Filomat 37:25 (2023), 8667–8673
https://doi.org/10.2298/FIL2325667A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Modulus of continuity of normal derivative of a harmonic functions at a boundary point

Miloš Arsenović[a], Miodrag Mateljević[a]

[a]*Faculty of mathematics, University of Belgrade, Studentski Trg 16, 11000 Belgrade, Serbia*

Abstract. We give sufficient conditions which ensure that harmonic extension $u = P[f]$ to the upper half space $\{(x, y) \mid x \in \mathbb{R}^n, y > 0\}$ of a function $f \in L^p(\mathbb{R}^n)$ satisfies estimate $\frac{\partial u}{\partial y}(x, y) \le C\omega(y)/y$ for every x in $E \subset \mathbb{R}^n$, where ω is a majorant. The conditions are expressed in terms of behaviour of the Riesz transforms $R_j f$ of f near points in E. We briefly investigate related questions for the cases of harmonic and hyperbolic harmonic functions in the unit ball.

1. Introduction and preliminaries

It is easily seen that if the majorant of $\varphi(x) \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is ω, and if $u(x, y)$ is the harmonic extension of φ to the upper half space, then for each $y > 0$ the function $\varphi_y(x) = u(x, y)$ has the same majorant as φ. This need not be true for functions $\varphi_x(y) = u(x, y)$ defined on $[0, +\infty)$. For example, if φ is Lipschitz continuous it is not necessarily true that u is also Lipschitz continuous. Therefore information on the vertical derivative of u is of interest in obtaining results on modulus of continuity of $u(x, y)$. We point out that in that respect hyperbolic Laplacian has better properties regarding preservation of Lipschitz continuity, see [5]. In addition, information on behaviour of normal derivatives is relevant when studying mappings which are at the same time harmonic and quasiconformal, see [4] for the case when the boundary is not flat.

In the case $n = 1$ important role is played by the harmonic conjugate of u and by the Hilbert transform of φ. In our general case the corresponding role is played by a conjugate system of harmonic functions, see (4) and by the Riesz transforms R_j, which are multi dimensional analogues of Hilbert transform.

We denote the upper half space by $\mathbb{H}^{n+1} = \{(x, y) \mid x \in \mathbb{R}^n, y > 0\}$, the boundary of $\mathbb{H}^{n+1}$ is identified with $\mathbb{R}^n$. The surface measure of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is $n\omega_n$, where ω_n is the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. The Poisson kernel for the upper half space is

$$P(x, y) = P_y(x) = c_n \frac{y}{(y^2 + |x|^2)^{(n+1)/2}}, \qquad x \in \mathbb{R}^n, \quad y > 0$$

where

$$c_n = \Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}.$$

2020 *Mathematics Subject Classification*. Primary 42B15; Secondary 42B30.
Keywords. Poisson kernel; Harmonic functions; Modulus of conitinuity.
Received: 25 Janaury 2023; Accepted: 03 April 2023
Communicated by Zoran Rakić and Mića Stanković
Email addresses: arsenovic@matf.bg.ac.rs (Miloš Arsenović), miodrag@matf.bg.ac.rs (Miodrag Mateljević)

The harmonic extension of a function φ on $\mathbb{R}^n$ to $\mathbb{H}^{n+1}$ is

$$P[\varphi](x,y) = c_n \int_{\mathbb{R}^n} \varphi(t) \frac{y}{(y^2 + |x-t|^2)^{(n+1)/2}}\, dt$$

Let R_j, $1 \le j \le n$, be the Riesz operators. They are defined by the following formula

$$R_j f(x) = \lim_{\varepsilon \to 0} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} \int_{|y| \ge \varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)dy, \quad f \in L^p(\mathbb{R}^n), \quad 1 \le p < \infty.$$

These operators are bounded linear operators on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

We say that a function $\omega : [0, +\infty) \to \mathbb{R}$ is a majorant if it is continuous, concave and increasing on $[0, +\infty)$, strictly positive on $(0, +\infty)$ and $\omega(0) = 0$.

Lemma 1.1. *Let ω be a majorant. Set $M_\beta = (\beta-1)^{-1} + (\beta-2)^{-1}$, where $\beta > 2$. Then*

$$\int_1^\infty \frac{\omega(ys)}{s^\beta} ds \le M_\beta \omega(y), \qquad y > 0. \tag{1}$$

Proof. Set, for $C > 0$ and $t > 0$, $\omega_{C,t}(x) = Cx$ if $0 \le x \le t$ and $\omega_{C,t}(x) = Ct$ if $x > t$. Then $\omega_{C,t}$ is a majorant and we have

$$\int_1^\infty \frac{\omega_{C,t}(ys)}{s^\beta} ds = Ct \int_1^\infty \frac{ds}{s^\beta} = \frac{Ct}{\beta-1} = \frac{1}{\beta-1}\omega(y), \qquad y \ge t.$$

If $0 < y < t$, then we have

$$\begin{aligned}
\int_1^\infty \frac{\omega_{C,t}(ys)}{s^\beta} ds &= \int_1^{t/y} \frac{\omega_{C,t}(ys)}{s^\beta} ds + \int_{t/y}^\infty \frac{\omega_{C,t}(ys)}{s^\beta} ds \\
&= Cy \int_1^{t/y} \frac{ds}{s^{\beta-1}} + Ct \int_{t/y}^\infty \frac{ds}{s^\beta} \\
&= Cy \frac{1}{\beta-2}\left(1 - \frac{y^{\beta-2}}{t^{\beta-2}}\right) + Ct \frac{1}{\beta-1} \frac{y^{\beta-1}}{t^{\beta-1}} \\
&\le Cy \frac{1}{\beta-2} + Cy \frac{1}{\beta-1} \frac{y^{\beta-2}}{t^{\beta-2}} \\
&< \left(\frac{1}{\beta-2} + \frac{1}{\beta-1}\right) Cy = M_\beta \omega(y).
\end{aligned}$$

We proved (1) for $\omega = \omega_{C,t}$, clearly (1) also holds for functions of the form $\omega_{C_1,t_1} + \cdots + \omega_{C_n,t_n}$, let us call them polygonal majorants. For arbitrary majorant ω there is a an increasing sequence ω_n of polygonal majorants ω_n which converges pointwise to ω. Then, by the Monotone Convergece Theorem and already proved estimate (1) for polygonal majorants, we have

$$\int_1^\infty \frac{\omega(ys)}{s^\beta} ds = \lim_{n\to\infty} \int_1^\infty \frac{\omega_n(ys)}{s^\beta} ds \le \lim_{n\to\infty} M_\beta \omega_n(y) = M_\beta \omega(y).$$

□

The case $\beta = 3$ is the only one that we need below.

2. An auxilliary result

Lemma 2.1. *Let $\varphi \in L^p(\mathbb{R}^n)$ for some $1 \le p \le \infty$. Assume*

$$|\varphi(t) - \varphi(x^0)| \le \omega(|t - x^0|), \qquad t \in \mathbb{R}^n \tag{2}$$

for some $x^0 \in \mathbb{R}^n$ and some majorant ω. Then the harmonic extension $g = P[\varphi]$ of φ satisfies the following estimate:

$$\left| \frac{\partial g}{\partial x_j}(x^0, y) \right| \le C(n) \frac{\omega(y)}{y}, \qquad 0 < y < +\infty, \quad 1 \le j \le n. \tag{3}$$

Proof. For all $(x, y) \in \mathbb{H}^{n+1}$ and all $j = 1, \dots, n$, we have

$$\begin{aligned} \frac{\partial g}{\partial x_j}(x, y) &= c_n \int_{\mathbb{R}^n} \varphi(t) \frac{\partial}{\partial x_j} \frac{y}{(y^2 + |x - t|^2)^{(n+1)/2}} dt \\ &= -(n+1) c_n \int_{\mathbb{R}^n} \varphi(t) y \frac{x_j - t_j}{(y^2 + |x - t|^2)^{\frac{n+3}{2}}} dt \\ &= -(n+1) c_n \int_{\mathbb{R}^n} [\varphi(t) - \varphi(x)] y \frac{x_j - t_j}{(y^2 + |x - t|^2)^{\frac{n+3}{2}}} dt. \end{aligned}$$

The last equality follows from the observation that $x_j - t_j$ is an odd function of the variable $x - t$. Therefore, using spherical coordinates centered at x^0, we obtain

$$\begin{aligned} \left| \frac{\partial g}{\partial x_j}(x^0, y) \right| &\le (n+1) c_n y \int_{\mathbb{R}^n} \omega(|x^0 - t|) \frac{|x^0 - t|}{(y^2 + |x^0 - t|^2)^{\frac{n+3}{2}}} dt \\ &= n(n+1) c_n \omega_n y \int_0^\infty \omega(r) \frac{r^n \, dr}{(y^2 + r^2)^{\frac{n+3}{2}}} \\ &= n(n+1) c_n \omega_n \frac{1}{y} \int_0^\infty \frac{\omega(ys) s^n}{(1 + s^2)^{\frac{n+3}{2}}} ds \end{aligned}$$

Let us denote the above integral by $I(y)$. Then we have

$$\begin{aligned} I(y) &= \int_0^1 \frac{\omega(ys) s^n}{(1 + s^2)^{\frac{n+3}{2}}} ds + \int_1^\infty \frac{\omega(ys) s^n}{(1 + s^2)^{\frac{n+3}{2}}} ds \\ &\le \int_0^1 \frac{\omega(y) s^n}{(1 + s^2)^{\frac{n+3}{2}}} ds + \int_1^\infty \frac{\omega(ys)}{s^3} ds \\ &\le (1 + M_3) \omega(y) \end{aligned}$$

and this proves desired estimate (3). □

The proof shows that one can take $C(n) = 5n(n+1) c_n \omega_n / 2$.

3. Main result

We recall that a system of harmonic functions u_j, $0 \le j \le n$, on $\mathbb{H}^{n+1}$ is called a conjugate system if it satisfies the following system of equations

$$\sum_{j=0}^n \frac{\partial u_j}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \tag{4}$$

where $x_0 = y$, see [9]. When $n = 2$ this reduces to the classical Cauchy - Riemann equations. Given a function $\varphi = \varphi(x_0, x_1, \dots, x_n)$ harmonic in $\mathbb{H}^{n+1}$ one gets a conjugate system by setting $u_j = \partial\varphi/\partial x_j$, $0 \le j \le n$. Conversely, given a conjugate system u_j, $0 \le j \le n$ of harmonic functions in $\mathbb{H}^{n+1}$, there is a (unique up to an additive constant) function φ on $\mathbb{H}^{n+1}$ such that $u_j = \partial\varphi/\partial x_j$ for $j = 0, 1, \dots, n$.

The above system allows one to infer estimates of $\partial u_0/\partial x_0 = \partial u_0/\partial y$ from the estimates of $\partial u_j/\partial x_j$ for $1 \le j \le n$, this is how one proves the following proposition.

Proposition 3.1. *Let ω be a majorant, $E \subset \mathbb{R}^n \cong \partial\mathbb{H}^{n+1}$ and let f_j, $0 \le j \le n$, be a system of conjugate functions on $\mathbb{H}^{n+1}$. Assume*

$$\left|\frac{\partial f_j}{\partial x_j}(x, y)\right| \le \frac{\omega(y)}{y} \qquad x \in E, \quad y > 0, \quad 1 \le j \le n.$$

Then we have

$$\left|\frac{\partial f_0}{\partial y}(x, y)\right| \le n\frac{\omega(y)}{y}, \qquad x \in E, \quad y > 0.$$

Theorem 3.2. *Let $f \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$ and set $f_j = R_j f$ for $1 \le j \le n$. Let $u = P[f]$ be the harmonic extension of f to the upper half space $\mathbb{H}^{n+1}$. Let ω be a majorant and let E be a subset of $\mathbb{R}^n \cong \partial\mathbb{H}^{n+1}$. Assume*

$$|f_j(t) - f_j(x)| \le \omega(|t - x|), \qquad x \in E, \quad t \in \mathbb{R}^n, \quad 1 \le j \le n. \tag{5}$$

Then there is a constant $C = C(n)$ such that

$$\left|\frac{\partial u}{\partial y}(x, y)\right| \le C(n)\frac{\omega(y)}{y}, \qquad y > 0, \quad x \in E. \tag{6}$$

Proof. Let $u_0 = u$ and $u_j = P[f_j]$ for $1 \le j \le n$. The system $(u_j)_{j=0}^n$ is a conjugate system of harmonic functions. By Lemma 2.1 and our assumptions we have

$$\left|\frac{\partial u_j}{\partial x_j}(x, y)\right| \le C(n)\frac{\omega(y)}{y}, \quad y > 0, \quad x \in E, \quad 1 \le j \le n.$$

Now the above propostion gives the desired estimate. □

The proof shows that one can take $C = 5n^2(n+1)c_n\omega_n/2$.

4. The unit ball setting: furhter resluts and remarks

Let $P_{\mathbb{B}}$ denote harmonic Poisson kernel for the unit ball $\mathbb{B}^n$ and also the corresponding extension operator with the same kernel. Let Λ_α denote the class of Hölder continuous function with exponent $0 < \alpha \le 1$ and set Lip $= \Lambda_1$. It is known that $P_{\mathbb{B}}$ maps $\Lambda_\alpha(\mathbb{S}^{n-1})$ into $\Lambda_\alpha(\mathbb{B}^n)$ whenever $0 < \alpha < 1$. However if $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$, then in general $P[f]$ is not in $\mathrm{Lip}(\mathbb{B}^n)$.

It is natural to consider the corresponding question for the hyperbolic Poisson kernel P_h for the unit ball and the corresponding extension operator.

Problem 4.1. *Are the partial derivatives of $P_h[f]$ bounded for every f in $\mathrm{Lip}(\mathbb{S}^{n-1})$?*

The answer is positive; see [1, 5]. More precisely, if $f \in \mathrm{Lip}(\mathbb{S}^{n-1})$, then $P_h[f]$ is in $\mathrm{Lip}(\mathbb{B}^n)$. This is not true in the standard (euclidean) harmonic case.

Let us introduce needed terminology and notation. If $x_0 \in G \subset \mathbb{R}^n$ is not an isolated point of G and $0 < \alpha \le 1$ we introduce, for $f : G \to \mathbb{R}^m$,

$$H_\alpha f(x_0) = \limsup_{G \ni x \to x_0} |f(x) - f(x_0)/|x - x_0|^\alpha$$

and write $Lf(x_0)$ instead of $H_1 f(x_0)$. We say that $f : G \to \mathbb{R}^m$ is locally Hölder (α -Hölder) continuous at $x_0 \in G$ if $H_\alpha f(x_0) < \infty$, for $\alpha = 1$ we use term locally Lipschitz continuous at x_0.

The following results appears in [7]:

Theorem 4.2 ([7]). *Assume $f : \mathbb{S}^{n-1} \to \mathbb{R}^n$ is locally Lipschitz continuous at $x_0 \in \mathbb{S}$, $f \in L^\infty(\mathbb{S}^{n-1})$ and set $h = P_{\mathbb{B}}[f]$. Then*

S1)

$$|h'(rx_0)T| \le M \tag{2'}$$

for every $0 \le r < 1$ and every unit vector T tangent to $r\mathbb{S}^{n-1}$ at rx_0, where M depends only on n, $\|f\|_\infty$ and $Lf(x_0)$.
If we suppose in addition that h is K-quasiregular (shortly K-qr) mapping along $[o, x_0)$, where o denotes the origin, then

S2)

$$|h'(rx_0)| \le KM \tag{3'}$$

for every $0 \le r < 1$.

The above result extends to the case of more general moduli functions which include $\omega(\delta) = \delta^\alpha$ $(0 < \alpha \le 1)$, and therefore includes earlier results on Hölder continuity (see [8]).

Let us review some results from [7]. In the proof of the next theorem we use Poisson integral representation of harmonic functions (see formula (12) below) by the Poisson kernel on the unit ball $\mathbb{B}^n$ which is given by

$$P_{\mathbb{B}}(x, \eta) = \frac{1 - |x|^2}{n\omega_n |x - \eta|^n}, \qquad x \in \mathbb{B}^n, \quad \eta \in \mathbb{S}^{n-1}.$$

Let $d\sigma$ denote positive Borel measure on $\mathbb{S}^{n-1}$ invariant with respect to orthogonal group $O(n)$ normalized such that $\sigma(S^{n-1}) = 1$.

For convenience of the reader we prove the following proposition which appeared in [7].

Proposition 4.3 ([7]). *Suppose that $0 < \alpha < 1$ and $x = re_n$, $0 < r < 1$. Then*

$$I_\alpha(re_n) =: \int_{\mathbb{S}^{n-1}} \frac{|e_n - t|^\alpha}{|x - t|^n} d\sigma(t) \ \le \ \frac{c_{\alpha,n}}{(1 - r)^{1-\alpha}}.$$

Proof. Since the integral is a continuous function of $0 \le r < 1$, it sufices to prove the estimate under additional assumption $1/2 \le r < 1$. The integrand depends only on the angle $\theta = \angle(t, e_n)$ so we can use integration in polar coordinates on the sphere $\mathbb{S}^{n-1}$. This gives

$$I_\alpha(re_n) \le c_n \int_0^\pi \frac{|\theta|^{n-2} |\theta|^\alpha}{((1 - r)^2 + \frac{4r}{\pi^2}\theta^2)^{n/2}} \, d\theta < \tag{7}$$

$$c_n \int_0^\infty \frac{\theta^{\alpha+n-2}}{\left((1 - r)^2 + \frac{4r}{\pi^2}\,\theta^2\right)^{n/2}} \, d\theta\,. \tag{8}$$

Next using $(1 + \frac{4r}{\pi^2}u^2)^{-1} \le C(1 + u^2)^{-1}$ for $\frac{1}{2} \le r < 1$ and a change of variable $\theta = (1 - r)u$, we find

$$I_\alpha(re_n) \le C(1 - r)^{\alpha-1} \int_0^\infty \frac{u^{\alpha+n-2}}{(1 + u^2)^{n/2}} \, du\,. \tag{9}$$

Since the above improper integral is convergent the proof is completed. □

If ω is a majorant satisfying the following condition

$$\int_0^\infty \frac{u^{n-2}}{(1 + u^2)^{n/2}} \omega(\delta u) du \le C\omega(\delta), \qquad 0 < \delta \le 1, \tag{10}$$

then one proves, by a similar argument, the following proposition.

Proposition 4.4.

$$I_\omega(re_n) =: \int_{\mathbb{S}^{n-1}} \frac{\omega(|e_n - t|)}{|x-t|^n} d\sigma(t) \le c \cdot \frac{\omega(1-r)}{1-r}, \qquad 0 \le r < 1.$$

Theorem 4.5. *Let ω be a majorant satisfying condition* (10). *Assume h is continuous on $\overline{\mathbb{B}}^n$, harmonic on $\mathbb{B}^n$ and let x_0 be a point in $\mathbb{S}^{n-1}$. In addition suppose the following estimate is valid:*

$$|h(x) - h(x_0)| \le C\omega(|x - x_0|), \qquad x \in \mathbb{S}^{n-1}. \tag{11}$$

Then there is a constant $M = M_{n,\omega}$ such that

$$|h'(rx_0)| \le MC\frac{\omega(1-r)}{1-r}, \qquad 0 \le r < 1.$$

Proof. Since h is harmonic on $\mathbb{B}^n$ and continuous on $\overline{\mathbb{B}}^n$ we have

$$h(x) = \int_{\mathbb{S}^{n-1}} P_{\mathbb{B}}(x,\eta)h(\eta)d\sigma(\eta), \qquad x \in \mathbb{B}^n. \tag{12}$$

Set $d := 1 - |x|^2$. By computation $\partial_{x_k} P_{\mathbb{B}}(x,t) = -(\frac{2x_k}{|x-t|^n} + dn\frac{x_k - t_k}{|x-t|^{n+2}})$. Since $d \le 2(1-|x|) \le 2|t-x|$ for all $t \in \mathbb{S}^{n-1}$ we obtain

$$|\partial_{x_k} P_{\mathbb{B}}(x,t)| \le \frac{c_n}{|x-t|^n} \qquad x \in \mathbb{B}^n, \quad t \in \mathbb{S}^{n-1}. \tag{13}$$

We can assume $x_0 = e_n$. Let $x = re_n$ and let θ be the angle between t and e_n. Note that $s := |x-t|^2 = 1 - 2r\cos\theta + r^2$ depends only on θ for fixed x. Next , since $\int_{\mathbb{S}^{n-1}} \partial_k P_{\mathbb{B}}(x,t)h(e_n)d\sigma(t) = 0$, we find

$$\partial_{x_k} h(x) = \int_{\mathbb{S}^{n-1}} \partial_k P_{\mathbb{B}}(x,t)\big(h(t) - h(e_n)\big)d\sigma(t)\,. \tag{14}$$

Hence by (13) and the hypothesis (11) we get

$$|\partial_{x_k} h(x)| \le c_n C \int_{\mathbb{S}^{n-1}} \frac{\omega(|e_n - t|)}{|x-t|^n} d\sigma(t) \tag{15}$$

and the result follows from Propoistion 4.4. □

Remark 4.6. *It is convenient to denote expressions that appear in formulae (7) and (8) without constants by $A(r,\alpha)$ and $B(r,\alpha)$ respectively. Note that $A(r,\alpha)$ is finite for $0 \le r < 1$ and $0 < \alpha \le 1$ and that $B(r,1) = +\infty$. In order to estimate $A(r,\alpha)$ we used a change of variable $\theta = (1-r)u$ and transformed the integral over $[0,\pi]$ into integral over $[0,a(r)]$ with respect to u, where $a(r) = \pi(1-r)^{-1}$. Since $a(r) \to \infty$ as $r \to 1$, it is convenient to estimate integral $A(r,\alpha)$ by integral $B(r,\alpha)$ over interval $[0,\infty)$.*

Remark 4.7. *The above proof breaks down for $\alpha = 1$ because $B(r,1) = \infty$. Moreover, for each $n = 2$, there is a Lipschitz continuous map $f : \mathbb{S}^{n-1} \to \mathbb{R}^n$ such that $u = P_{\mathbb{B}}[f]$ is not Lipschitz continuous. In the planar case, consider $f = u + iv$ such that $zf' = -\log(1-z)$. u'_θ is bounded while its harmonic conjugate ru'_r is not bounded. In the spatial case, consider $U(x_1, x_2, ...x_n) = u(x_1 + ix_2, x_3, ...x_n)$.*

References

[1] Chen J., Huang M., Rasila A., Wang X., *On Lipschitz continuity of solutions of hyperbolic Poisson's equation,* Calc. Var. Partial Differ. Equ., **57**, no. 1, 2018, p. 1–32.

[2] Ma L., *Hölder continuity of hyperbolic Poisson integral and hyperbolic Green integral*, Monatshefte für Mathematik, **199,** 2022,

[3] M. Mateljević, *The Lower Bound for the Modulus of the Derivatives and Jacobian of Harmonic Injective Mappings.* - Filomat 29:2, 2015, 221-244.

[4] M. Mateljević, V. Božin, M. Knežević: *Quasiconformality of harmonic mappings between Jordan domains*, Filomat, Vol 24, No 3, 2010, 111-124.

[5] M. Mateljević, N. Mutavdzić: *On Lipschitz continuity and smoothness up to the boundary of solutions of hyperbolic Poisson's equation*, arXiv:2208.06197v1 [math.CV] 12 Aug 2022

[6] M. Mateljević, N. Mutavdžić, The Boundary Schwarz lemma for harmonic and pluriharmonic mappings and some generalizations, submitted for publication, accepted in Bulletin of the Malaysian Mathematical Sciences Society, June 2022

[7] Mateljević M., Salimov R. and Sevost'yanov E., *Hölder and Lipschitz continuity in Orlicz-Sobolev classes, distortion and harmonic mappings,* Filomat, **36** no. 16 p. 5361–5392, 2022 .

[8] Nodler, C.A. and D.M. Oberlin: *Moduli of continuity and a Hardy-Littlewood theorem.* - Lecture Notes in Math. 1351, p. 265-272, Springer-Verlag, Berlin etc., 1988.

[9] E. M. Stein: *Singular Integrals and Differentiability Properties of Functions*,, Princeton University Press, 1970.

[10] E. M. Stein, G. Weiss: *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, 1971.