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Existence and uniqueness results for a semilinear fuzzy fractional elliptic equation

Aziz El Ghazouani[a], Amale Talhaoui[a], M'hamed Elomari[a], Said Melliani[a]

[a]*Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, P.O. Box 523, Beni Mellal, 23000, Morocco*

Abstract. The purpose of this study is to look at a family of starting value problem for semilinear fuzzy fractional elliptic equation with fractional Caputo derivatives. Firstly, we are going to extend the definition of laplacian operator under generalized H-differentiability in the Fuzzy systems. Secondly, the fuzzy integral equation are founded. Then, the existence and uniqueness of a fuzzy solution are etablished utilizing the Banach fixed point assessment method under Lipschitz conditions. Finally, we conclude our work by a conclusion.

1. Introduction

It is common knowledge that fuzzy mathematics elegantly simulates unpredictable processes [1, 2] as well as investigated in discourse analysis, psychology, information science, choice, and other relevant industrial and applied scientific domains; This is due to its incredible versatility and usefulness (see [2]). Because there is still the potential of uncertainty in real life, fuzzy ambiguity must be considered in way to properly adapt theory to practice [3]. One of the fundamental properties of fuzzy sets is the use of membership functions over realistic facts to mitigate information loss [4].

In recent years, fraction differential operators, a type of absolute operator, have provided a larger degree of flexibility [5–8]. Caputo, as each of us aware, invented the notion of Caputo fractional derivative in 1967. A fewer-known thrut is that the notion of fractional derivative was established 20 years before Caputo by the Russian mathematician Gerasimov. As a result, it's also known as the Gerasimov-Caputo derivative [9]. Moreover, fractional order differential equations combine and properly characterize difficulties [4] and collect all function information in a weighted version [10]. Consequently, fractional-order differential equations are frequently employed in modelling viscous-elastic, chaotic, non - linear physiological functions, as well as other real world processes, particularly in explaining Memories and heredity features, and help to advance vital fields such as biochemistry and physics [11]. That is, real-world situations may be completely explained theoretically using fractional PDEs, and they might aid us in reaching more precise information [12].

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Email addresses: aziz.elghazouani@usms.ac.ma (Aziz El Ghazouani), ml.talhaoui@gmail.com (Amale Talhaoui), m.elomari@usms.ma (M'hamed Elomari), s.melliani@usms.ma (Said Melliani)

Iqbal and Niazi investigated a type of equations of Caputo's fuzzy fractional evolution in 2021 and came to several noteworthy discoveries, including the controllability and the existence and uniqueness of mild solutions (see [13, 14]). Around ten years previously, Arshad and Lupulescu [16], and Agarwal et al. [15] used a variety of techniques to demonstrate the existence and uniqueness of solutions to fuzzy FODE.

Inspired by the above studies, We explore the existence and uniqueness of fuzzy solutions to the underlying semi-linear fuzzy fractional elliptic problem in this research.

$$ {}_{gH}^{C}D_t^q u \oplus \Delta u = F(y,t,u(y,t)), \quad (y,t) \in \Omega \times J = (0,T) =: \triangle_T, \tag{1} $$

with the following conditions:

$$ \begin{cases} u(y,t) = \tilde{0}, & (y,t) \in \partial\Omega \times J, \\ u(y,0) = f(y), & y \in \Omega \\ u_t(y,0) = g(y), & y \in \Omega \end{cases} \tag{2} $$

where $\Omega \subset \mathbb{R}$ is a bounded space with a smooth boundary $\partial\Omega$, and $T > 0$ is a predetermined number. In 1, $q \in (1,2)$ is the fractional order and ${}_{gH}^{C}D_t^q$ signifies the fuzzy Caputo fractional derivative with regard to t.

The manuscript is laid out as pursuing: Following this introduction, We have offered some options related to fuzzy sets theory in section 2. we expand the concept of laplacian operator under extended H-differentiability in Section 3. In Section 4, we introduce the fuzzy nonlinear integral equation satisfied by the solution of the dilemma (1)–(2). We provide our key conclusions, the existence and uniqueness of a fuzzy solution for the semilinear fuzzy differential equation using the Banach fixed point theorem, in Section 5.

2. Preliminaries

In this section we recall fundamental tools of fuzzy theory that will be used through this research.

Let E^n represents the set of all fuzzy numbers Φ in $\mathbb{R}^n$, in particular, E^1 reflects the set of all fuzzy numbers Φ over *mathbbR*.

Definition 1. *[17]* $\Phi : \mathbb{R} \to [0,1]$ *A fuzzy membership function is alluded to as a fuzzy number if and only if the very next cases are met:*

(1) Φ *is normal. This indicates that there's a* ξ *such as* $\Phi(\xi) = 1$.

(2) Φ *is fuzzy convex.*

(3) Φ *is upper semi continuous.*

(4) $\mathrm{Supp}(\Phi) = \{\rho \in \mathbb{R} \mid \Phi(\rho) > 0\}$ *is a compact set as a support set.*

If Φ is a fuzzy number on $\mathbb{R}$, therefore, the α-cut of Φ is $[\Phi]^\alpha = \{s \in \mathbb{R} \mid \Phi(s) \geq \alpha\}$, for $\alpha \in (0,1]$.

Since $[\Phi]^\alpha$ is a compact set of all $\alpha \in [0,1]$, then we can represent $[\Phi]^\alpha$ by $[\Phi_l(\alpha), \Phi_r(\alpha)]$.

Definition 2. *[17] Assume that* Υ *and* Ψ *are two level-wise fuzzy sets. The generalized Hukuhara difference* $\Upsilon \ominus g\Psi$ *is established as described in the following:*

$$ \Upsilon \ominus_g \Psi = \varpi \Leftrightarrow \begin{cases} \Upsilon = \Psi + \varpi & (i) \\ or \quad \Psi = \Upsilon + (-1)\varpi, & (ii) \end{cases} \tag{3} $$

For the α*-levels,*

$$ \left(\Upsilon \ominus_g \Psi\right)^\alpha = [\min\{\Upsilon_l(\alpha) - \Psi_l(\alpha), \Upsilon_r(\alpha) - \Psi_r(\alpha)\}, \max\{\Upsilon_l(\alpha) - \Psi_l(\alpha), \Upsilon_r(\alpha) - \Psi_r(\alpha)\}] $$

Lemma 1. *(See [18].) If $u, v \in E^n$, thus for $\alpha \in (0,1]$,*

$$[u+v]^\alpha = \left[u_l^\alpha + v_l^\alpha, u_r^\alpha + v_r^\alpha\right],$$
$$[u \cdot v]^\alpha = \left[\min\left\{u_i^\alpha v_j^\alpha\right\}, \max\left\{u_i^\alpha v_j^\alpha\right\}\right], \quad i,j = l,r,$$
$$[u-v]^\alpha = \left[u_l^\alpha - v_r^\alpha, u_r^\alpha - v_l^\alpha\right].$$

Let ι signify an element in $\mathbb{R}^n$ and $\mathcal{S}$ symbolize a non-empty subsoace of $\mathbb{R}^n$. The distance $d(\iota, \mathcal{S})$ separating ι and $\mathcal{S}$ is given as

$$d(\iota, \mathcal{S}) = \inf\{\|\iota - \vartheta\| : \vartheta \in \mathcal{S}\}$$

Next consider $\mathcal{S}$ and $\mathcal{X}$ to be non-empty subspace of $\mathbb{R}^n$. The Hausdorff split of $\mathcal{X}$ and $\mathcal{S}$ is stated as

$$d_H^*(\mathcal{X}, \mathcal{S}) = \sup\{d(\nu, \mathcal{S}) : \nu \in \mathcal{X}\}$$

In practice,

$$d_H^*(\mathcal{S}, \mathcal{X}) \neq d_H^*(\mathcal{X}, \mathcal{S}).$$

The Hausdorff length of nonempty subsets of $\mathcal{S}$ and $\mathcal{X}$ of $\mathbb{R}^n$ is given as

$$d_H(\mathcal{S}, \mathcal{X}) = \max\left\{d_H^*(\mathcal{S}, \mathcal{X}), d_H^*(\mathcal{X}, \mathcal{S})\right\}. \tag{4}$$

It is now symmetrical in $\mathcal{S}$ and $\mathcal{X}$. Additionally,

(1) $d_H(\mathcal{S}, \mathcal{X}) \geq 0$ with $d_H(\mathcal{S}, \mathcal{X}) = 0$ iff $\bar{\mathcal{S}} = \bar{\mathcal{X}}$,

(2) $d_H(\mathcal{S}, B) = d_H(\mathcal{X}, \mathcal{S})$

(3) $d_H(\mathcal{S}, \mathcal{X}) \leq d_H(\mathcal{S}, \mathcal{Y}) + d_H(\mathcal{Y}, \mathcal{X})$

for all non-empty subsets of R^n's $\mathcal{S}, \mathcal{X}$, and $\mathcal{Y}$. The Hausdorff measure (4) is a kind of metric.

The supremum measure $d\infty$ on E^n is characterized as:

$$d_\infty(\phi, \varphi) = \sup\left\{d_H\left([\phi]^\alpha, [\varphi]^\alpha\right) : \alpha \in (0,1]\right\},$$

for all $\phi, \varphi \in E^n$, and is clearly distance on E^n.

The supremum measure H_1 on $C(J, E^n)$ is characterized as follows:

$$H_1(\psi, \Theta) = \sup\left\{d_\infty(\psi(s), \Theta(s)) : s \in J\right\},$$

for each $\psi, \Theta \in C(J, E^n)$.

Definition 3. *[19] Let $f \in L^{E^1}(J)$. The fuzzy Riemann Liouville integral of f with order $0 < q$ is stated as:*

$$I_{RL}^q f(s) = \frac{1}{\Gamma(q)} \odot \int_0^s (s-t)^{q-1} \odot f(t) ds, \tag{5}$$

where Γ is the Gamma function defined as

$$\Gamma(t) = \int_0^\infty s^{t-1} e^{-s} ds$$

Definition 4. *[20] Let $f \in L^{E^1}(J)$. The definition of Caputo gH derivative of $f(t)$ is as follow*

$$ {}_{gH}^{C}D^q f(s) = \begin{cases} \dfrac{1}{\Gamma(n-q)} \odot \displaystyle\int_0^s (s-\tau)^{n-q-1} \odot f_{gH}^{(n)}(\tau) d\tau, & n-1 < q < n \\ \left(\dfrac{d}{ds}\right)^{n-1} f(s) & , \quad q = n-1 \end{cases} \tag{6}$$

Lemma 2. *Let* $\Phi \in L^{E^1}(J)$ *and* $\forall q \in (n-1, n)$ *we gain*

$$(i) \quad {}^{C}D^{q} I^{q} \Phi(s) = \Phi(s)$$

$$(ii) \quad I^{q\,C}D^{q}\Phi(s) = \Phi(s) \ominus_{gH} \Phi(0) \ominus_{gH} (s) \odot \Phi'(0) \ominus_{gH} \cdots \ominus_{gH} \frac{(s)^{(n-1)}}{(n-1)!} \odot \Phi^{(n-1)}(0).$$

Definition 5. *[21] Let* $g : [0, \infty) \to Y \subset \mathbb{R}_{\mathcal{F}}$ *be continuous such that* $e^{-s\tau} \odot g(\tau)$ *is integrable. The fuzzy Laplace transform of* g*, indicated by* $\boldsymbol{L}[g(\tau)]$*, is therefore computed as*

$$\boldsymbol{L}[g(\tau)] := G(s) = \int_0^\infty e^{-s\tau} \odot g(\tau) d\tau, s > 0.$$

Proposition 1. *If* Φ *is a fuzzy peacewise continuous function on* $[0, \infty]$ *with exponential levels a, so*

$$\boldsymbol{L}((\Phi \star \Psi)(x)) = \boldsymbol{L}(\Phi(x)) \odot \boldsymbol{L}(\Psi(x))$$

where Ψ *is a peacewise continuous real function on* $[0, \infty)$.

Proof.

$$\begin{aligned} \boldsymbol{L}(\Phi(x)) \odot \boldsymbol{L}(\Psi(x)) &= \left(\int_0^\infty e^{-s\tau} \odot \Phi(\tau) d\tau\right) \odot \left(\int_0^\infty e^{-s\sigma} \odot \Psi(\sigma) d\sigma\right) \\ &= \int_0^\infty \left(\int_0^\infty e^{-s(\tau+\sigma)} \odot \Phi(\tau) d\tau\right) \odot \Psi(\sigma) d\sigma \end{aligned}$$

Holding τ constant in the inside integral, We gain by replacing $x = \tau + \sigma$ and $d\sigma = dt$

$$\begin{aligned} \boldsymbol{L}(\Phi(x)) \odot \boldsymbol{L}(\Psi(x)) &= \int_0^\infty \left(\int_\sigma^\infty e^{-sx} \odot \Phi(\tau) \odot \Psi(x-\tau) dx\right) d\tau \\ &= \int_0^\infty \int_\sigma^\infty e^{-sx} \odot \Phi(\tau) \odot \Psi(x-\tau) dx d\tau \\ &= \int_0^\infty e^{-sx} \odot \left(\int_0^x \Phi(x-\sigma) \odot \Psi(\sigma) d\tau\right) d\sigma \\ &= \boldsymbol{L}((\Phi \star \Psi)(x)) \end{aligned}$$

□

Proposition 2. *For every* $\alpha > 0$*, we get the subsequent outcome*

$$\int_0^t E_{\alpha,1}(As^\alpha) ds = t E_{\alpha,2}(At^\alpha)$$

Proof.

$$\begin{aligned} \int_0^t E_{\alpha,1}(As^\alpha) ds &= \int_0^t \sum_{n=0}^\infty \frac{s^{n\alpha}}{\Gamma(n\alpha+1)} A^n ds \\ &= \sum_{n=0}^\infty \frac{\int_0^t s^{n\alpha} ds}{\Gamma(n\alpha+1)} A^n \\ &= \sum_{n=0}^\infty \frac{t^{n\alpha+1}}{(n\alpha+1)\Gamma(n\alpha+1)} A^n \\ &= \sum_{n=0}^\infty \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)} A^n \\ &= t E_{\alpha,2}(At^\alpha) \end{aligned}$$

□

Lemma 3. *For all $\alpha \in [1,2]$ and $s > 0$,*
1. $s^{\alpha-1}(s^{\alpha}-A)^{-1} = \mathscr{L}(E_{\alpha,1}(At^{\alpha}))(s)$,
2. $s^{\alpha-2}(s^{\alpha}-A)^{-1} = \mathscr{L}(tE_{\alpha,2}(At^{\alpha}))(s)$,
3. $(s^{\alpha}-A)^{-1} = \frac{1}{\Gamma(\alpha-1)}\mathscr{L}\left(\int_0^t (t-s)^{\alpha-2}E_{\alpha,1}(As^{\alpha})\,ds\right)$.

Proof. 1. For $s > 0$,

$$\begin{aligned}\mathscr{L}(E_{\alpha,1}(At^{\alpha}))(s) &= \mathscr{L}\left(\sum_{n=0}^{+\infty}\frac{t^{\alpha n}A^n}{\Gamma(\alpha n+1)}\right)\\ &= \sum_{n=0}^{+\infty}\mathscr{L}(t^{\alpha n})\frac{A^n}{\Gamma(\alpha n+1)}\\ &= \sum_{n=0}^{+\infty}\frac{1}{s^{n\alpha+1}}A^n\\ &= s^{\alpha-1}(s^{\alpha}-A)^{-1}\end{aligned}$$

2. For $s > 0, s^{\alpha-1}(s^{\alpha}-A)^{-1} = \mathscr{L}(E_{\alpha,1}(At^{\alpha}))(s)$, then

$$\begin{aligned}s^{\alpha-2}(s^{\alpha}-A)^{-1} &= s^{-1}s^{\alpha-1}(s^{\alpha}-A)^{-1}\\ &= \mathscr{L}(1)(s)\mathscr{L}(E_{\alpha,1}(At^{\alpha}))(s)\\ &= \mathscr{L}(1 * E_{\alpha,1}(At^{\alpha}))(s)\\ &= \mathscr{L}\left(\int_0^t E_{\alpha,1}(At^{\alpha})\right)(s)\\ &= \mathscr{L}(tE_{\alpha,2}(t^{\alpha}A))(s)\end{aligned}$$

3. From (1), we get

$$\begin{aligned}(s^{\alpha}-A)^{-1} &= s^{1-\alpha}\mathscr{L}(E_{\alpha,1}(At^{\alpha}))(s)\\ &= \mathscr{L}\left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\right)\mathscr{L}(E_{\alpha,1}(At^{\alpha}))(s)\\ &= \mathscr{L}\left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * E_{\alpha,1}(At^{\alpha})\right)(s)\\ &= \mathscr{L}\left(\int_0^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}E_{\alpha,1}(A\delta^{\alpha})\,d\delta\right)(s),\end{aligned}$$

As a consequence, the intended goal. □

Lemma 4. *[21]*
(1) Allow $\varphi, \phi : [0,\infty) \to Y \subset \mathbb{R}_{\mathcal{F}}$ to be continuous functions, $c_1, c_2 \in \mathbb{R}^+$. Therefore

$$\mathcal{L}\Big[c_1 \odot \varphi(s) + c_2 \odot \phi(s)\Big] = c_1 \odot \mathcal{L}[\varphi(s)] + c_2 \odot \mathcal{L}[\phi(s)]$$

(2) Allow $\varphi : [0,\infty) \to Y \subset \mathbb{R}_{\mathcal{F}}$ to be a continuous function. Thus

$$\mathcal{L}\left[e^{as} \odot \varphi(t)\right] = \boldsymbol{\varphi}(t-c), t-c > 0.$$

Definition 6. *[22] Assume $\mathcal{G}$ to be a vector set in $\mathbb{R}$. A fuzzy inner product on $\mathcal{G}$ is a mapping $\langle .,. \rangle : \mathcal{G} \times \mathcal{G} \longrightarrow E^1$ such as $\forall \Theta, \Upsilon, \Xi \in \mathcal{G}$ and $\lambda \in \mathbb{R}$, we obtain:*

1) $\langle \Theta + \Upsilon, \Xi \rangle = \langle \Theta, \Xi \rangle \oplus \langle \Upsilon, \Xi \rangle$,

2) $\langle \lambda\Theta, \Upsilon \rangle = \tilde{\lambda}\langle \Theta, \Upsilon \rangle$,

3) $\langle\Theta,\Upsilon\rangle = \langle\Upsilon,\Theta\rangle$,

4) $\langle\Theta,\Theta\rangle \geq \tilde{0}$,

5) $\inf_{\alpha\in(0,1]}\langle\Theta,\Theta\rangle_\alpha^- > 0$ *if* $\Theta \neq 0$,

6) $\langle\Theta,\Theta\rangle = \tilde{0}$ *iff* $\Theta = 0$.

A fuzzy inner product set is a vector space $\mathcal{G}$ *that admire a fuzzy inner product.*

3. Fuzzy Laplacian operator

In this section, we are going to extend the definition of laplacian operator under generalized H-differentiability in the fuzzy theory.

Definition 7. *[23] Take* $\Phi : J \to \mathbb{R}_{\mathcal{F}}$ *and fix* $s_0 \in J$. *We say* Φ *is (i)-differentiable at* s_0, *if there's an element* $\Phi'(s_0) \in \mathbb{R}_{\mathcal{F}}$ *such as* $\forall h > 0$ *enough close to 0 , exist* $\Phi(s_0+h)\ominus\Phi(s_0), \Phi(s_0)\ominus\Phi(s_0-h)$ *and the limits*

$$\lim_{h\to0^+}\frac{\Phi(s_0+h)\ominus\Phi(s_0)}{h} = \lim_{h\to0^+}\frac{\Phi(s_0)\ominus\Phi(s_0-h)}{h} = \Phi'(t_0).$$

In this case we denote $\Phi'(s_0)$ *by* $\partial_1^1\Phi(s_0)$. *And* Φ *is (ii)-differentiable if for all* $h > 0$ *enough close to 0 , exist* $\Phi(s_0+h)\ominus\Phi(s_0), \Phi(s_0)\ominus\Phi(s_0-h)$ *and the limits*

$$\lim_{h\to0^+}\frac{\Phi(s_0)\ominus\Phi(s_0+h)}{-h} = \lim_{h\to0^+}\frac{\Phi(s_0-h)\ominus\Phi(s_0)}{-h} = \Phi'(t_0).$$

This derivative is indicated in this instance by $\partial_2^1\Phi(s_0)$.

Here we remember certain notions and proofs for the initial order derivative [23] and 2nd order derivatives [24] dependent on the selection of derivative kind in every stage of differentiating.

Theorem 1. *Allow* $\Phi : J \to \mathbb{R}_F$ *be fuzzy function, where* $[\Phi(s)]^\beta = [\Phi_l(s,\beta), \Phi_r(s,\beta)]$ *for any* $\beta \in [0,1]$. *Then*

(1) If Φ *is (i)-differentiable thus* $\Phi_l(s,\beta)$ *and* $\Phi_l(s,\beta)$ *are differentiable functions and* $\left[\partial_{(i)}^1\Phi(s)\right]^\beta = \left[\Phi_l'(s,\beta), \Phi_r'(s,\beta)\right]$.

(2) If Φ *is (ii)-differentiable thus* $\Phi_l(s,\beta)$ *and* $\Phi_l(s,\beta)$ *are differentiable functions and* $\left[\partial_{(ii)}^1\Phi(s)\right]^\beta = \left[\Phi_r'(s,\beta), \Phi_l'(s,\beta)\right]$.

Proof. See [23]. □

Definition 8. *Allow* $\Phi : J \to \mathbb{R}_{\mathcal{F}}$ *and* $n, m = (i), (ii)$. *We declare* Φ *is* (n,m)*-differentiable at* $s_0 \in J$, *if* $\partial_n^1\Phi$ *occur in the near of* s_0 *as a fuzzy function and it's* (m)*-differentiable at* s_0. *The 2nd derivatives of* Φ *is noted by* $\partial_{n,m}^2\Phi(t_0)$ *for* $n, m = (i), (ii)$.

According to Definition 7 we have:

Theorem 2. *Let* $\partial_{(i)}^1\Phi : J \to \mathbb{R}_{\mathcal{F}}$ *or* $\partial_{(ii)}^1\Phi : J \to \mathbb{R}_{\mathcal{F}}$ *be fuzzy functions, with* $[\Phi(s)]^\beta = [\Phi_l(s,\beta), \Phi_r(s,\beta)]$. *Then*

(1) if $\partial_{(i)}^1\Phi$ *is (i)-differentiable, then* $\Phi_l'(s,\beta)$ *and* $\Phi_r'(s,\beta)$ *are differentiable functions and* $\left[\partial_{(i),(i)}^2\Phi(s)\right]^\beta = \left[\Phi_l''(s,\beta), \Phi_r''(s,\beta)\right]$.

(2) if $\partial_{(i)}^1\Phi$ *is (ii)-differentiable, then* $\Phi_l'(s,\beta)$ *and* $\Phi_r'(s,\beta)$ *are differentiable functions and* $\left[\partial_{(i),(ii)}^2\Phi(s)\right]^\beta = \left[\Phi_r''(s,\beta), \Phi_l''(s,\beta)\right]$.

(3) if $\partial_{(ii)}^1\Phi$ *is (i)-differentiable, then* $\Phi_l'(s,\beta)$ *and* $\Phi_r'(s,\beta)$ *are differentiable functions and* $\left[\partial_{(ii),(i)}^2\Phi(s)\right]^\beta = \left[\Phi_r''(s,\beta), \Phi_l''(s,\beta)\right]$.

(4) if $\partial_{(ii)}^1\Phi$ *is (ii)-differentiable, then* $\Phi_l'(s,\beta)$ *and* $\Phi_l'(s,\beta)$ *are differentiable functions and* $\left[\partial_{(ii),(ii)}^2\Phi(s)\right]^\beta = \left[\Phi_l''(s,\beta), \Phi_r''(s,\beta)\right]$.

Proof. See the Proof of the Theorem 3.9 in [24]. □

Definition 9. *The fuzzy Laplace operator of Φ is the summation of all the fuzzy 2nd partial derivatives in Cartesian coordinate system t_j:*

$$\Delta\Phi = \sum_{j=1}^{n} \frac{\partial^2 \Phi}{\partial t_j^2}$$

The fuzzy Laplace operator, as a 2nd fuzzy differential operator, transfers C^k fuzzy functions to C^{k-2} fuzzy functions with $k \geq 2$. It is a linear operator $\mathbf{\Delta} : C^k(\mathbf{R}^n) \to C^{k-2}(\mathbf{R}^n)$, or more broadly, an operator $\mathbf{\Delta} : C^k(\Omega) \to C^{k-2}(\Omega)$ for every open set $\Omega \subseteq \mathbf{R}^n$.

Theorem 3. *Let Φ and Φ' be differentiable fuzzy value functions, and if α-cut representation of f is denoted by $[\Phi]^\alpha = \left[f\Phi_l^\alpha, \Phi_r^\alpha\right]$, then the fuzzy Laplacian operator denoted $\mathbf{\Delta}$ defined as*

(1) if $\partial^1_{(i)}\Phi$ is (i)-differentiable or if $\partial^1_{(ii)}\Phi$ is (ii)-differentiable, thus $\Phi_l'(.,\alpha)$ and $\Phi_r'(.,\alpha)$ are differentiable functions and $[\mathbf{\Delta}\Phi]^\alpha = \left[\Delta\Phi_l^\alpha, \Delta\Phi_r^\alpha\right]$.

(2) if $\partial^1_{(i)}\Phi$ is (ii)-differentiable or if $\partial^1_{(ii)}\Phi$ is (i)-differentiable, thus $\Phi_l'(.,\alpha)$ and $\Phi_r'(.,\alpha)$ are differentiable functions and $[\mathbf{\Delta}\Phi]^\alpha = \left[\Delta\Phi_r^\alpha, \Delta\Phi_l^\alpha\right]$.

where Δ is the usuel laplacian operator.

Proof. We just offer the specifics for scenario (1) because the other situations are comparable.

If $h > 0$ and $\alpha \in [0, 1]$, we obtain

$$\left[\partial_1^{(1)}\Phi(t_j + h) \ominus \partial_1^{(1)}\Phi(t_j)\right]^\alpha = \left[\Phi_l'(t_j + h, \alpha) - \Phi_l'(t_j, \alpha), \Phi_r'(t_j + h, \alpha) - \Phi_r'(t_j, \alpha)\right],$$

and then multiply by $1/h$, we get

$$\frac{\left[\partial_1^{(1)}\Phi(t_j + h) \ominus \partial_1^{(1)}\Phi(t_j)\right]^\alpha}{h} = \left[\frac{\Phi_l'(t_j + h, \alpha) - \Phi_l'(t_j, \alpha)}{h}, \frac{\Phi_r'(t_j + h, \alpha) - \Phi_r'(t_j, \alpha)}{h}\right].$$

Similarly, we obtain

$$\frac{\left[\partial_1^{(1)}\Phi(t_j) \ominus \partial_1^{(1)}\Phi(t_j - h)\right]^\alpha}{h} = \left[\frac{\Phi_l'(t_j, \alpha) - \Phi_l'(t_j - h, \alpha)}{h}, \frac{\Phi_r'(t_j, \alpha) - \Phi_r'(t_j - h, \alpha)}{h}\right].$$

Getting to the limits, we gain

$$\left[\partial_{1,1}^{(2)}\Phi(t_j)\right]^\alpha = \left[\partial^{(2)}\Phi_l(t_j, \alpha), \partial^{(2)}\Phi_r(t_j, \alpha)\right].$$

by applying the sum, we get

$$\left[\sum_{j=1}^{n} \partial_{1,1}^{(2)}\Phi(t_j)\right]^\alpha = \left[\sum_{j=1}^{n} \partial^{(2)}\Phi_l(t_j, \alpha), \sum_{j=1}^{n} \partial^{(2)}\Phi_r(t_j, \alpha)\right].$$

therefore,

$$[\mathbf{\Delta}\Phi]^\alpha = \left[\Delta\Phi_l^\alpha, \Delta\Phi_r^\alpha\right].$$

This concludes the theorem's demonstration. □

4. The fuzzy integral equation

Consider the following fuzzy eigenvalue problem for the fuzzy Laplacian on a bounded domain Ω.

$$\begin{cases} \ominus\boldsymbol{\Delta}\phi_j(y) = \lambda_j \odot \phi_j(y), & y \in \Omega \\ \phi_j(y) = \tilde{0}, & y \in \partial\Omega, \end{cases}$$

where $\tilde{0}$ is a fuzzy number. Then, the above equation is expanded in accordance with its left and right functions as follows:

$$\left(-\Delta\phi_{j,l}(y), -\Delta\phi_{j,r}(y)\right) = \lambda_j \odot \left(\phi_{j,l}(y), \phi_{j,r}(y)\right)$$
$$\left(\phi_{j,l}(y), \phi_{j,r}(y)\right) = (0,0)\,,$$

Now, we look at these equations according to the two following cases. The equation with lower functions is

$$\begin{cases} -\Delta\phi_{j,l}(y,\alpha) = \lambda_j\phi_{j,l}(y,\alpha), & y \in \Omega \\ \phi_{j,l}(y) = 0, & y \in \partial\Omega, \end{cases} \tag{7}$$

and with upper functions is

$$\begin{cases} -\Delta\phi_{j,r}(y,\alpha) = \lambda_j\phi_{j,r}(y,\alpha), & y \in \Omega \\ \phi_{j,r}(y) = 0, & y \in \partial\Omega, \end{cases} \tag{8}$$

The boundary value problems (7) and (8) is the Dirichlet problems for the Helmholtz system, and thus λ_j is classified as a Dirichlet eigen-value for Ω.

By using the theorem of compact self-adjoint operator spectral one can demonstrate that the eigen-spaces are size limitations and that the Dirichlet eigen-values λ_j are real, positive, and unbounded. As a result, they may be ordered in ascending order :

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \cdots \ \texttt{and}\ \lambda_j \to \infty \ \texttt{as}\ j \to \infty.$$

Consider the related eigen-functions $\phi_j \in H_0^1(\Omega)$.

Next, assume that the difficulty (1) has a solution u in the kind

$$u(y,t) = \sum_{j=1}^{\infty} u_j(t) \odot \phi_j(y).$$

Thus, $u_j(t)$ solves the pursuing fuzzy fractional ordinary differential equation with conditions:

$$\begin{cases} {}_{gH}^{C}D_t^q u_j \ominus \lambda_j \odot u_j(t) & = \left\langle F(y,t,u(y,t)), \phi_j \right\rangle, \quad t \in J, \\ u_j(0) & = \left\langle f, \phi_j \right\rangle, \\ \frac{du_j}{dt}(0) & = \left\langle g, \phi_j \right\rangle, \end{cases} \tag{9}$$

where $\langle\cdot,\cdot\rangle$ indicate the fuzzy inner product in $L^2(\Omega)$. By using the lemmas (2), (3) and the proposition (1), we gain the solution of (9) as below:

(i) If u is Caputo $(i) - gH$ differentiable, then

$$\begin{aligned} u_j(t) =& E_{q,1}\left(\lambda_j \odot t^q\right) \odot \left\langle f, \phi_j \right\rangle \oplus t \odot E_{q,2}\left(\lambda_j \odot t^q\right) \odot \left\langle g, \phi_j \right\rangle \\ &\oplus \int_0^t \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(\lambda_j \odot (\tau - s)^q\right) \odot \left\langle F(\cdot, s, u(\cdot,s)), \phi_j \right\rangle d\tau ds \end{aligned} \tag{10}$$

(ii) If u is Caputo $(ii)-gH$ differentiable, therefore

$$\begin{aligned} u_j(t) =& E_{q,1}\left(\lambda_j \odot t^q\right) \odot \left\langle f, \phi_j \right\rangle \oplus t \odot E_{q,2}\left(\lambda_j \odot t^q\right) \odot \left\langle g, \phi_j \right\rangle \\ & \ominus (-1) \odot \int_0^t \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(\lambda_j \odot (\tau - s)^q\right) \odot \left\langle F(\cdot, s, u(\cdot, s)), \phi_j \right\rangle d\tau ds \end{aligned} \tag{11}$$

therefore, the problem (1) solution is as follow

(i) If u is Caputo $(i)-gH$ differentiable, then

$$\begin{aligned} u(y,t) =& \sum_{j=1}^{\infty} \left[E_{q,1}\left(\lambda_j \odot t^q\right) \odot \left\langle f, \phi_j \right\rangle \oplus t \odot E_{q,2}\left(\lambda_j \odot t^q\right) \odot \left\langle g, \phi_j \right\rangle\right] \odot \phi_j(y) \\ & \oplus \sum_{j=1}^{\infty} \left[\int_0^t \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(\lambda_j \odot (\tau - s)^q\right) \odot \left\langle F(y, s, u(y, s)), \phi_j \right\rangle d\tau ds\right] \odot \phi_j(y). \end{aligned} \tag{12}$$

(ii) If u is Caputo $(ii)-gH$ differentiable, then

$$\begin{aligned} u(y,t) =& \sum_{j=1}^{\infty} \left[E_{q,1}\left(\lambda_j \odot t^q\right) \odot \left\langle f, \phi_j \right\rangle \oplus t \odot E_{q,2}\left(\lambda_j \odot t^q\right) \odot \left\langle g, \phi_j \right\rangle\right] \odot \phi_j(y) \\ & \ominus (-1) \odot \sum_{j=1}^{\infty} \left[\int_0^t \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(\lambda_j \odot (\tau - s)^q\right) \odot \left\langle F(y, s, u(y, s)), \phi_j \right\rangle d\tau ds\right] \odot \phi_j(y). \end{aligned} \tag{13}$$

5. Existence and uniqueness results

Consider the points that follow.

(H1) The inhomogeneous term $F : \triangle_T \times C(\triangle_T, L^2(\triangle_T)) \to C(\triangle_T, L^2(\triangle_T))$ is a continuous function that meets the globally Lipschitz criterion

$$d_H\left([f(y,s,\xi(y,s)]^\alpha, [f(y,s,\zeta(y,s)]^\alpha\right) \leq K d_H\left([\xi(y,s)]^\alpha, [\zeta(y,s)]^\alpha\right),$$

for all $\xi(y,s), \zeta(y,s) \in C(\triangle_T, L^2(\triangle_T))$, and a constant $K > 0$.

(H2) $Q(t)$ is a fuzzy set that is appropriate for $u \in C\left(\triangle_T; L^2(\triangle_T)\right)$, the equation

$$Q(t-s)F(u)(y,s) := \sum_{j=1}^{\infty} \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(\lambda_j \odot (\tau - s)^q\right) \odot \left\langle F(u)(y,s), \phi_j \right\rangle \odot \phi_j(y) d\tau$$

such as

$$[Q(t)]^\alpha = \left[Q_l^\alpha(t), Q_r^\alpha(t)\right],$$

and $Q_i^\alpha(t) (i = l, r)$ is continuous. That is, there's a constant $M > 0$ such as $\left|Q_i^\alpha(t)\right| \leq M \quad \forall t \in J$

Theorem 4. *Asume that assumptions (H1)-(H2) are correct. Thus, for any $f, g \in L^2(\Omega)$, the fuzzy initial value problem* (1), (2) *has a unique solution $u \in C\left(\triangle_T; L^2(\triangle_T)\right)$.*

Proof. Denote

$$\mathcal{S}(t)f := \sum_{j=1}^{\infty} E_{\alpha,1}\left(\lambda_j \odot t^q\right) \odot \left\langle f, \phi_j \right\rangle \odot \phi_j, \quad \mathcal{P}(t)g := \sum_{j=1}^{\infty} t \odot E_{\alpha,2}\left(\lambda_j \odot t^q\right) \odot \left\langle g, \phi_j \right\rangle \odot \phi_j,$$

The solutions is then defined as meeting the equation as

$$u(y,t) = \mathcal{S}(t)f(y) \oplus \mathcal{P}(t)g(y) \oplus \int_0^t \mathcal{Q}(t-s)F(u)(y,s)ds \tag{14}$$

where

$$F(u)(y,s) := F(y,s,u(y,s)),$$

$$\mathcal{Q}(t-s)F(u)(y,s) := \sum_{j=1}^{\infty} \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(\lambda_j \odot (\tau - s)^q\right) \odot \left\langle F(u)(y,s), \phi_j \right\rangle \odot \phi_j(y) d\tau.$$

For each $\xi(t) \in C(\triangle_T, L^2(\triangle_T)), t, y \in \triangle_T$ define

$$(\Phi\xi)(y,t) = \mathcal{S}(t)f(y) \oplus \mathcal{P}(t)g(y) \oplus \int_0^t \mathcal{Q}(t-s)F(u)(y,s)ds.$$

Thus, $(\Phi\xi)(y,t) : \triangle_T \to C(\triangle_T, L^2(\triangle_T))$ is continuous in regard to t, and $\Phi : C\left(\triangle_T; L^2(\triangle_T)\right) \to C\left(\triangle_T; L^2(\triangle_T)\right)$. It is obvious that fixed points of Φ are solutions to the initial value problem (1),(2). For $\xi(t), \zeta(t) \in C\left(\triangle_T; L^2(\triangle_T)\right)$, we have

$$\begin{aligned}
& d_H\left([(\Phi\xi)(y,t)]^\alpha, [(\Phi\zeta)(y,t)]^\alpha\right) \\
&= d_H\left(\left[\mathcal{S}(t)f(y) \oplus \mathcal{P}(t)g(y) \oplus \int_0^t \mathcal{Q}(t-s)F(\xi)(y,s)ds\right]^\alpha,\right. \\
&\left.\left[\mathcal{S}(t)f(y) \oplus \mathcal{P}(t)g(y) \oplus \int_0^t \mathcal{Q}(t-s)F(\zeta)(y,s)ds\right]^\alpha\right) \\
&\le d_H\left([\mathcal{S}(t)f(y)]^\alpha \oplus [\mathcal{P}(t)g(y)]^\alpha \oplus \left[\int_0^t \mathcal{Q}(t-s)F(\xi)(y,s)ds\right]^\alpha,\right. \\
&\left.[\mathcal{S}(t)f(y)]^\alpha \oplus [\mathcal{P}(t)g(y)]^\alpha \oplus \left[\int_0^t \mathcal{Q}(t-s)F(\zeta)(y,s)ds\right]^\alpha\right) \\
&\le d_H\left(\left[\int_0^t \mathcal{Q}(t-s)F(\xi)(y,s)ds\right]^\alpha, \left[\int_0^t \mathcal{Q}(t-s)F(\zeta)(y,s)ds\right]^\alpha\right) \\
&\le \int_0^t d_H\left(\left[\mathcal{Q}_l^\alpha(t-s)F_l^\alpha(\xi)(y,s), \mathcal{Q}_r^\alpha(t-s)F_r^\alpha(\xi)(y,s)\right], \left[\mathcal{Q}_l^\alpha(t-s)F_l^\alpha(\zeta)(y,s), \mathcal{Q}_r^\alpha(t-s)F_r^\alpha(\zeta)(y,s)\right]\right)ds \\
&\le \int_0^t \max\left(|\, \mathcal{Q}_l^\alpha(t-s)\left[F_l^\alpha(\xi)(y,s) - F_l^\alpha(\zeta)(y,s)\right] |, |\, \mathcal{Q}_r^\alpha(t-s)\left[F_r^\alpha(\xi)(y,s) - F_r^\alpha(\zeta)(y,s)\right] |\right)ds \\
&\le M \int_0^t \max\left(|\left[F_l^\alpha(\xi)(y,s) - F_l^\alpha(\zeta)(y,s)\right]|, |\left[F_r^\alpha(\xi)(y,s) - F_r^\alpha(\zeta)(y,s)\right]|\right)ds \\
&\le M \int_0^t \max\left(\left[F_l^\alpha(\xi)(y,s), F_r^\alpha(\xi)(y,s)\right], \left[F_l^\alpha(\zeta)(y,s), F_r^\alpha(\zeta)(y,s)\right]\right)ds \\
&\le M \int_0^t d_H\left([F(\xi)(y,s)]^\alpha, [F(\zeta)(y,s)]^\alpha\right)ds \\
&\le MK \int_0^t d_H\left([\xi(y,s)]^\alpha, [\zeta(y,s)]^\alpha\right)ds
\end{aligned}$$

Thus,

$$\begin{aligned} d_\infty((\Phi\xi)(y,t),(\Phi\zeta)(y,t)) &= \sup_{\alpha\in(0,1]} d_H\left([(\Phi\xi)(y,t)]^\alpha,[(\Phi\zeta)(y,t)]^\alpha\right) \\ &\le MK\int_0^t \sup_{\alpha\in(0,1]} d_H\left([\xi(y,s)]^\alpha,[\zeta(y,s)]^\alpha\right) ds \\ &= MK\int_0^t d_\infty(\xi(y,s),\zeta(y,s))ds. \end{aligned}$$

Consequently,

$$\begin{aligned} H_1(\Phi\xi,\Phi\zeta) &= \sup_{t\in J} d_\infty((\Phi\xi)(y,t),(\Phi\zeta)(y,t)) \\ &\le MK\sup_{t\in J}\int_0^t d_\infty(\xi(y,s),\zeta(y,s))ds \\ &\le MKTH_1(\xi(y,s),\zeta(y,s)). \end{aligned}$$

Pick T so that $T < \frac{1}{MK}$. Hence, Φ is a contraction mapp. According to the Banach fixed point theorem, the semilinear fuzzy fractional elliptic equation has a unique fixed point $u \in C\left(\triangle_T; L^2(\triangle_T)\right)$. $\square$

6. Conclusion

The goal of this study is to look at a family of starting value issues for semilinear fuzzy fractional elliptic equations with fractional Caputo derivatives. Before that, In the fuzzy theory, we will broaden the definition of the laplacian operator under extended H-differentiability. The fuzzy integral equation is first established, and then the existence and uniqueness of a fuzzy solution are established by utilising the Banach fixed point assessment technique under Lipschitz conditions.

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