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On the Roter type of generalised Wintgen ideal Legendrian submanifolds

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Abstract. I. Mihai obtained an inequality relating intrinsic normalised scalar curvature and extrinsic squared mean curvature and normalised normal curvature of Legendrian submanifolds M^n in Sasakian space forms $\widetilde{M}^{2n+1}(c)$. In this paper, for the class of generalised Wintgen ideal Legendrian submanifolds M^n of Sasakian space form $\widetilde{M}^{2n+1}(c)$, we study relationship between some properties concerning their Deszcz symmetry and their Roter type.

1. Preliminaries ([2], [3], [16], [17])

Let M^n be an n-dimensional Riemannian manifold with metric tensor g (g is positive definite $(0,2)$-tensor). With R we denote $(0,4)$-Riemann–Christofel curvature tensor and with S the $(0,2)$-Ricci tensor on M^n. The Ricci tensor S is symmetric and all its eigenvalues are real and S determines an orthogonal set of eigendirections on M^n, which are the intrinsic (Ricci) principal directions on M^n.

For two $(0,2)$ tensors t and r, we denote with $\wedge$ Nomizu–Kulkarni product defined by

$$(t \wedge r)(X, Y, Z, W) = t(X, W)r(Y, Z) + t(Y, Z)r(X, W) - t(X, Z)r(Y, W) - t(Y, W)r(X, Z),$$

whereby X, Y, Z, W are tangent vector fields on M^n. Now, for plane π spanned by tangent vector fields X and Y, the sectional curvature $K(\pi)$ is given by

$$K(\pi) = \frac{R(X, Y, Y, X)}{\frac{1}{2}(g \wedge g)(X, Y, Y, X)}.$$

Riemannian manifold with constant sectional curvature c, $K = c$, is called a real space form of curvature c, denoted with $M^n(c)$.

If Ricci tensor S is proportional to metric tensor g on manifold M^n ($S = \lambda q$, λ is some function), we say that M^n is an Einstein space and every 3-dimensional Einstein space has constant sectional curvature. If the Ricci tensor S has an eigenvalue of multiplicity $\geq n-1$ on M^n, we say that M^n is quasi-Einstein.

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The Weyl conformal curvature tensor C is defined by

$$C = R - \left(\frac{1}{n-2}\, g \ \wedge \ S + \frac{\tau}{(n-1)(n-2)}\, g \ \wedge \ g\right).$$

For manifold M^n, $n = 3$, it is known that $C \equiv 0$. If $C \equiv 0$ for $n \geq 4$, we say that M^n is conformally flat.

A Riemannian manifold M^n ($n \geq 3$) is called a Roter space when its Riemann–Christofel curvature tensor R satisfies the equality

$$R = \widetilde{\lambda}\,(g \wedge g) + \widetilde{\mu}\,(g \wedge S) + \widetilde{\nu}\,(S \wedge S), \tag{1}$$

for some functions $\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu} : M^n \to \mathbb{R}$, [9]. The Roter spaces, from an algebraic point of view, may be considered as the simplest Riemannian manifolds, which are the next to the real space forms [5].

It is obvious that the real space forms $M^n(c)$ are Roter spaces for which $\widetilde{\lambda} = \frac{c}{2}$ and $\widetilde{\mu} = \widetilde{\nu} = 0$. Einstein Roter spaces are real space forms. Also, all 3-dimensional Riemannian manifolds and all conformally flat Riemannian manifolds M^n ($n \geq 4$) are Roter spaces for which

$$\widetilde{\lambda} = \frac{\tau}{2(n-1)(n-2)}, \quad \widetilde{\mu} = \frac{1}{n-2}, \quad \widetilde{\nu} = 0.$$

The Deszcz symmetric spaces, from a geometric point of view, may be considered to be the simplest Riemannian manifolds next to the real space form [5]. The Riemannian spaces M^n ($n \geq 3$) are Deszcz symmetric if $(0,6)$-tensors $R \circ R$ and $Q(g,R)$ are proportional, i.e.,

$$R \circ R = L\, Q(g,R), \tag{2}$$

for some function $L : M^n \to \mathbb{R}$, [8]. $(0,6)$-tensor $R \circ R$ is defined by

$$\begin{aligned}(R \circ R)(X_1, X_2, X_3, X_4; X, Y) &= (R(X,Y) \circ R)(X_1, X_2, X_3, X_4)\\ &= -R(R(X,Y)X_1, X_2, X_3, X_4) - R(X_1, R(X,Y)X_2, X_3, X_4)\\ &\quad -R(X_1, X_2, R(X,Y)X_3, X_4) - R(X_1, X_2, X_3, R(X,Y)X_4),\end{aligned}$$

and Tachibana $(0,6)$-tensor $Q(g,R)$ is given by

$$\begin{aligned}Q(g,R)(X_1, X_2, X_3, X_4; X, Y) &= ((X \wedge_g Y) \circ) R)(X_1, X_2, X_3, X_4)\\ &= -R((X \wedge_g Y)X_1, X_2, X_3, X_4) - R(X_1, (X \wedge_g Y)X_2, X_3, X_4)\\ &\quad -R(X_1, X_2, (X \wedge_g Y)X_3, X_4) - R(X_1, X_2, X_3, (X \wedge_g Y)X_4),\end{aligned}$$

where $\wedge_g$ is metric endomorphisam defined by

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y.$$

For Riemannian manifold M^n ($n \geq 3$) we say that it is Ricci pseudo-symmetric if

$$(R \circ S)(p) = L_s(p)\, Q(g,S)(p), \quad \forall p \in M^n,$$

where $R \circ S$ and $Q(g,S)$ are $(0,4)$-tensors given by

$$(R \circ S)(X,Y,Z,W) = -S(R(Z,W)X, Y) - S(X, R(Z,W)Y),$$

$$Q(g,S) = -S((Z \wedge_g W)X, Y) - S(X, (Z \wedge_g W)Y),$$

for $X, Y, Z, W \in TM^n$.

Similarly, Riemannian manifold M^n ($n \geq 4$) has pseudo-symmetric Weyl tensor C if

$$C \circ C = L_C\ Q(g,C),$$

for some functions $L_C : M^n \to \mathbb{R}$ (on the open part of M^n where $Q(g,C) \neq 0$) for $(0,6)$-tensors $C \circ C$ and $Q(g,C)$.

It is known ([6], [11], [10], [13], [14]) that:

a) the open submanifold $\mathcal{U}$ of a Riemannian manifold M^n of Roter type is Deszcz symmetric and has pseudo-symmetric Weyl conformal tensor C;
b) the open submanifold $\mathcal{U}$ of a Deszcz symmetric space with pseudo-symmetric Weyl tensor C is a space of Roter type.

Let M^n be n-dimensional Riemannian submanifold in $(m+n)$-dimensional real space form $\widetilde{M}^{n+m}(c)$, and let g, ∇ and $\widetilde{g}$, $\widetilde{\nabla}$ be the metric (Riemannian) and the corresponding Levi–Civita connection on M^n and $\widetilde{M}^{n+m}(c)$, respectively. Let $X, Y, Z, \ldots$ be the tangent vector fields on M^n and $\xi, \eta, \ldots$ be the normal vector fields on $\widetilde{M}^{n+m}(c)$. Then we have the formula of Gauss ($\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$) and Weingarten ($\widetilde{\nabla}_X \xi = -A_\xi(x) + \nabla^\perp_X \xi$) which decompose the vector fields $\widetilde{\nabla}_X Y$ and $\widetilde{\nabla}_X \xi$ on their tangential ($\nabla_X Y$ and $A_\xi(X)$) and normal ($h(X,Y)$ and $\nabla^\perp_X \xi$) components along M^n in $\widetilde{M}^{n+m}(c)$, respectively. With h and A_ξ we denote the second fundamental form and the shape operator of M^n with respect to ξ (normal vector field), such that

$$g(h(X,Y),\xi) = g(A_\xi(X), Y).$$

$\nabla^\perp$ denote the connection in the normal bundle.

Let $\{E_1, E_2, \ldots, E_n, \xi_1, \ldots, \xi_m\}$ be any local orthonormal frame field on M^n in $\widetilde{M}^{n+m}(c)$. Then the mean curvature vector field of M^n in $\widetilde{M}^{n+m}(c)$ is defined by

$$\vec{H} = \frac{1}{n}\operatorname{tr} h = \frac{1}{n}\sum_{i=1}^{n} h(E_i, E_i) = \frac{1}{n}\sum_{\alpha=1}^{m} (\operatorname{tr} A_\alpha)\,\xi_\alpha.$$

For submanifold M^n in $\widetilde{M}^{n+m}(c)$ we say that it is:

(i) totally geodesic when $h = 0$,
(ii) totally umbilical when $h = g\vec{H}$,
(iii) minimal when $\vec{H} = 0$,
(iv) pseudo-umbilical when $A_{\vec{H}} = \lambda I_d$ (where I_d denote identity operator on TM and λ is some real function on M^n).

The normalised scalar curvature od M^n is given by

$$\rho = \frac{2}{n(n-1)}\sum_{i<j}^{n} R(E_i, E_j, E_j, E_i),$$

where

$$R(X,Y,Z,W) = \widetilde{g}(h(Y,Z), h(X,W)) - \widetilde{g}(h(X,Z), h(Y,W)) + c(g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W))$$

is Riemann–Christofel curvature tensor of M^n in $\widetilde{M}^{n+m}(c)$.

The normalised normal scalar curvature function of M^n at a point p is given by

$$\rho^\perp(p) = \frac{2}{n(n-1)}\sqrt{\sum_{i<j}^{n}\sum_{\alpha<\beta}^{m} R^\perp(E_i, E_j, \xi_\alpha, \xi_\beta)^2},$$

where $R^\perp$ is the curvature tensor of normal space and $\{\xi_1, \ldots, \xi_m\}$ is an orthonormal frame field of that space, $R^\perp(X,Y;\xi,\eta) = g([A_\xi, A_\eta]X, Y)$, whereby $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$.

We further will be concerned with Wintgen ideal submanifolds. The original inequality is obtained for surfaces M^2 in $\mathbb{E}^4$ by Wintgen in 1979. He proved that, intrinsic invariant of M^2, Gauss curvature K and extrinsic invariants, the squared of mean curvature H^2 and normal curvature $K^\perp$, satisfy the inequality $K \leq H^2 - K^\perp$, and also characterised equality case [21]. After that, Rouxel [19], Rodriguez–Guadalupe [15], De Smet, Dillen, Verstraelen and Vrancken [7] gave some generalizations of this results. And, finally, Choi and Lu [4] and Ge–Tang [12] proved that for submanifolds M^n in a real space form $\widetilde{M}^{n+m}(c)$ holds inequality

$$\rho \leq H^2 - \rho^{\perp} + c. \tag{3}$$

They also proved that equality in inequality (3) holds if the shape operators of the submanifold take the special forms for suitable adapted orthonormal frame $\{E_1, \dots, E_n, \xi_1, \dots, \xi_m\}$ on M^n in $\widetilde{M}^{n+m}(c)$.

The submanifolds M^n in $\widetilde{M}^{n+m}(c)$ for which hold equality in inequality (3) are called Wintgen ideal submanifolds. In [5] the authors studied Wintgen ideal submanifolds M^n ($n \geq 4$) in real space forms $\widetilde{M}^{n+m}(c)$ which are Roter spaces and proved that such submanifold is Deszcz symmetric if and only if it is Roter space.

2. Generalised Wintgen inequality for Legendrian submanifolds

A $(2m+1)$-dimensional Riemannian manifold $(\widetilde{M}^{2m+1}(c), g)$ is Sasakian manifold if the triple (ϕ, ξ, η) (ϕ is an endomorphism of tangent bundle of $T\widetilde{M}^{2m+1}(c)$; η is 1-form and ξ is vector field called characteristic vector field) satisfy:

$$\phi^2 = -I_d + \eta \otimes \xi, \ \eta(\xi) = -1, \ \phi\ \xi = 0, \ \eta \circ \phi = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi),$$
$$(\widetilde{\nabla}_X \phi)Y = -g(X, Y)\,\xi + \eta(Y)X, \ \widetilde{\nabla}_X \xi = \phi X,$$

where X and Y are vector fields on $\widetilde{M}^{2m+1}(c)$ and $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g. If a plane π is spanned by X and ϕX, then a plane section π in $T_p\ \widetilde{M}^{2m+1}$ is called a ϕ-section, where X is a unit tangent vector which is orthogonal to ξ. The sectional curvature of a ϕ-section is called ϕ-sectional curvature and a Sasakian manifold with constant ϕ-sectional curvature c is called a Sasakian space form $\widetilde{M}^{2m+1}(c)$. On a Sasakian space form $\widetilde{M}^{2m+1}(c)$ the curvature tensor $\widetilde{R}$ is given by, [1]

$$\begin{aligned} \widetilde{R}(X,Y)Z \ &= \ \frac{c+3}{4}\Big\{g(Y,Z)X - g(X,Z)Y\Big\} + \frac{c-1}{4}\Big\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\Big\}, \end{aligned}$$

for the tangent vector fields X, Y, Z on $\widetilde{M}^{2m+1}(c)$.

Let M^n be an n-dimensional submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$. Then the Gauss equation is given by

$$\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

whereby R and h are the Riemann curvature tensor and second fundamental form, respectively, of M^n, and X, Y, Z, W are vectors tangent to M^n. For every $p \in M^n$, the mean curvature is given by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(E_i, E_i),$$

where $\{E_1, E_2, \dots, E_n, \dots, E_{2m+1}\}$ is an orthonormal basis of $T_p\ \widetilde{M}^{2m+1}$.

C-totally real submanifold is a submanifold M^n normal to ξ in a Sasakian manifold, i.e. $\phi(T_p\ M^n) \subset T_p^{\perp}\ M^n$, for every $p \in M^n$. If $m \equiv n$, then M^n is called Legendrian submanifold.

Let M^n be an n-dimensional Legendrian submanifold of a Sasakian space form $\widetilde{M}^{2n+1}(c)$ and $\{E_1, \dots, E_n\}$ an orthonormal frame on M^n and $\{E_{n+1}, \dots, E_{2n}, E_{2n+1} = \xi\}$ an orthonormal frame in the normal bundle $T^{\perp}\ M^n$. The Gauss equation is given by

$$\begin{aligned} R(X,Y,Z,W) \ &= \ \frac{c+3}{4}\Big\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\Big\} + \\ &+ g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)). \end{aligned}$$

I. Mihai in [18] established a generalised Wintgen inequality for Legendrian submanifolds in Sasakian space forms.

Theorem 2.1 ([18]). *Let M^n be an n-dimensional Legendrian submanifold of a Sasakian space form $\widetilde{M}^{2n+1}(c)$. Then*

$$(\rho^{\perp})^2 \leq \left(\|H\|^2 - \rho + \frac{c+3}{4}\right)^2 + \frac{4}{n(n-1)}\left(\rho - \frac{c+3}{4}\right)\frac{c-1}{4} + \frac{(c-1)^2}{8n(n-1)}, \tag{4}$$

and equality holds if and only if with respect to suitable orthonormal frames $\{E_1, \ldots, E_n\}$ and $\{E_{n+1}, \ldots, E_{2n}, E_{2n+1} = \xi\}$, the shape operators of M^n in $\widetilde{M}^{2n+1}(c)$ are given by:

$$A_{E_{n+1}} = \begin{bmatrix} \lambda_1 & \mu & 0 & \cdots & 0 \\ \mu & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{bmatrix}, \quad A_{E_{n+2}} = \begin{bmatrix} \lambda_2 + \mu & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 - \mu & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 \end{bmatrix},$$

$$A_{E_{n+3}} = \begin{bmatrix} \lambda_3 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_3 \end{bmatrix}, \quad A_{E_{n+4}} = \cdots = A_{E_{2n}} = A_{E_{2n+1}} = 0,$$

whereby $\lambda_1, \lambda_2, \lambda_3$ and μ are real functions on M^n.

Legendrian submanifolds M^n in a Sasakian space forms $\widetilde{M}^{2n+1}(c)$ satisfying equality in generalised Wintgen inequality (4) are called generalized Wintgen ideal Legendrian submanifolds. A frame $\{E_1, E_2, \ldots, E_n, E_{n+1}, \ldots, E_{2n+1}\}$ from Theorem 2.1 is called Choi–Lu frame on such M^n in $\widetilde{M}^{2n+1}(c)$.

3. Main result

From Theorem 2.1, using Gauss equation, we obtain, [20], that all components of $(0,4)$ curvature tensor R of generalised Wintgen ideal Legendrian submanifold M^n of a Sasakian space form $\widetilde{M}^{2n+1}(c)$ are zero, except these:

$$R_{1221} = 2\mu^2 - c_1, \quad R_{1kk1} = -\lambda_2\mu - c_1, \ k \geq 3, \quad R_{2kk2} = \lambda_2\mu - c_1, \ k \geq 3,$$

$$R_{1kk2} = -\lambda_1\mu, \ k \geq 3, \quad R_{kllk} = -c_1, \ k \neq l, \ k, l \geq 3,$$

whereby $c_1 = \frac{c+3}{4} + \lambda_1^2 + \lambda_2^2 + \lambda_3^2$.

The nontrivial components of $(0,2)$-Ricci tensor S of such submanifold are, [20]:

$$S_{11} = 2\mu^2 - (n-1)c_1 - (n-2)\lambda_2\mu,$$

$$S_{22} = 2\mu^2 - (n-1)c_1 + (n-2)\lambda_2\mu,$$

$$S_{12} = -(n-2)\lambda_1\mu,$$

$$S_{kk} = -(n-1)c_1, \ k \geq 3.$$

The equation (1) in local components looks like:

$$R_{ijkl} = \widetilde{\lambda}(g_{il}g_{jk} - g_{ik}g_{jl}) + \widetilde{\mu}(g_{il}S_{jk} + g_{jk}S_{il} - g_{ik}S_{jl} - g_{jl}S_{ik}) + \widetilde{\nu}(S_{il}S_{jk} - S_{ik}S_{jl}). \tag{5}$$

The condition (5) for M^n to be a Roter space is equivalent to the following system of linear equations:

$$\left.\begin{array}{rcl} 2\mu^2 - c_1 &=& \widetilde{\lambda} - 2(n-1)c_1\widetilde{\mu} + ((2\mu^2 - (n-1)c_1)^2 - (n-2)^2\mu^2(\lambda_1^2 + \lambda_2^2))\widetilde{\nu}, \\ -\lambda_2\mu - c_1 &=& \widetilde{\lambda} + (2\mu^2 - 2(n-1)c_1 - (n-2)\lambda_2\mu)\widetilde{\mu} + \\ && +(n-1)c_1(-2\mu^2 + (n-1)c_1 + (n-2)\lambda_2\mu)\widetilde{\nu}, \\ \lambda_1\mu &=& -(n-2)\lambda_1\mu\widetilde{\mu} + (n-1)(n-2)\lambda_1\mu c_1\widetilde{\nu}, \\ \lambda_2\mu - c_1 &=& \widetilde{\lambda} + (2\mu^2 - 2(n-1)c_1 + (n-2)\lambda_2\mu)\widetilde{\mu} + \\ && +(n-1)c_1(-2\mu^2 + (n-1)c_1 - (n-2)\lambda_2\mu)\widetilde{\nu}, \\ -c_1 &=& \widetilde{\lambda} - (n-1)c_1\widetilde{\mu} + (n-1)^2c_1^2\widetilde{\nu}. \end{array}\right\} \quad (6)$$

For the Deszcz symmetric generalised Wintgen ideal Legendrian submanifold M^n in a Sasakian space form $\widetilde{M}^{2n+1}(c)$ the system (6) of linear equations is valid if and only if

(i) $\mu = 0$ or
(ii) $\mu \neq 0$ and $\lambda_1 = \lambda_2 = 0$.

In case (i), we have that M^n is itself a space form and hence a Roter space. In case (ii) from system (6), we obtain

$$\widetilde{\lambda} = \left(\frac{c+3}{4} + \lambda_3^2\right)\left(2\mu^2 - (n-1)^2\left(\frac{c+3}{4} + \lambda_3^2\right)\right), \quad \widetilde{\mu} = \frac{(n-1)\left(\frac{c+3}{4} + \lambda_3^2\right)}{2\mu^2}, \quad \widetilde{\nu} = -\frac{1}{4\mu^2},$$

as its unique solution. We thus obtained the following result:

Theorem 3.1. *Let M^n be a generalised Wintgen ideal Legendrian submanifold of a Sasakian space form $\widetilde{M}^{2n+1}(c)$, $n \geq 4$. Then M^n is Deszcz symmetric if and only if it is a Roter space.*

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