Filomat 37:25 (2023), 8661–8666
https://doi.org/10.2298/FIL2325661D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Almost Kähler structures on complex hyperbolic space

Andrijana Dekić[a]

[a] *Mathematical Institute SANU, Serbia*

Abstract. Complex hyperbolic space has the structure of a solvable Lie group. Based on the recent classification of Riemannian left-invariant metrics we describe all left-invariant almost Kähler structures on that group. The only Kähler structure is the standard structure of the complex hyperbolic space and all others are strictly almost Kähler. We calculated the Chern connection and obtained the characterization by its torsion tensor. It is proved that the only SCF-soliton is the Kähler one.

Introduction

The great progress in the research of symplectic manifolds in this century is one of the major motivations for the renewed interest in almost Kähler geometry. Almost Kähler manifolds are almost Hermitian manifolds that carry the symplectic structure given by their closed Kähler 2-form. They are not necessarily complex (if complex, they are Kähler). A Kähler manifold with a non-integrable almost complex structure is called *strictly almost Kähler*. Almost Kähler structures naturally arise as one of the sixteen classes of the Gray-Hervella classification of almost Hermitian structures [6]. Every Kähler manifold is also an almost Kähler manifold. However, the converse does not hold in general, not even for compact manifolds.

The first example of strictly almost Kähler structure on a compact four-dimensional manifold was given by Thurston [17] in 1976. He considered strictly almost Kähler structure on 2-torus bundle over the 2-torus and verified its non-Kähler status using the oddness of its first Betti number. The research of strictly almost Kähler structures on compact manifolds is related to Goldberg's conjecture stating that the almost complex structure of a compact Einstein almost Kähler manifold is integrable. A partial positive answer is given in the case of non-negative scalar curvature with certain restrictions [13, 14], but the case of negative scalar curvature remains unresolved. One reason may be the paucity of examples of compact strictly almost Kähler manifolds. Cordero, Fernandez, and Leon in [4] generalized Thurston construction for every dimension $2n > 4$. Their examples do not admit any Kähler structure. Jelonek [8] constructed many strictly almost Kähler structures on the manifold $M \times T$ where M is any almost Kähler manifold and $T = S^1 \times S^1$.

A class of examples of non-compact strictly almost Kähler manifolds was obtained as tangent bundles of non-flat Riemannian manifolds [16]. Here, the flatness gives the integrability condition of the almost complex structure. Using results of Olszak [12] and Thurston [17], Watson [19] found many examples of non-compact strictly almost Kähler structures. He gave an example of 10-dimensional compact strictly

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 53B355, 53D30.
Keywords. Complex hyperbolic space; Almost Kähler structures.
Received: 13 Feburary 2023; Accepted: 27 March 2023
Communicated by Zoran Rakić and Mića Stanković
This research was supported by the Serbian Ministry of of Science, Technological Development and Innovation through Mathematical Institute of Serbian Academy of Sciences and Arts.
URL: andrijanadekic@mi.sanu.ac.rs (Andrijana Dekić)

almost Kähler manifold. This was the basis for the construction of (6n + 4)-dimensional spaces with similar properties. Sekigawa and Oguro [10] constructed an example of strictly almost Kähler structure on $\mathbb{H}^3 \times \mathbb{R}$, a product of 3-dimensional hyperbolic space $\mathbb{H}^3$ and real line $\mathbb{R}$. Furthermore, they proved that $2n-$dimensional ($n \geq 2$) hyperbolic space could not admit almost Kähler structure in [11].

Cartan was the first to introduce the notion of torsion of a connection in 1922. He defined metric connections uniquely characterized by the algebraic type of the corresponding torsion tensors [3]. The space of all possible torsion tensors splits under the action of $O(n)$ into the sum of three irreducible representations $\mathcal{T} \cong TM^n \oplus \Lambda^3(M^n) \oplus \mathcal{T}'$. Consequently, there are eight classes of linear connections defined by the possible components of their torsions in these spaces, as given by Tricerri and Vanhecke in [18]. It turns out that the Chern connection of every almost Kähler structure on the complex hyperbolic space belongs to a distinguished class of metric connections (denoted by $\mathfrak{T}_2$ in [18], Table I), as its torsion belongs only to the third component of the above-mentioned decomposition (Section 3). A nice overview of this topic can also be found in Ilka Agricola's beautiful lectures [1].

Street and Tian [15] took a geometric approach to studying symplectic manifolds. Specifically, they introduced a natural way to evolve an almost Kähler manifold, known as the symplectic curvature flow (i.e., SCF). Using this flow, Lauret and Will [9] showed a way how to search for SCF-solitons. An almost Kähler structure is SCF-soliton if the symplectic Lie group admits a compatible metric.

Complex hyperbolic space is a non-compact rank-one symmetric space of negative sectional curvature $\mathbb{C}H^n = SU(1,n)/S(U(1) \times U(n))$. Therefore, it has a structure of a solvable Lie group [7], and the standard Kähler metric is left invariant. All possible left invariant metrics on this group, denoted by $\mathcal{CH}^n$, were classified recently in [5]. In this paper, we found all almost Kähler structures on Lie group $\mathcal{CH}^n$ and investigated their properties.

This paper is organized as follows.

In Section 1 we introduce basic notation and give a brief overview of almost Kähler geometry.

Theorem 2.1 of Section 2 is the main result of this paper. All left-invariant almost Kähler structures on the metric Lie group $(\mathcal{CH}^n, g)$ are obtained. In particular, we show the existence of strictly almost Kähler structures on $\mathcal{CH}^n$.

In Section 3 we calculate explicitly the Chern connection and intrinsic torsion. Cartan showed that a metric connection can be uniquely characterized by its torsion. In this sense, we give the characterization of Chern connection of every almost Kähler structure on $\mathcal{CH}^n$. Additionally, we prove that the only SCF-soliton is the standard Kähler structure of the complex hyperbolic space.

1. Preliminaries

$\mathbb{C}H^n$ is a symmetric space of negative sectional curvature, hence it is a solvmanifold, i.e., it can be represented as a connected solvable Lie group with a left-invariant metric [7]. This group is a semidirect product of the abelian and the nilpotent part (Heisenberg group) of the Iwasawa decomposition of its isometry group, i.e., $\mathcal{CH}^n = \mathbb{R} \ltimes H^{2n-1}$. The Lie algebra of the Lie group $\mathcal{CH}^n$ is the semidirect product of abelian and Heisenberg algebra $ch_n = \mathbb{R} \ltimes \mathfrak{h}_{2n-1}$. It is spanned by vectors $X, Y_1, \ldots, Y_{n-1}, Z_1, \ldots, Z_{n-1}, W$ with nonzero commutators:

$$[X, Y_i] = \frac{1}{2}Y_i, \quad [X, Z_i] = \frac{1}{2}Z_i, \quad [X, W] = W, \quad [Z_j, Y_i] = \delta_{ij}W, \quad i, j \in \{1, \ldots, n-1\}. \tag{1}$$

Using the identification $\mathbb{C}^n \cong \mathbb{R}^{2n} : (z_1, \ldots, z_n) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n)$, $z_k = x_k + iy_k$, $k \in \{1, \ldots, n\}$, the multiplication by i on $\mathbb{C}^n$ induces the standard complex structure on $\mathbb{R}^{2n}$, given by the matrix $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Symplectic form is a closed non-degenerate skew-symmetric bilinear form. The standard symplectic form in vector space $\mathbb{R}^{2n}$ is $\omega(u, v) = u^T J_n v$, $u, v \in \mathbb{R}^{2n}$. An *almost complex structure* on a manifold M is an automorphism $J : TM \to TM$ satisfying $J^2 = -Id$. If g is a Riemannian metric on M, then J is *Hermitian* if $g(X, Y) = g(JX, JY)$, $X, Y \in TM$ and we say that J and g are compatible. The space of Hermitian orientation preserving almost complex structures on $2n-$dimensional vector space with a given positive definite inner product is the symmetric space $SO(2n)/U(n)$. If J and g are compatible the almost complex structure is called *almost Hermitian structure*. An almost Hermitian structure (g, J) must satisfy that $J^2 = -Id$ and *compatibility condition* $\omega = g(J\cdot, \cdot)$, i.e.,

$$g = \omega(\cdot, J\cdot). \tag{2}$$

Classification of all non-isometric left-invariant Riemannian metrics [5] enables us to find all almost Kähler structures on $\mathcal{CH}^n$. Every left-invariant metric on a Lie group is determined by an inner product of its Lie algebra. Therefore, the following theorem describes all non-isometric left-invariant Riemannian metrics on $\mathcal{CH}^n$.

Theorem 1.1. *[5] All positive definite inner products on the Lie algebra ch_n in some basis with commutators (1) are represented by the matrices*

$$S(p, x, \sigma, \beta) = \begin{pmatrix} p & x^T & 0 & 0 \\ x & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \tag{3}$$

where $p, \beta > 0$, $x = (x_1, \ldots, x_{n-1})^T \in \mathbb{R}^{n-1}$, $x_i \geq 0$, $\sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_{n-2}, 1)$, $\sigma_1 \geq \cdots \geq \sigma_{n-2} \geq 1$. *If all eigenvalues of σ are distinct then all inner products (3) are non-isometric. If $m_1, \ldots, m_{k+1}$ are the multiplicities of the eigenvalues of the matrix σ, i.e.,* $\sigma = \mathrm{diag}(\underbrace{\hat{\sigma}_1, \ldots, \hat{\sigma}_1}_{m_1}, \ldots \underbrace{\hat{\sigma}_k, \ldots, \hat{\sigma}_k}_{m_k}, \underbrace{1, \ldots, 1}_{m_{k+1}})$, $m_1 + \cdots + m_{k+1} = n - 1$, *then non-isometric inner products are represented by (3) with* $x = (\hat{x}_1, \underbrace{0, \ldots, 0}_{m_1-1}, \hat{x}_2, \underbrace{0, \ldots, 0}_{m_2-1} \ldots, \hat{x}_{k+1}, \underbrace{0, \ldots, 0}_{m_{k+1}-1})^T \in \mathbb{R}^{n-1}$, $\hat{x}_i \geq 0$.

2. Almost Kähler structure on $\mathcal{CH}^n$

It is difficult to decide whether $\mathcal{CH}^n$ with an arbitrary left-invariant metric g admits any compatible almost Kähler structure at all. We seek the answer by checking for the existence of an almost complex structure J satisfying condition (2), i.e., compatibility with metric and symplectic form. It turns out that only special metrics satisfying $\sigma_i = 1$, $i \in \{1, \ldots, n-1\}$ admit almost Kähler structures.

Theorem 2.1. *(Classification theorem) All left-invariant almost Kähler structures on $\mathcal{CH}^n$ are determined by the metrics represented by the inner products (3) with $\sigma_i = 1$, $\beta(p - x^T x) = 1$, $x^T = (\hat{x}, 0, \ldots, 0)$, $\hat{x} \geq 0$ and almost complex structures J represented by the matrices*

$$J = \pm \begin{pmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & -I & -x\beta \\ x & I & 0 & 0 \\ -\frac{1}{\beta} & 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

Proof. The symplectic form ω is the non-degenerate skew-symmetric bilinear form. It is represented by an antisymmetric matrix Ω. Let's denote with $(e_1, e_2, \ldots, e_{2n})$ the left-invariant basis $(X, Y_1, \ldots, Y_{n-1}, Z_1, \ldots, Z_{n-1}, W)$ of ch_n with commutators (1). In terms of exterior algebra, $\omega = \sum_{i=1}^{2n} \sum_{i<j} a_{ij} e^i \wedge e^j$ where $(e^1, e^2, \ldots, e^{2n})$ is a dual basis. The exterior derivative has the properties that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, α is a p-form and $de^i(e_j, e_k) = e^i[e_j, e_k]$, $e^i(e_j) = \delta_{ij}$, $e^i(e_j) = \delta_{ij}$.

Using the condition that the symplectic form on ch_n is closed, i.e., $d\omega = 0$, the matrix Ω becomes:

$$\Omega = \begin{pmatrix} 0 & A^T & \alpha \\ -A & -\alpha J_{n-1} & 0 \\ -\alpha & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_1^T & A_2^T & \alpha \\ -A_1 & 0 & -\alpha I_{n-1} & 0 \\ -A_2 & \alpha I_{n-1} & 0 & 0 \\ -\alpha & 0 & 0 & 0 \end{pmatrix},$$

where $A^T = (A_1^T, A_2^T) \in \mathbb{R}^{2(n-1)}$, $\alpha = a_{1(2n)} \in \mathbb{R}$.

If the metric g is given by the inner product $S(p, x, \sigma, \beta)$, the compatibility condition (2) becomes $S = \Omega J$. Together with J being an almost complex structure, i.e. $J^2 = -Id$, it imposes the following restrictions on metric: $\beta(p - x^T x) = 1$, $\sigma_i = 1$, $i \in \{1, ..., n-2\}$ and symplectic structure: $A_1 = \vec{0}$, $A_2 = -\alpha x$, $\alpha^2 = 1$. Therefore, for $\alpha = -1$ we obtain

$$J = \begin{pmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & -I & -x\beta \\ x & I & 0 & 0 \\ -\frac{1}{\beta} & 0 & 0 & 0 \end{pmatrix}.$$

For $\alpha = 1$ the almost complex structure is equal to $-J$.

Since all σ_i are equal 1, by Theorem 1.1 vector x has the simple form $x^T = (\hat{x}, 0, \ldots, 0)$, where $\hat{x} \geq 0$. □

Remark 2.2. *If the metric of the complex hyperbolic space $\mathbb{C}H^n$ is standard (Einstein), i.e., $\beta > 0$, $x = 0$, $p\beta = 1$, $\sigma_i = 1$, the almost Kähler structure is Kähler. Otherwise, for all the metrics from Theorem 2.1 the structures are strictly almost Kähler.*

Remark 2.3. *Based on the condition in [6] for nearly Kähler structure one can check that there is no nearly Kähler structure on $\mathcal{CH}^n$.*

3. Chern connection, torsion tensor and SCF-soliton

Let ∇ be the Levi-Civita connection for the metric g. The set of metric connections is an affine space modeled on the space of sections of $\Lambda^1 \otimes so(2n)$, where $so(2n)$ is the bundle of skew-symmetric endomorphisms of TM. This means that if $\overline{\nabla}$ is a metric connection, the difference $\nabla - \overline{\nabla}$ is a $so(2n)$-valued 1-form. The Hermitian connection is metric connection ($\overline{\nabla} g = 0$) satisfies the condition that $\overline{\nabla} J = 0$. The difference $\nabla - \overline{\nabla}$ decomposes into

$$\Lambda^1 \otimes so(2n) = \Lambda^1 \otimes u(n)^{\perp} \oplus \Lambda^1 \otimes u(n),$$

concerning the decomposition

$$\nabla - \overline{\nabla} = \eta + \xi,$$

where the tensor η is independent of the choice of the Hermitian connection. Let $\overline{\nabla}$ be the unique Hermitian connection such that $\xi = 0$, i.e.,

$$\nabla - \overline{\nabla} = \eta.$$

Definition 3.1. *[2] We call $\overline{\nabla}$ the **intrinsic connection**, or the canonical Hermitian connection, of (M, g, J) and η the **intrinsic torsion** of the $U(n)$-structure on the almost Hermitian manifold.*

In the literature, the intrinsic connection also is called the Chern connection. Hence, the Chern connection is the unique connection which is Hermitian (i.e., $\overline{\nabla}\omega = 0$, $\overline{\nabla} g = 0$, $\overline{\nabla} J = 0$) and its torsion satisfies $T^{1,1} = 0$. Here T denotes the torsion tensor of $\overline{\nabla}$ and $T^{1,1}$ refers to the (1,1) component of the torsion of $\overline{\nabla}$ thought of as a section of $\Lambda^2 \otimes TM$. Directly from the definitions, lengthly calculations give us the following Lemma.

Lemma 3.2. *The Chern connection $\overline{\nabla}$ and intrinsic torsion η on almost Kähler manifold $\mathcal{CH}^n$ are given by:*

$$\begin{aligned}
\overline{\nabla}_X &= \frac{\beta\hat{x}}{4}(\hat{x}(E_{11} - E_{22}) + E_{12} - pE_{21} + E_{(n+1)(2n)} - \frac{1}{\beta}E_{(2n)(n+1)}), \\
\overline{\nabla}_{Y_1} &= \frac{1}{2}(\hat{x}\beta(E_{11} - E_{22}) + \beta E_{12} - (\hat{x}^2\beta + 1)E_{21} + \beta E_{(n+1)(2n)} - E_{(2n)(n+1)}), \\
\overline{\nabla}_{Y_i} &= \frac{1}{2}(\beta E_{1(n+i)} - \hat{x}\beta E_{2(n+i)} - E_{(i+1)1} + \beta E_{(n+i)(2n)} - E_{(2n)(n+i)}), \\
\overline{\nabla}_{Z_1} &= \frac{1}{2}(\beta E_{1(n+1)} - \beta\hat{x}E_{2(n+1)} + \hat{x}\beta^2 E_{1(2n)} - (\hat{x}^2\beta^2 + \beta)E_{2(2n)} - E_{(n+1)1} + E_{(2n)2}, \\
\overline{\nabla}_{Z_i} &= \frac{1}{2}(\beta E_{1(n+i)} - \beta\hat{x}E_{2(n+i)} - \beta E_{(i+1)(2n)} - E_{(n+i)1} + E_{(2n)(i+1)}),
\end{aligned}$$

$$\begin{aligned}
\overline{\nabla}_W = \quad & \frac{1}{2}\Big(\frac{\hat{x}\beta^2}{2}E_{1(n+1)} + 2\beta^2 E_{1(2n)} - \big(\frac{\beta^2\hat{x}^2}{2}+\beta\big)E_{2(n+1)} - \frac{5\hat{x}\beta^2}{2}E_{2(2n)} + \beta\sum_{i=2}^{n-1}(E_{(n-1+i)i} - E_{(i+1)(n+i)}) + \\
& \frac{\hat{x}\beta}{2}E_{(n+1)1} + \beta E_{(n+1)2} + \big(\frac{\hat{x}^2\beta}{2}-2\big)E_{(2n)1} + \frac{\hat{x}\beta}{2}E_{(2n)2}\Big),\\
\eta_X = \quad & \frac{\beta\hat{x}}{4}\Big(\hat{x}(E_{11}-E_{22}) + E_{12} - pE_{21} - E_{(n+1)(2n)} + \frac{1}{\beta}E_{(2n)(n+1)}\Big), \qquad \eta_{Y_j} = \eta_{Z_j} = 0,\\
\eta_W = \quad & \frac{\hat{x}\beta}{4}(\beta E_{1(n+1)} - \beta\hat{x}E_{2(n+1)} + \beta E_{2(2n)} - E_{(n+1)1} - \hat{x}E_{(2n)1} - E_{(2n)2}),
\end{aligned}$$

where $i \in \{2,\dots,n-1\}$, $j \in \{1,\dots,n-1\}$ and E_{ij} is the elementary matrix.

Another way to explicitly find the intrinsic torsion is by the formula $\eta_X = \frac{1}{2}J \circ (\nabla_X J)$. We used it to double-check the previous calculations. The torsion T of a canonical connection $\overline{\nabla}$ and the intrinsic torsion η are related by $T(X,Y) = \eta_X Y - \eta_Y X$.

Cartan [3] considered eight classes of metric connections according to the algebraic type of the corresponding torsion tensors. Let us denote with the same symbol the $(3,0)$-tensor obtained from the $(2,1)$-tensor via metric: $T(X,Y,Z) = g(T(X,Y),Z)$. If we identify TM^n with $(TM^n)^*$, then the space of all possible torsion tensors is:

$$\mathcal{T} := \{T \in \otimes^3 TM^n | T(X,Y,Z) = -T(Y,X,Z)\} \cong \Lambda^2 TM^n \otimes TM^n.$$

A connection is metric if and only if η belongs to the space

$$\mathcal{A}^g := TM^n \otimes (\Lambda^2 TM^n) = \{\eta \in \otimes^3 TM^n | \eta(X,V,W) + \eta(X,W,V) = 0\}.$$

The metric connections can be characterized uniquely by their torsions [3].

Proposition 3.3. *([1, 3]) The spaces $\mathcal{T}$ and $\mathcal{A}^g$ are isomorphic as $O(n)$ representations, an equivariant bijection being $T(X,Y,Z) = \eta(X,Y,Z) - \eta(Y,X,Z)$, $2\eta(X,Y,Z) = T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y)$. For $n \geq 3$, they split under the action of $O(n)$ into the sum of three irreducible representations, $\mathcal{T} \cong TM^n \oplus \Lambda^3(M^n) \oplus \mathcal{T}'$. The last module will also be denoted $\mathcal{A}'$ if viewed as a subspace of $\mathcal{A}^g$ and is equivalent to the Cartan product of representations $TM^n \otimes \Lambda^2 TM^n$,*

$$\mathcal{T}' = \{T \in \mathcal{T} \mid \overset{X,Y,Z}{\mathfrak{S}} T(X,Y,Z) = 0, \sum_{i=1}^{n} T(X,e_i,e_i) = 0 \ \forall X,Y,Z\} \tag{5}$$

for any orthonormal frame $e_1,\dots,e_n$. For $n = 2$, $\mathcal{T} \cong \mathcal{A}^g \cong \mathbb{R}^2$ is $O(2)$-irreducible.

Metric connections whose torsions belong to exactly one of the irreducible components of the representation $\mathcal{T} \cong TM^n \oplus \Lambda^3(M^n) \oplus \mathcal{T}'$ are particularly interesting. If the torsion belongs to the TM^n component only, the connection is said to have *vectorial torsion*. If it lies only in $\Lambda^3(M^n)$ then it is said to have *skew-symmetric torsion*. In the case of Chern connection on $\mathcal{CH}^n$, torsion belongs to the third component:

Remark 3.4. *The torsion of canonical connection $\overline{\nabla}$ of every almost Kähler manifold $(\mathcal{CH}^n, g, \omega, J)$ from Theorem 2.1 belongs to the component $\mathcal{T}'$.*

This remark is a direct consequence of Theorem 8.1 from [18].

In order to search for SCF-solitons, we apply the procedure outlined in [9]. The transpose of a linear map $A : \mathfrak{g} \to \mathfrak{g}$ with respect to g and ω are respectively given by

$$g(A\cdot,\cdot) = g(\cdot, A^t\cdot), \ \ \omega(A\cdot,\cdot) = \omega(\cdot, A^{t\omega}\cdot).$$

If $p : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a bilinear map, then their *complexified* (or J-invariant) and *anti-complexified* (or anti-J-invariant) components are defined by

$$A = A^c + A^{ac}, \ \ A^c := \frac{1}{2}(A - JAJ), \ \ A^{ac} := \frac{1}{2}(A + JAJ).$$

The *Chern-Ricci form* $p = p(\omega, g)$ of a left-invariant almost-hermitian structure (ω, g, J) on a Lie group with Lie algebra $\mathfrak{g}$ is given by

$$p(X, Y) = -\frac{1}{2}\operatorname{tr}(J\operatorname{ad}[X, Y]) + \frac{1}{2}\operatorname{tr}(\operatorname{ad}J[X, Y]), \quad X, Y \in \mathfrak{g}.$$

Remarkably, p only depends on J. The *Chern-Ricci operator* P is defined by

$$p = \omega(P\cdot, \cdot).$$

Since form p is exact, there exists a $Z \in \mathfrak{g}$ such that $p(X, Y) = g([X, Y], JZ) = \omega(Z, [X, Y])$ and $P = adZ + (adZ)^{t_\omega}$.

An almost Kähler structure (ω, g) on Lie algebra $\mathfrak{g}$ is a *SCF-soliton* (i.e., symplectic curvature flow-solitons) if for some $c \in \mathbb{R}$ and $D \in Der(\mathfrak{g})$,

$$\begin{cases} P = cI + \frac{1}{2}(D - JD^tJ), \\ P^c + Ric^{ac} = cI + \frac{1}{2}(D + D^t). \end{cases} \tag{6}$$

Theorem 3.5. *The almost Kähler structure on complex hyperbolic space is SCF-soliton only if it is Kähler structure with the standard metric.*

Proof. From the Classification Theorem 2.1, we have the almost complex structure J of the form (4), inner product S of the form (3), and symplectic form ω depending on parameters $\hat{x}$ and β. We calculate Chern-Ricci form p and operator P directly from definitions. Now consider an arbitrary derivation D. After a very long computation, condition $D([X, Y]) = [D(X), Y] + [X, D(Y)]$, $X, Y \in \mathfrak{g}$ together with the system (6) give $\hat{x} = 0$. In other words, it turns out that system (6) has a solution only in the case of Kähler structure of $\mathbb{C}H^n$. □

References

[1] I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Math., 42:5, 5-84 (2006)
[2] J-B. Butruille, *Twistors and 3-symmetric spaces*, Proc. of the Lon. Math. Soc. 96, 738–766 (2008)
[3] E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (deuxième partie)*, Ann. Ec. Norm. Sup. 42 (1925), 17–88, part two. English transl. of both parts by A. Magnon and A. Ashtekar, *On manifolds with an connection and the theory of general relativity*, Napoli: Bibliopolis (1986)
[4] L.A. Cordero, M. Fernandez, M. de Leon, *Examples of compact non-Kähler almost Käler manifolds*, Proc. Amer. Math. Soc., 95, 282-286 (1985)
[5] A. Dekić, M. Babić, S. Vukmirović, *Classification of Left Invariant Riemannian metrics on Complex hyperbolic space*, Mediterr. J. Math., 19:232, (2022)
[6] A. Gray, L.M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., 123, 35-58 (1980)
[7] E. Heintze, *On Homogeneous Manifolds of Negative Curvature*, Math. Ann., 211, 23-34, (1974)
[8] W. Jelonek, *Some simple examples of almost Kählere non-Kählre strustures*, Math. Ann. 305, 639-649 (1996)
[9] J. Lauret, C. Will, *On the symplectic curvature flow for locally homogeneous manifolds*, J. Symplectic Geom., 15:1, 1-49, (2017)
[10] T. Oguro, K. Sekigawa, *Almost Kähler structures on the Riemannian product of a 3-dimensional Hyperbolic space ana a Real line*, Tsukuba J. Math., 20:1, 151-161 (1996)
[11] T. Oguro, K. Sekigawa, *Non-existence of almost Kähler structure on hyperbolic spaces of dimension* $(2n \geq 4)$, Math. Ann. 300, 317-329 (1994)
[12] Z. Olszak, *On almost cosymplectic manifolds*, Kodai Math. J. 4, 239-285 (1981)
[13] K. Sekigawa, *On some 4-dimensional compact Einstein almost Kähler manifolds*, Math. Ann. 271, 333-337 (1985)
[14] K. Sekigawa, *On some compact Einstein almost Kähler manifolds*, J. Math. Soc. Japan 39, 677-684 (1987)
[15] J. Streets, G. Tian, *Symplectic curvature flow*, J. reine angew. Math, Vol. 2014, Issue 696, 143-185
[16] S. Tachibana and M. Okumura,*On the almost complex structure of tangent bundles of Riemannian spaces*, Tôhoku Math. J. 14, 156-161, (1962)
[17] W. P. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc., (1976)
[18] F. Tricerri, L. Vanhecke, *Homogeneous Structures on Riemannian Manifolds*, London Math. Soc. Lecture Note Ser., (1983)
[19] B. Watson, *New examples of strictly almost Kähler manifolds*, Proc. Am. Math. Soc., Vol. 88, 541-544, (1983)