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Fractional midpoint-type inequalities for twice-differentiable functions

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Abstract. In this research article, we obtain an identity for twice differentiable functions whose second derivatives in absolute value are convex. By using this identity, we prove several left Hermite–Hadamard-type inequalities for the case of Riemann–Liouville fractional integrals. Furthermore, we provide our results by using special cases of obtained theorems.

1. Introduction

The theory of inequalities has an important place in the literature. One of the most famous inequalities for the case of convex functions is the Hermite–Hadamard inequality. Hence, a considerable number of mathematicians has investigated Hermite–Hadamard-type inequalities and related inequalities such as trapezoid, midpoint, and Simpson's inequality.

Hermite–Hadamard-type inequalities were first investigated by C. Hermite and J. Hadamard for the case of convex functions. Let $f : I \to \mathbb{R}$ denote a convex function on the interval I of real numbers and $a_1, a_2 \in I$ with $a_1 < a_2$. Then, the double inequality

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)dx \leq \frac{f(a_1) + f(a_2)}{2} \tag{1}$$

is valid. Let us consider that f is concave. Then, both inequalities in (1) hold in the reverse direction.

In the last two decades, many papers have been considered for midpoint and trapezoid type inequalities, which give bounds for the left-hand side and right-hand side of the inequality (1), respectively. For instance, Dragomir and Agarwal first established trapezoid inequalities to the case of convex functions in [8]. Kirmacı first obtained midpoint inequalities for the case of convex functions in [16]. Sarikaya et al. generalized (1) for the case of fractional integrals and the authors also investigated new trapezoid type inequalities in [24]. Some fractional midpoint type inequalities for the case of convex functions given in [13]. For further information about these kinds of inequalities, we refer to [6, 7, 14] and the references therein.

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In the literature, many researchers have focused on twice differentiable functions to obtain many important inequalities. For example, Barani et al. proved inequalities for the case of twice-differentiable convex functions, which are associated with Hermite–Hadamard inequalities in [4]. In [18], some new generalized fractional integral inequalities of trapezoid and midpoint type for the case of twice-differentiable convex functions are obtained. In [23], the authors proved some new inequalities of Hermite–Hadamard-type and Simpson for the case of functions whose absolute values of derivatives are convex. In addition, J. Park [20] has obtained new estimates on generalizations of Hadamard, Ostrowski and Simpson type inequalities for functions whose second derivatives in absolute value at certain powers are convex and quasi-convex functions. Furthermore, [5] established some trapezoid and midpoint type inequalities for the case of functions whose second derivatives in absolute value are convex. For results connected with these types of inequalities involving twice-differentiable functions, one can see [2, 11, 12, 21, 22, 26, 29, 30].

Now, we present mathematical preliminaries of fractional calculus theory, which are used further in the sequel of this paper.

For $0 < x, y < \infty$ and $x, y \in \mathbb{R}$, the well-known *Gamma function* and *Beta function* are defined

$$\Gamma(x) := \int_0^\infty \xi^{x-1} e^{-\xi} d\xi$$

and

$$\beta(x, y) := \int_0^1 \xi^{x-1} (1-\xi)^{y-1} d\xi = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively.

Definition 1.1. *Let $f \in L_1[a_1, a_2]$. The Riemann–Liouville integrals $J^\alpha_{a_1+}f$ and $J^\alpha_{a_2-}f$ of order $\alpha > 0$ with $a_1 \geq 0$ are defined by*

$$J^\alpha_{a_1+}f(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x} (x-\xi)^{\alpha-1} f(\xi) d\xi, \quad x > a_1$$

and

$$J^\alpha_{a_2-}f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{a_2} (\xi - x)^{\alpha-1} f(\xi) d\xi, \quad x < a_2,$$

respectively. Let us note that $J^0_{a_1+}f(x) = J^0_{a_2-}f(x) = f(x)$.

Remark 1.2. *In the case of $\alpha = 1$, the fractional integral becomes the classical integral.*

Many authors have studied fractional integral inequalities and applications by using Riemann–Liouville fractional integrals. For instance, a variant of Hermite–Hadamard inequalities in Riemann–Liouville fractional integral forms was proved in [27]. Moreover, Tomar et al. [28] obtained several new Hermite–Hadamard-type inequalities for Riemann–Liouville fractional integrals on twice differentiable functions. The reader is referred to [9, 10, 15, 17] and the references therein for further information and properties of Riemann–Liouville fractional integrals. While a considerable number of mathematicians has investigated Hermite–Hadamard inequalities for Riemann–Liouville fractional integrals, some authors have also studied Hermite–Hadamard inequalities for the case of other types of fractional integrals such as k-fractional integrals, Hadamard fractional integrals, conformable fractional integrals, etc. For instance, we refer the reader to [1, 3, 19] and the references cited therein.

Sarikaya et al. [24] first presented the following interesting integral inequalities of Hermite–Hadamard-type involving Riemann–Liouville fractional integrals.

Theorem 1.3. *Let $0 \leq a_1 < a_2$ and $\alpha > 0$. If $f \in L_1([a_1, a_2], \mathbb{R})$ is positive and convex, then*

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(a_2 - a_1)^{\alpha}}\left[J^{\alpha}_{a_1+}f(a_2) + J^{\alpha}_{a_2-}f(a_1)\right] \leq \frac{f(a_1) + f(a_2)}{2}. \tag{2}$$

Remark 1.4. *For $\alpha = 1$, (2) becomes (1).*

Sarıkaya and Yıldırım also presented [25] the following Hermite–Hadamard-type inequality for the case of Riemann–Liouville fractional integrals.

Theorem 1.5. *Let $0 \leq a_1 < a_2$ and $\alpha > 0$. If $f \in L_1([a_1, a_2], \mathbb{R})$ is positive and convex, then*

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(a_2 - a_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a_1+a_2}{2}\right)+}f(a_2) + J^{\alpha}_{\left(\frac{a_1+a_2}{2}\right)-}f(a_1)\right] \leq \frac{f(a_1) + f(a_2)}{2}.$$

The main purpose of this paper is to prove left Hermite–Hadamard-type inequalities for the case of Riemann–Liouville fractional integrals using an identity obtained for fractional integrals. The entire research structure takes three sections, including the introduction. In Section 2, we establish an identity for the case of twice differentiable functions. By utilizing this equality, we prove midpoint-type inequalities for mappings whose second derivatives are convex. We also give some remarks. Some conclusions of research are given in Section 3.

2. Main results

In this section, we present several fractional midpoint-type inequalities for the case of twice-differentiable functions.

Lemma 2.1. *If*

$$f : [a_1, a_2] \to \mathbb{R} \text{ is absolutely continuous on } (a_1, a_2) \text{ and } f'' \in L_1([a_1, a_2]), \tag{H}$$

then

$$\frac{\Gamma(\alpha + 1)}{2(a_2 - a_1)^{\alpha}}\left[J^{\alpha}_{a_1+}f(a_2) + J^{\alpha}_{a_2-}f(a_1)\right] - f\left(\frac{a_1 + a_2}{2}\right) = \frac{(a_2 - a_1)^2}{2(\alpha + 1)}\sum_{k=1}^{4} I_k, \tag{3}$$

where

$$\begin{cases} I_1 = \int\limits_0^{\frac{1}{2}} \xi^{\alpha+1} f''(\xi a_2 + (1 - \xi)a_1)\, d\xi, \\ I_2 = \int\limits_0^{\frac{1}{2}} \xi^{\alpha+1} f''(\xi a_1 + (1 - \xi)a_2)\, d\xi, \\ I_3 = \int\limits_{\frac{1}{2}}^{1} \left(\xi^{\alpha+1} - (1 + \alpha)\xi + \alpha\right) f''(\xi a_2 + (1 - \xi)a_1)\, d\xi, \\ I_4 = \int\limits_{\frac{1}{2}}^{1} \left(\xi^{\alpha+1} - (1 + \alpha)\xi + \alpha\right) f''(\xi a_1 + (1 - \xi)a_2)\, d\xi. \end{cases}$$

Proof. Using integration by parts, we obtain

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} \xi^{\alpha+1} f''(\xi a_2 + (1-\xi) a_1)\, d\xi = \xi^{\alpha+1} \frac{f'(\xi a_2 + (1-\xi) a_1)}{a_2 - a_1}\bigg|_0^{\frac{1}{2}} \\
&\quad - \frac{\alpha+1}{a_2 - a_1} \int_0^{\frac{1}{2}} \xi^{\alpha} f'(\xi a_2 + (1-\xi) a_1)\, d\xi \\
&= \frac{1}{2^{\alpha+1}(a_2 - a_1)} f'\left(\frac{a_1+a_2}{2}\right) - \frac{\alpha+1}{a_2-a_1}\left[\frac{\xi^{\alpha} f(\xi a_2 + (1-\xi) a_1)}{a_2 - a_1}\bigg|_0^{\frac{1}{2}} \right. \\
&\quad \left. - \frac{\alpha}{a_2 - a_1} \int_0^{\frac{1}{2}} \xi^{\alpha-1} f(\xi a_2 + (1-\xi) a_1)\, d\xi \right] \\
&= \frac{1}{2^{\alpha+1}(a_2 - a_1)} f'\left(\frac{a_1+a_2}{2}\right) - \frac{\alpha+1}{2^{\alpha}(a_2-a_1)^2} f\left(\frac{a_1+a_2}{2}\right) \\
&\quad + \frac{\alpha(\alpha+1)}{(a_2-a_1)^2} \int_0^{\frac{1}{2}} \xi^{\alpha-1} f(\xi a_2 + (1-\xi) a_1)\, d\xi.
\end{aligned} \tag{4}
$$

Similarly, we have

$$
\begin{aligned}
I_2 &= -\frac{1}{2^{\alpha+1}(a_2 - a_1)} f'\left(\frac{a_1+a_2}{2}\right) - \frac{\alpha+1}{2^{\alpha}(a_2-a_1)^2} f\left(\frac{a_1+a_2}{2}\right) \\
&\quad + \frac{\alpha(\alpha+1)}{(a_2-a_1)^2} \int_0^{\frac{1}{2}} \xi^{\alpha-1} f(\xi a_1 + (1-\xi) a_2)\, d\xi,
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
I_3 &= \frac{\frac{\alpha-1}{2} - \frac{1}{2^{\alpha+1}}}{a_2 - a_1} f'\left(\frac{a_1+a_2}{2}\right) + \frac{\frac{\alpha+1}{2^{\alpha}} - 1 - \alpha}{(a_2-a_1)^2} f\left(\frac{a_1+a_2}{2}\right) \\
&\quad + \frac{\alpha(\alpha+1)}{(a_2-a_1)^2} \int_{\frac{1}{2}}^{1} \xi^{\alpha-1} f(\xi a_2 + (1-\xi) a_1)\, d\xi,
\end{aligned} \tag{6}
$$

and

$$I_4 = \frac{\frac{1}{2^{\alpha+1}} - \frac{\alpha-1}{2}}{a_2 - a_1} f'\left(\frac{a_1 + a_2}{2}\right) + \frac{\frac{\alpha+1}{2^{\alpha}} - 1 - \alpha}{(a_2 - a_1)^2} f\left(\frac{a_1 + a_2}{2}\right)$$

$$+ \frac{\alpha(\alpha+1)}{(a_2 - a_1)^{\alpha+2}} \int_{\frac{1}{2}}^{1} \xi^{\alpha-1} f(\xi a_1 + (1-\xi) a_2)\, d\xi. \tag{7}$$

Adding (4)–(7), we get

$$\sum_{k=1}^{4} I_k = \frac{\alpha(\alpha+1)}{(a_2 - a_1)^2}\left[\int_0^1 \xi^{\alpha-1} f(\xi a_2 + (1-\xi) a_1)\, d\xi + \int_0^1 \xi^{\alpha-1} f(\xi a_1 + (1-\xi) a_2)\, d\xi\right] \tag{8}$$

$$- \frac{2(\alpha+1)}{(a_2 - a_1)^2} f\left(\frac{a_1 + a_2}{2}\right)$$

$$= \frac{\alpha(\alpha+1)\Gamma(\alpha)}{(a_2 - a_1)^{\alpha+2}}\left[\frac{1}{\Gamma(\alpha)} \int_{a_1}^{a_2} (x - a_1)^{\alpha-1} f(x)\, dx + \frac{1}{\Gamma(\alpha)} \int_{a_1}^{a_2} (a_2 - x)^{\alpha-1} f(x)\, dx\right]$$

$$- \frac{2(\alpha+1)}{(a_2 - a_1)^2} f\left(\frac{a_1 + a_2}{2}\right)$$

$$= \frac{(\alpha+1)\Gamma(\alpha+1)}{(a_2 - a_1)^{\alpha+2}}\left[J_{a_2-}^{\alpha} f(a_1) + J_{a_1+}^{\alpha} f(a_2)\right] - \frac{2(\alpha+1)}{(a_2 - a_1)^2} f\left(\frac{a_1 + a_2}{2}\right).$$

Multiplying both sides of (8) by $\frac{(a_2-a_1)^2}{2(\alpha+1)}$, we obtain (3). This completes the proof. □

Theorem 2.2. *If* (H) *holds and* $|f''|$ *is convex on* $[a_1, a_2]$*, then*

$$\left|\frac{\Gamma(\alpha+1)}{2(a_2 - a_1)^{\alpha}}\left[J_{a_1+}^{\alpha} f(a_2) + J_{a_2-}^{\alpha} f(a_1)\right] - f\left(\frac{a_1 + a_2}{2}\right)\right| \tag{9}$$

$$\leq \frac{(a_2 - a_1)^2}{2(\alpha+1)}\left(\frac{1}{\alpha+2} + \frac{\alpha-3}{8}\right)\left[|f''(a_1)| + |f''(a_2)|\right].$$

Proof. Let us take modulus in Lemma 2.1. Then, we obtain

$$\left|\frac{\Gamma(\alpha+1)}{2(a_2 - a_1)^{\alpha}}\left[J_{a_1+}^{\alpha} f(a_2) + J_{a_2-}^{\alpha} f(a_1)\right] - f\left(\frac{a_1 + a_2}{2}\right)\right| \tag{10}$$

$$\leq \frac{(a_2-a_1)^2}{2(\alpha+1)}\left[\int_0^{\frac{1}{2}}\left|\xi^{\alpha+1}\right|\left|f''(\xi a_2+(1-\xi)a_1)\right|d\xi+\int_0^{\frac{1}{2}}\left|\xi^{\alpha+1}\right|\left|f''(\xi a_1+(1-\xi)a_2)\right|d\xi\right.$$

$$+\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|\left|f''(\xi a_2+(1-\xi)a_1)\right|d\xi$$

$$\left.+\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|\left|f''(\xi a_1+(1-\xi)a_2)\right|d\xi\right].$$

With the help of the convexity of $|f''|$, we obtain

$$\left|\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^\alpha}\left[J_{a_1+}^{\alpha}f(a_2)+J_{a_2-}^{\alpha}f(a_1)\right]-f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq \frac{(a_2-a_1)^2}{2(\alpha+1)}\left[\int_0^{\frac{1}{2}}\xi^{\alpha+1}\left[\xi\left|f''(a_2)\right|+(1-\xi)\left|f''(a_1)\right|\right]d\xi+\int_0^{\frac{1}{2}}\xi^{\alpha+1}\left[\xi\left|f''(a_1)\right|+(1-\xi)\left|f''(a_2)\right|\right]d\xi\right.$$

$$+\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)\left[\xi\left|f''(a_2)\right|+(1-\xi)\left|f''(a_1)\right|\right]d\xi$$

$$\left.+\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)\left[\xi\left|f''(a_1)\right|+(1-\xi)\left|f''(a_2)\right|\right]d\xi\right]$$

$$=\frac{(a_2-a_1)^2}{2(\alpha+1)}\left[\int_0^{\frac{1}{2}}\xi^{\alpha+1}d\xi+\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)d\xi\right]\left(\left|f''(a_1)\right|+\left|f''(a_2)\right|\right)$$

$$=\frac{(a_2-a_1)^2}{2(\alpha+1)}\left(\frac{1}{\alpha+2}+\frac{\alpha-3}{8}\right)\left(\left|f''(a_1)\right|+\left|f''(a_2)\right|\right).$$

This finishes the proof. □

Remark 2.3. *If we let $\alpha=1$ in Theorem 2.2, then we obtain the midpoint-type inequality*

$$\left|\frac{1}{(a_2-a_1)}\int_{a_1}^{a_2}f(\xi)d\xi-f\left(\frac{a_1+a_2}{2}\right)\right|\leq\frac{(a_2-a_1)^2}{48}\left(\left|f''(a_1)\right|+\left|f''(a_2)\right|\right),$$

which is given in [21, Theorem 5].

Example 2.4. *Let us consider a function $f : [a_1, a_2] = [0, 1] \to \mathbb{R}$ given by $f(x) = \frac{x^5}{20}$. Then, the left-hand side of* (9) *reduces to*

$$\left|\frac{\Gamma(\alpha+1)}{2}\left[J_{0+}^{\alpha} f(1) + J_{1-}^{\alpha} f(0)\right] - f\left(\frac{1}{2}\right)\right|$$

$$= \left|\frac{\alpha}{2}\left[\int_0^1 (1-t)^{\alpha-1}\frac{t^5}{20}dt + \int_0^1 t^{\alpha-1}\frac{t^5}{20}dt\right] - \frac{1}{640}\right|$$

$$= \left|\frac{\alpha}{40}\left[\beta(6,\alpha) + \frac{1}{\alpha+5}\right] - \frac{1}{640}\right|$$

$$= \left|\frac{3}{(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)} + \frac{\alpha}{40(\alpha+5)} - \frac{1}{640}\right|.$$

The right hand-side of (9) *becomes*

$$\frac{\alpha^2-\alpha+2}{16(\alpha+2)(\alpha+1)}.$$

Consequently, we have the inequality

$$\left|\frac{3}{(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)} + \frac{\alpha}{40(\alpha+5)} - \frac{1}{640}\right| \le \frac{\alpha^2-\alpha+2}{16(\alpha+2)(\alpha+1)}.$$

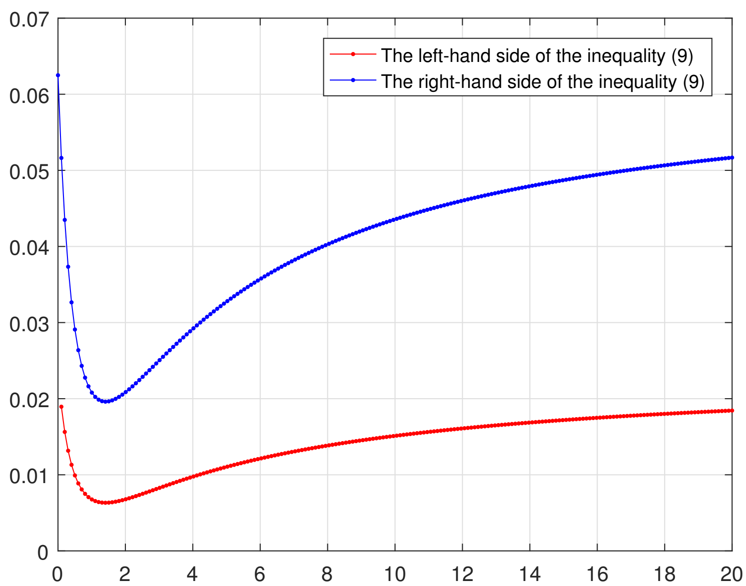

Figure 1: Graph of both sides of (9) in Example 2.4, depending on α, computed and plotted with MATLAB.

As one can see in Figure 1, the left-hand side of (9) *in Example 2.4 is always below the right-hand side of this equation, for all values of* $\alpha \in (0, 20]$.

Theorem 2.5. *If* (H) *holds and* $\left|f''\right|^{q}$*,* $q>1$*, is convex on* $[a_1,a_2]$*, then*

$$\left|\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^{\alpha}}\left[J_{a_1+}^{\alpha}f(a_2)+J_{a_2-}^{\alpha}f(a_1)\right]-f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq\frac{(a_2-a_1)^2}{2(\alpha+1)}\left[\left(\frac{1}{2^{(p(1+\alpha)+1)}(p(1+\alpha)+1)}\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|^{p}d\xi\right)^{\frac{1}{p}}\right]$$

$$\times\left[\left(\frac{3\left|f''(a_2)\right|^{q}+\left|f''(a_1)\right|^{q}}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|f''(a_1)\right|^{q}+\left|f''(a_2)\right|^{q}}{8}\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{(a_2-a_1)^2}{2^{\frac{3}{q}-1}(\alpha+1)}\left[\left(\frac{1}{2^{(p(1+\alpha)+1)}(p(1+\alpha)+1)}\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|^{p}d\xi\right)^{\frac{1}{p}}\right]\left(\left|f''(a_2)\right|+\left|f''(a_1)\right|\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying Hölder's inequality in (10), we obtain

$$\left|\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^{\alpha}}\left[J_{a_1+}^{\alpha}f(a_2)+J_{a_2-}^{\alpha}f(a_1)\right]-f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq\frac{(a_2-a_1)^2}{2(\alpha+1)}\left[\left(\int_{0}^{\frac{1}{2}}\left|\xi^{\alpha+1}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left|f''(\xi a_2+(1-\xi)a_1)\right|^{q}d\xi\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{0}^{\frac{1}{2}}\left|\xi^{\alpha+1}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left|f''(\xi a_1+(1-\xi)a_2)\right|^{q}d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|f''(\xi a_2+(1-\xi)a_1)\right|^{q}d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|f''(\xi a_1+(1-\xi)a_2)\right|^{q}d\xi\right)^{\frac{1}{q}}\Bigg].$$

By using convexity of $\left|f''\right|^{q}$, we get

$$\left|\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^{\alpha}}\left[J_{a_1+}^{\alpha}f(a_2)+J_{a_2-}^{\alpha}f(a_1)\right]-f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq\frac{(a_2-a_1)^2}{2(\alpha+1)}\Bigg[\left(\int_{0}^{\frac{1}{2}}\xi^{(\alpha+1)p}d\xi\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left[\xi\left|f''(a_2)\right|^{q}+(1-\xi)\left|f''(a_1)\right|^{q}\right]d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{0}^{\frac{1}{2}}\xi^{(\alpha+1)p}d\xi\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left[\xi\left|f''(a_1)\right|^{q}+(1-\xi)\left|f''(a_2)\right|^{q}\right]d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left[\xi\left|f''(a_2)\right|^{q}+(1-\xi)\left|f''(a_1)\right|^{q}\right]d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left[\xi\left|f''(a_1)\right|^{q}+(1-\xi)\left|f''(a_2)\right|^{q}\right]d\xi\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(a_2-a_1)^2}{2(\alpha+1)}\left[\left(\frac{1}{2^{(p(1+\alpha)+1)}(p(1+\alpha)+1)}\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|^{p}d\xi\right)^{\frac{1}{p}}\right]$$

$$\times\left[\left(\frac{3\left|f''(a_2)\right|^{q}+\left|f''(a_1)\right|^{q}}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|f''(a_1)\right|^{q}+\left|f''(a_2)\right|^{q}}{8}\right)^{\frac{1}{q}}\right].$$

For the proof of the second inequality, let $c_1=\left|f''(a_1)\right|^{q}$, $c_2=3\left|f''(a_2)\right|^{q}$, $d_1=3\left|f''(a_1)\right|^{q}$ and $d_2=\left|f''(a_2)\right|^{q}$. Using the facts that

$$\sum_{k=1}^{n}(c_k+d_k)^{s}\leq\sum_{k=1}^{n}c_k^{s}+\sum_{k=1}^{n}d_k^{s},\ 0\leq s<1$$

and $1+3^{\frac{1}{q}}\leq 4$, the desired result can be achieved directly. This completes the proof. □

Remark 2.6. *If we let $\alpha = 1$ in Theorem 2.5, then we obtain the inequalities*

$$\left|\frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} f(\xi)\,d\xi - f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq \frac{(a_2-a_1)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|f''(a_2)\right|^q+\left|f''(a_1)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f''(a_1)\right|^q+\left|f''(a_2)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(a_2-a_1)^2}{16}\left(\frac{4}{2p+1}\right)^{\frac{1}{p}}\left(\left|f''(a_2)\right|+\left|f''(a_1)\right|\right),$$

which are given in [5, Corollary 4.8].

Theorem 2.7. *If* (H) *holds and $\left|f''\right|^q$, $q \geq 1$, is convex on $[a_1, a_2]$, then*

$$\left|\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^\alpha}\left[J_{a_1+}^{\alpha}f(a_2)+J_{a_2-}^{\alpha}f(a_1)\right]-f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq \frac{(a_2-a_1)^2}{2(\alpha+1)}\left[\left(\frac{1}{2^{\alpha+2}(\alpha+2)}\right)\left[\left(\frac{(\alpha+2)\left|f''(a_2)\right|^q+(\alpha+4)\left|f''(a_1)\right|^q}{2(\alpha+3)}\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\frac{(\alpha+2)\left|f''(a_1)\right|^q+(\alpha+4)\left|f''(a_2)\right|^q}{2(\alpha+3)}\right)^{\frac{1}{q}}\right]+(\Omega_1(\alpha))^{1-\frac{1}{q}}$$

$$\times\left[\left[(\Omega_2(\alpha))\left|f''(a_2)\right|^q+(\Omega_1(\alpha)-\Omega_2(\alpha))\left|f''(a_1)\right|^q\right]^{\frac{1}{q}}\right.$$

$$\left.\left.+\left[(\Omega_2(\alpha))\left|f''(a_1)\right|^q+(\Omega_1(\alpha)-\Omega_2(\alpha))\left|f''(a_2)\right|^q\right]^{\frac{1}{q}}\right]\right].$$

Here,

$$\begin{cases} \Omega_1(\alpha) = \frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)} + \frac{\alpha-3}{8}, \\ \Omega_2(\alpha) = \frac{2^{\alpha+3}-1}{2^{\alpha+3}(\alpha+3)} + \frac{2\alpha-7}{24}. \end{cases}$$

Proof. By applying the power–mean inequality in (10), we have

$$\left|\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^\alpha}\left[J_{a_1+}^{\alpha}f(a_2)+J_{a_2-}^{\alpha}f(a_1)\right]-f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq \frac{(a_2-a_1)^2}{2(\alpha+1)}\left[\left(\int_0^{\frac{1}{2}}\left|\xi^{\alpha+1}\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\xi^{\alpha+1}\right|\left|f''(\xi a_2+(1-\xi)a_1)\right|^q d\xi\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{2}}\left|\xi^{\alpha+1}\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\xi^{\alpha+1}\right|\left|f''\left(\xi a_1+(1-\xi)a_2\right)\right|^q d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|d\xi\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|\left|f''\left(\xi a_2+(1-\xi)a_1\right)\right|^q d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|d\xi\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_{\frac{1}{2}}^{1}\left|\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right|\left|f''\left(\xi a_1+(1-\xi)a_2\right)\right|^q d\xi\right)^{\frac{1}{q}}\right].$$

Since $\left|f''\right|^q$ is convex, we obtain

$$\left|\frac{\Gamma(\alpha+1)}{2(a_2-a_1)^\alpha}\left[J_{a_1+}^{\alpha}f(a_2)+J_{a_2-}^{\alpha}f(a_1)\right]-f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq\frac{(a_2-a_1)^2}{2(\alpha+1)}\left[\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}\left[\xi\left|f''(a_2)\right|^q+(1-\xi)\left|f''(a_1)\right|^q\right]d\xi\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}\left[\xi\left|f''(a_1)\right|^q+(1-\xi)\left|f''(a_2)\right|^q\right]d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)d\xi\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)\left[\xi\left|f''(a_2)\right|^q+(1-\xi)\left|f''(a_1)\right|^q\right]d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)d\xi\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)\left[\xi\left|f''(a_1)\right|^q+(1-\xi)\left|f''(a_2)\right|^q\right]d\xi\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(a_2-a_1)^2}{2(\alpha+1)}\left[\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}d\xi\right)^{1-\frac{1}{q}}\left[\left(\left|f''(a_2)\right|^q\int_0^{\frac{1}{2}}\xi^{\alpha+2}d\xi+\left|f''(a_1)\right|^q\int_0^{\frac{1}{2}}\xi^{\alpha+1}(1-\xi)d\xi\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\left|f''(a_1)\right|^q\int_0^{\frac{1}{2}}\xi^{\alpha+2}d\xi+\left|f''(a_2)\right|^q\int_0^{\frac{1}{2}}\xi^{\alpha+1}(1-\xi)d\xi\right)^{\frac{1}{q}}\right]+\left(\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)d\xi\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\left|f''(a_2)\right|^q\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)\xi d\xi+\left|f''(a_1)\right|^q\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)(1-\xi)d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\left|f''(a_1)\right|^q\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)\xi d\xi+\left|f''(a_2)\right|^q\int_{\frac{1}{2}}^{1}\left(\xi^{\alpha+1}-(1+\alpha)\xi+\alpha\right)(1-\xi)d\xi\right)^{\frac{1}{q}}\right]\right]$$

$$=\frac{(a_2-a_1)^2}{2(\alpha+1)}\left[\left(\frac{1}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{(\alpha+2)\left|f''(a_2)\right|^q+(\alpha+4)\left|f''(a_1)\right|^q}{2(\alpha+3)}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{1}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{(\alpha+2)\left|f''(a_1)\right|^q+(\alpha+4)\left|f''(a_2)\right|^q}{2(\alpha+3)}\right)^{\frac{1}{q}}$$

$$+\left(\Omega_1(\alpha)\right)^{1-\frac{1}{q}}\left[\left(\Omega_2(\alpha)\right)\left|f''(a_2)\right|^q+\left(\Omega_1(\alpha)-\Omega_2(\alpha)\right)\left|f''(a_1)\right|^q\right]^{\frac{1}{q}}$$

$$+\left(\Omega_1(\alpha)\right)^{1-\frac{1}{q}}\left[\left(\Omega_2(\alpha)\right)\left|f''(a_1)\right|^q+\left(\Omega_1(\alpha)-\Omega_2(\alpha)\right)\left|f''(a_2)\right|^q\right]^{\frac{1}{q}}\Bigg].$$

Thus, we obtain the desired result. □

Remark 2.8. *If we let $\alpha = 1$ in Theorem 2.7, then we have the midpoint-type inequality*

$$\left|\frac{1}{(a_2-a_1)}\int_{a_1}^{a_2} f(\xi)\,d\xi - f\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq \frac{(a_2-a_1)^2}{48}\left[\left(\frac{3\left|f''(a_2)\right|^q+5\left|f''(a_1)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|f''(a_1)\right|^q+5\left|f''(a_2)\right|^q}{8}\right)^{\frac{1}{q}}\right],$$

which is given in [23, Proposition 5].

3. Conclusion

In this paper, we derive an identity for the case of twice-differentiable functions whose second derivatives are convex. By using this equality, we establish midpoint type inequalities for the case of Riemann–Liouville fractional integrals. Moreover, our results generalize known results from the literature. In future studies, improvements or generalizations of our results can be investigated by using different kinds of convex function classes or other types of fractional integral operators.

Author contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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