circ\phi}) \to 1.$$

In addition, for all $a \in I$, $f \in I^*$ and $\alpha \in \mathfrak{A}$, $h_{F\cdot a} = h_F \cdot a = f \cdot a$, $h_{F\cdot\alpha} = h_F \cdot \alpha = f \cdot \alpha$ and

$$\begin{aligned}
|\langle a \cdot n_i - \phi|_I(a) \cdot n_i, f\rangle| &= |\langle n_i, f \cdot a\rangle - \phi(a) \cdot \langle n_i, f\rangle| \\
&= |\langle m_i, h_{F\cdot a}\rangle - \phi(a) \cdot \langle m_i, h_F\rangle| \\
&= |\langle m_i, h_F \cdot a\rangle - \phi(a) \cdot \langle m_i, h_F\rangle| \\
&= |\langle a \cdot m_i - \phi(a) \cdot m_i, h_F\rangle| \\
&\le \|a \cdot m_i - \phi(a) \cdot m_i\| \|h_F\|,
\end{aligned}$$

so $\|a \cdot n_i - \phi|_I(a) \cdot n_i\| \to 0$, and

$$\begin{aligned}
\|a \cdot n_i - \phi|_I(a) \cdot n_i\| &\le L\|a\|K\|F\| \\
&\le LK\|f\|\|a\|,
\end{aligned}$$

also we have

$$\begin{aligned}
|\langle \alpha \cdot n_i - \varphi(\alpha) n_i, f\rangle| &= |\langle n_i, f \cdot \alpha\rangle - \phi(\alpha)\langle n_i, f\rangle| \\
&= |\langle m_i, h_{F\cdot\alpha}\rangle - \varphi(\alpha)\langle m_i, h_F\rangle| \\
&= |\langle m_i, h_F \cdot \alpha\rangle - \varphi(\alpha)\langle m_i, h_F\rangle| \\
&\le \|\alpha \cdot m_i - \varphi(\alpha) m_i\| \|h_F\|,
\end{aligned}$$

so $\|\alpha \cdot n_i - \varphi(\alpha) n_i\| \to 0$, and

$$\begin{aligned}
\|\alpha \cdot n_i - \varphi(\alpha) n_i\| &\le L'\|\alpha\|K\|F\| \\
&\le L'K\|f\|\|\alpha\|.
\end{aligned}$$

These show that $(n_i) \subset I^{**}$ is a right $b \cdot app \cdot m \cdot (\phi|_I, \varphi)$-mean, thus I is right $b \cdot app \cdot m \cdot (\phi|_I, \varphi)$-am. □

6. Projective tensor product and l^p-direct sum of Banach algebras

In this section, A and B are Banach $\mathfrak{A}$-bimodules. The projective tensor product $A\hat{\otimes}B$ of A and B is a Banach $\mathfrak{A}\hat{\otimes}\mathfrak{A}$-bimodule with following actions

$$(\alpha \otimes \beta) \cdot (a \otimes b) =: (\alpha \cdot a) \otimes (\beta \cdot b),$$
$$(a \otimes b) \cdot (\alpha \otimes \beta) =: (a \cdot \alpha) \otimes (b \cdot \beta) \quad (a \in A \ , \ b \in B \ ; \ \alpha, \beta \in \mathfrak{A}).$$

For $\phi_1 \in \Omega_A$, $\phi_2 \in \Omega_B$ and $\varphi_1, \varphi_2 \in \sigma(\mathfrak{A})$, consider

$$\phi_1 \otimes \phi_2 : A\hat{\otimes}B \to \mathfrak{A}\hat{\otimes}\mathfrak{A} \left(\phi_1 \otimes \phi_2(a \otimes b) =: \phi_1(a) \otimes \phi_2(b)\right),$$

and

$$\varphi_1 \otimes \varphi_2 : \mathfrak{A}\hat{\otimes}\mathfrak{A} \to \mathbb{C} \left(\varphi_1 \otimes \varphi_2(\alpha \otimes \beta) =: \varphi_1(\alpha)\varphi_2(\beta)\right).$$

Clearly, $\phi_1 \otimes \phi_2 \in \Omega_{A\hat{\otimes}B}$ and $\varphi_1 \otimes \varphi_2 \in \sigma(\mathfrak{A}\hat{\otimes}\mathfrak{A})$.

Proposition 6.1. *If $A\hat{\otimes}B$ is right [left] $b \cdot app \cdot m \cdot (\phi_1 \otimes \phi_2, \varphi_1 \otimes \varphi_2)$-am. then A is right [left] $b \cdot app \cdot m \cdot (\phi_1, \varphi_1)$-am. and B is right [left] $b \cdot app \cdot m \cdot (\phi_2, \varphi_2)$-am.*

Proof. By Proposition 2.5, there exist a net $(m_i) \subset (A\hat{\otimes}B)^{**}$ and $L, L^{'} > 0$ such that $m_i\Big((\varphi_1 \otimes \varphi_2) \circ (\phi_1 \otimes \phi_2)\Big) = 1$, and for all $w = (a \otimes b) \in A\hat{\otimes}B$ and $\omega = (\alpha \otimes \beta) \in \mathfrak{A}\hat{\otimes}\mathfrak{A}$

$$\begin{array}{ccc} w \cdot m_i - (\phi_1 \otimes \phi_2)(w) \cdot m_i \to 0 & , & \|w \cdot m_i - (\phi_1 \otimes \phi_2)(w) \cdot m_i\| \leq L\|w\|, \\ \omega \cdot m_i - (\varphi_1 \otimes \varphi_2)(\omega) m_i \to 0 & , & \|\omega \cdot m_i - (\varphi_1 \otimes \varphi_2)(\omega) m_i \leq L^{'}\|\omega\|. \end{array}$$

We choose $a_0 \in A$, $b_0 \in B$ and $\alpha_0, \beta_0 \in \mathfrak{A}$ such that $\big(\varphi_1 \circ \phi_1\big)(a_0) = \big(\varphi_2 \circ \phi_2\big)(b_0) = 1$, and $\varphi_1(\alpha_0) = \varphi_2(\beta_0) = 1$. Define $(\bar{m}_i) \subset A^{**}$ by $\bar{m}_i(f) =: m_i\Big(f \otimes (\varphi_2 \circ \phi_2)\Big)$ $(f \in A^*)$. We have

$$\begin{aligned} \bar{m}_i(\varphi_1 \circ \phi_1) &= m_i\Big((\varphi_1 \circ \phi_1) \otimes (\phi_2 \circ \phi_2)\Big) \\ &= m_i\Big((\varphi_1 \otimes \varphi_2) \circ (\phi_1 \otimes \phi_2)\Big) \\ &= 1. \end{aligned}$$

Similar to the proof of Proposition 5.10 in [5] for each $a \in A$, $\alpha \in \mathfrak{A}$ and $f \in A^*$, we have

$$\begin{aligned} & \lim_i \langle \alpha \cdot \bar{m}_i - \varphi_1(\alpha)\bar{m}_i, f\rangle \\ = \;& \lim_i \langle m_i, (f \cdot \alpha) \otimes (\varphi_2 \circ \phi_2)\rangle - \varphi_1(\alpha) \lim_i \langle m_i, f \otimes (\varphi_2 \circ \phi_2)\rangle \\ = \;& \lim_i \langle (\alpha_0 \otimes \beta_0) \cdot m_i, (f \cdot \alpha) \otimes (\varphi_2 \circ \phi_2)\rangle - \varphi_1(\alpha) \lim_i \langle m_i, f \otimes (\varphi_2 \circ \phi_2)\rangle \\ = \;& \varphi_1(\alpha) \lim_i \langle m_i, f \otimes (\varphi_2 \circ \phi_2)\rangle - \varphi_1(\alpha) \lim_i \langle m_i, f \otimes (\varphi_2 \circ \phi_2)\rangle = 0. \end{aligned}$$

On the other hand,

$$\lim_i \langle (\alpha_0 \otimes \beta_0) \cdot m_i, \Big(f \cdot \alpha - \varphi_1(\alpha) f\Big) \otimes (\varphi_2 \circ \phi_2)\rangle = 0,$$

hence there exists a $K > 0$ such that

$$\begin{aligned} |\langle (\alpha_0 \otimes \beta_0) \cdot m_i, \Big(f \cdot \alpha - \varphi_1(\alpha) f\Big) \otimes (\varphi_2 \circ \phi_2)\rangle| &\leq K\|\Big(f \cdot \alpha - \varphi_1(\alpha) f\Big) \otimes (\varphi_2 \circ \phi_2)\| \\ &\leq K(1 + \|\varphi_1\|)\|\varphi_2 \circ \phi_2\|\|\alpha\|\|f\| \\ &= K^{'}\|f\|, \end{aligned}$$

in which $K^{'} = K(1 + \|\varphi_1\|)\|\varphi_2 \circ \phi_2\|\|\alpha\|$, therefore

$$\begin{aligned} |\langle \alpha \cdot \bar{m}_i - \varphi_1(\alpha)\bar{m}_i, f\rangle| &= |\langle m_i - (\alpha_0 \otimes \beta_0) m_i, (f \cdot \alpha) \otimes (\varphi_2 \circ \phi_2)\rangle \\ &\quad + \langle (\alpha_0 \otimes \beta_0) m_i, (f \cdot \alpha) \otimes (\varphi_2 \circ \phi_2)\rangle \\ &\quad - \langle m_i - (\alpha_0 \otimes \beta_0) m_i, \varphi_1(\alpha) f \otimes (\varphi_2 \circ \phi_2)\rangle \\ &\quad - \langle (\alpha_0 \otimes \beta_0) m_i, \varphi_1(\alpha) f \otimes (\varphi_2 \circ \phi_2)\rangle| \\ &\leq \|m_i - (\alpha_0 \otimes \beta_0) m_i\|\|\Big(f \cdot \alpha - \varphi_1(\alpha) f\Big) \otimes (\varphi_2 \circ \phi_2)\| \\ &\quad + |\langle (\alpha_0 \otimes \beta_0) m_i, \Big(f \cdot \alpha - \varphi_1(\alpha) f\Big) \otimes (\varphi_2 \circ \phi_2)\rangle| \\ &\leq L^{'}\|\alpha_0 \otimes \beta_0\|(1 + \|\varphi_1\|)\|\alpha\|\|f\|\|\varphi_2 \circ \phi_2\| \\ &\quad + K^{'}(1 + \|\varphi_1\|)\|\alpha\|\|f\|\|\varphi_2 \circ \phi_2\| \\ &= L^{''}\|\alpha\|\|f\|, \end{aligned} \tag{1}$$

where, $L'' = \Big(L'\|\alpha_0 \otimes \beta_0\| + K'\Big)(1 + \|\varphi_1\|)\|\varphi_2 \circ \phi_2\|$, we conclude that $\|\alpha \cdot \bar{m}_i - \varphi_1(\alpha)\bar{m}_i\| \le L''\|\alpha\|$. Moreover, by similar calculation, $\lim_i \|a \cdot \bar{m}_i - \phi_1(a) \cdot \bar{m}_i\| = 0$, the proof of the boundedness of this part is similar to calculations in (1). Thus $(\bar{m}_i)$ is a right $b \cdot app \cdot m \cdot (\phi_1, \varphi_1)$-mean for A, so A is right $b \cdot app \cdot m \cdot (\phi_1, \varphi_1)$-am. by Proposition 2.5. Similarly, B is right $b \cdot app \cdot m \cdot (\phi_2, \varphi_2)$-am. □

Now let $\phi \in \Omega_A$, $\psi \in \Omega_B$, $\varphi \in \sigma(\mathfrak{A})$ and $1 \le p \le +\infty$. The *l^p-direct sums* $A \oplus_\infty B$ and $A \oplus_p B$ are Banach algebras with respect to multiplication defined by

$$(a, b)(c, d) =: (ac, bd) \qquad (a, c \in A \ \ , \ \ b, d \in B),$$

and norms

$$\|(a, b)\|_\infty =: \max\{\|a\|, \|b\|\} \quad , \quad \|(a, b)\|_p = \Big(\|a\|^p + \|b\|^p\Big)^{1/p} \qquad (a \in A \ \ , \ \ b \in B).$$

Furthermore, $A \oplus_\infty B$ and $A \oplus_p B$ are Banach $\mathfrak{A}$-bimodules under the following $\mathfrak{A}$-module actions

$$\alpha \cdot (a, b) =: (\alpha \cdot a, \alpha \cdot b) \quad , \quad (a, b) \cdot \alpha =: (a \cdot \alpha, b \cdot \alpha) \qquad (a \in A \ \ , \ \ b \in B \ \ , \ \ \alpha \in \mathfrak{A}).$$

We define

$$(\phi, 0) : A \oplus_p B \to \mathfrak{A} \quad , \quad (\phi, 0)(a, b) =: \phi(a),$$
$$(0, \psi) : A \oplus_p B \to \mathfrak{A} \quad , \quad (0, \psi)(a, b) =: \psi(b),$$

for $(a, b) \in A \oplus_p B$ and $1 \le p \le +\infty$. Then $(0, \psi), (\phi, 0) \in \Omega_{A \oplus_p B}$ for $1 \le p \le +\infty$, and $(\phi, 0)\ |_A = \phi$, $(0, \psi)\ |_B = \psi$.

Proposition 6.2. *Let A and B be Banach algebras and $\mathfrak{A}$-bimodules, $\phi \in \Omega_A$, $\psi \in \Omega_B$, $\varphi \in \sigma(\mathfrak{A})$ and $1 \le p \le +\infty$. Then*

(i) $A \oplus_p B$ is right [left] $b \cdot app \cdot m \cdot \Big((\phi, 0), \varphi\Big)$-am. if and only if A is right [left] $b \cdot app \cdot m \cdot (\phi, \varphi)$-am.

(ii) $A \oplus_p B$ is right [left] $b \cdot app \cdot m \cdot \Big((0, \psi), \varphi\Big)$-am. if and only if B is right [left] $b \cdot app \cdot m \cdot (\psi, \varphi)$-am.

Proof. These are consequences of Propositions 5.3 and 5.4. □

7. Examples

We start this section with following definitions.

Definition 7.1. *[1] A discrete semigroup S is called an inverse semigroup if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of all idempotents of S is denoted by E. It is easy to see that E is a commutative subsemigroup of S and $l^1(E)$ is a subalgebra of $l^1(S)$. Suppose that $l^1(S)$ is a $l^1(E)$-bimodule by following actions, that is multiplication from right and trivially from left*

$$\delta_e \cdot \delta_s =: \delta_s \quad , \quad \delta_s \cdot \delta_e =: \delta_{se}\Big(= \delta_s * \delta_e\Big) \qquad (s \in S \ , \ e \in E).$$

We denote $J_{l^1(s)}$ by J that is the closed ideal of $l^1(s)$ generated by $\Big\{\delta_{set} - \delta_{st} \ \ : \ \ s, t \in S \ , \ e \in E\Big\}$.

Next, we consider the congruence relation $\backsim$ on S by

$$s \backsim t \ \Leftrightarrow \ \exists\, e \in E \ \ : \ \ se = te \qquad (s, t \in S).$$

The quotient semigroup $G_S := S/\backsim$ is a group. Furthermore, $l^1(G_S)$ is a quotient of $l^1(S)$ by Lemma 3.2 in [1]. Indeed $l^1(G_S) \cong l^1(S)/J$, and by lifting the $l^1(E)$-module actions on $l^1(S)$ to $l^1(G_S)$ it becomes a Banach $l^1(E)$-bimodule. But, the right and left $l^1(E)$-module actions on $l^1(G_S)$ are trivial, so we have

$$l^1(G_S)\hat{\otimes}_{l^1(E)}l^1(G_S) \cong l^1(G_S)\hat{\otimes}l^1(G_S),$$

see Lemma 3.3 in [1].

Now we are ready to show the main results of this section.

Proposition 7.2. *Let S be an inverse semigroup with idempotents E. Consider $l^1(S)$ as a Banach $l^1(E)$-bimodule with multiplication right action and the trivial left action. Then*

(i) $l^1(S)$ is $app \cdot m \cdot char \cdot am.$ if and only if S is amenable.
(ii) $l^1(S)^{**}$ is $b \cdot app \cdot m \cdot char \cdot am$ if and only if G_S is finit.

Proof. Part (i) and part (ii) without boundedness are true by Theorem 5.6 in [5]. For proving the boundedness in part (ii), since G_S is finit, then $L^1(G_S)^{**} \cong l^1(G_S)^{**}$ is $b \cdot app \cdot char$·am. by Example 4.4 in [15]. Thus, $l^1(S)^{**}$ is $b \cdot app \cdot char$·am. by Proposition 5.1. On the other hand, $l^1(S)^{**}$ is $app \cdot m \cdot char$·am by Theorem 5.6 in [5]. Finally, we conclude that $l^1(S)^{**}$ is $b \cdot app \cdot m \cdot char$·am. □

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