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Some extended fractional integral inequalities with applications

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Abstract. Here, an extended fractional integral identity has been established to construct some extended Simpson-type inequalities for differentiable convex functions and differentiable concave functions connected to Hermite-Hadamard inequality. Some applications to means, $f-$divergence measure, probability density function and approximate error to some quadrature rules are given.

1. Introduction

Fractional calculus, as a very useful tool, shows its significance to implement differentiation and integration of real or complex number orders. This topic has attracted much attention from researchers who focus on the study of partial differential equations during the last few decades. The fractional integral provides several functional tools for various problems involving special functions of mathematical science as well as their extensions and generalizations in one and more variables [3, 4, 14, 17, 26, 27]. For recent results related to this subject, we refer to some studies by Sohail et al. [32], Hameed et al. [13], and Khan et al. [18, 19]. Among a lot of the fractional integral operators growed, the Riemann Liouville fractional integral operator has been extensively studied, because of applications in many fields of sciences, such as differential equations, differential geometry and physics science. An important generalization of Riemann Liouville fractional integrals was considered by Raina [31]. One of the important applications of fractional integrals is Hermite-Hadamard integral inequality [16, 34]. It provides a lower and an upper estimation for the integral average of any convex function defined on a closed interval, involving the midpoint and the endpoints of the domain. The following inequality is known as classical Hermite-Hadamard inequality [12] for a convex function f on a real interval I

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2},$$

for $a, b \in I$. Hadamard's inequality is not merely a consequence of convexity, but also characterizes it if a continuous function satisfies either its left or its right hand side on any subinterval of the domain, then the function is convex. The Hermite-Hadamard's inequality is significant in its rich geometry and hence there are many studies on it to demonstrate its new proofs, refinements, extensions and generalizations [1, 2, 9, 21, 23–25, 28, 29]. Integral inequalities have played an important role to calculate error of some quadrature rules. For instance, Simpson's inequality [15, 20] provides an error bound for the Simpson's

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rule [8]. Midpoint inequality provides an error bound for the midpoint rule [6] and Trapezoidal inequality provides an error bound for the trapezoidal rule [7]. The Simpson type inequalities have been the subject of intensive research since many important inequalities can be obtained from the Simpson inequality. The following inequality is well known as Simpson's inequality.

Theorem 1.1. *[8] Let $f : [a, b] \to \mathbb{R}$ be a four times differentiable function on (a, b) and $\|f^{(4)}\|_\infty := \sup_{x\in(a,b)} |f^{(4)}|(x) < \infty$, then the following inequality holds*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{\|f^{(4)}\|_\infty (b-a)^4}{2880}.$$

The only drawback of Simpson's inequality was it can not be applied on derivatives of less than fourth order but Dragomir reduced this difficulty by expressing the Simpson's inequality in terms of derivatives lower than the fourth one.

Theorem 1.2. *[10] Suppose $f : [a, b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a, b) and $f' \in L[a, b]$, then*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{3}\|f'\|_1$$

provided that: $\|f'\|_1 = \int_a^b |f'(x)|dx < \infty$.

Theorem 1.3. *[11] Let $f : [a; b] \to \mathbf{R}$ be a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{8}[|f'(a)|+|f'(b)|]$$

which is the trapezoid inequality provided that $|f'|$ is convex on $[a, b]$.

Theorem 1.4. *[11] Let $f : [a, b] \to \mathbf{R}$ be a differentiable function on (a, b) and $q \ge 1$. If the mapping $|f'|^q$ is convex on $[a, b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\sqrt[q]{\frac{|f'(a)|^q+|f'(b)|^q}{2}}$$

which is the midpoint inequality provided $|f'|^q$ is convex on $[a, b]$.

The main purpose of this paper is to establish some extended fractional Simpson's type inequality, providing some applications to some special means, quadrature rules, $f-$divergence measures and probability density. This paper is organized in the following way: After this Introduction in Section 2 some basic concepts and assumptions are discussed, in Section 3 main results relating to the topic and in Section 4 applications of the derived results are discussed.

2. Preliminaries and Assumptions

Definition 2.1. *[22] Let $I \subseteq (0, \infty)$ be a real interval and $0 \ne p \in \mathbf{R}$. A function $f : I \to \mathbf{R}$ is said to be $p-$convex function, if*

$$f\left(\sqrt[p]{tx^p+(1-t)y^p}\right) \le tf(x)+(1-t)f(y),$$

provided that $x, y \in I$ and $t \in [0, 1]$. If the inequality is reversed, the f is said to be $p-$concave function. The function, $f : (0, \infty) \to (0, \infty)$ defined by $f(x) = x^p$ for $p > 0$, is $p-$convex.

Definition 2.2. *[29] Let $[a, b]$ be a finite interval on the real axis and $f \in [a, b]$. The right-hand side and the left-hand side Riemann-Liouville fractional integrals $\mathcal{J}^{\alpha}_{a+}f$ and $\mathcal{J}^{\alpha}_{b-}f$ of order $\alpha > 0$, respectively, are defined by:*

$$\left(\mathcal{J}^{\alpha}_{a+}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ x > a, \tag{1}$$

$$\left(\mathcal{J}^{\alpha}_{b-}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ x < b. \tag{2}$$

Raina [31] introduced a class of functions as follows:

$$\mathfrak{F}^{\sigma}_{\rho,\lambda}(x) = \mathfrak{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k,\ \ \rho,\lambda \in \mathbf{R}^+; |x| < \mathbf{R}, \tag{3}$$

where the coefficients $\sigma(k) \in \mathbf{R}^+$, $k \in \mathbb{N}_0$ form a bounded sequence. By using (3) Raina and Agarwal et al. [5, 31] defined, respectively, the left-side and right-sided fractional integral operators:

$$\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,a+;w}\phi\right)(x) = \int_a^x (x-t)^{\lambda-1}\mathfrak{F}^{\sigma}_{\rho,\lambda}[w(x-t)^{\rho}]\phi(t)dt,\ x > a, \tag{4}$$

$$\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,b-;w}\phi\right)(x) = \int_x^b (t-x)^{\lambda-1}\mathfrak{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\phi(t)dt,\ x < b, \tag{5}$$

where $w \in \mathbf{R}$ and ϕ is a function such that the integrals on right hand sides exit. It is easy to verify that $\mathfrak{J}^{\sigma}_{\rho,\lambda,a+;w}\phi(x)$ and $\mathfrak{J}^{\sigma}_{\rho,\lambda,b-;w}\phi(x)$ are bounded integral operators on $L(a, b)$, provided that $\mathfrak{M} := \mathfrak{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] < \infty$. In fact, for $\phi \in L(a, b)$, we have

$$\left\|\mathfrak{J}^{\sigma}_{\rho,\lambda,a+;w}\phi\right\|_1 \le \mathfrak{M}(b-a)^{\lambda}\|\phi\|_1;\ \ \left\|\mathfrak{J}^{\sigma}_{\rho,\lambda,b-;w}\phi\right\|_1 \le \mathfrak{M}(b-a)^{\lambda}\|\phi\|_1.$$

By setting $\lambda = \alpha$; $\sigma(0) = 1$ and $w = 0$ in (4) and (5), respectively, (1) and (2) are recaptured. Before starting the main results, we consider some following assumptions to make the representation easier and compact. Throughout the discussion, let $0 \neq p \in \mathbf{R}$ and $\epsilon \geq 0$

$$\mathfrak{L}_{1,\epsilon}(x,y,a,b) := \int_{a^p}^{b^p} |\kappa_{\epsilon}(t)|\frac{b^p - t}{b^p - a^p}t^{\frac{1-p}{p}}dt;\ \ \mathfrak{L}_{2,\epsilon}(x,y,a,b) := \int_{a^p}^{b^p} |\kappa_{\epsilon}(t)|\frac{t - a^p}{b^p - a^p}t^{\frac{1-p}{p}}dt.$$

$$\kappa_{\epsilon}(t) := \begin{cases} (t-a^p)^{\beta} \\ \times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(x^p-a^p\right)^{\rho}\left(t-a^p\right)^{\rho}\right]t^{\epsilon}, & a^p \le t < x^p; \\ (t-\frac{\alpha b^p-\alpha a^p+2a^p}{2})^{\beta} \\ \times\mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(\frac{a^p+b^p-2x^p}{2})^{\rho}(t-\frac{\alpha b^p-\alpha a^p+2a^p}{2})^{\rho}]t^{\epsilon}, & x^p \le t < \frac{a^p+b^p}{2}; \\ -(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-t)^{\beta} \\ \times\mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(\frac{2y^p-a^p-b^p}{2})^{\rho}(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-t)^{\rho}]t^{\epsilon}, & \frac{a^p+b^p}{2} \le t < y^p; \\ -(b^p-t)^{\beta} \\ \times\mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(b^p-y^p)^{\rho}(b^p-t)^{\rho}]t^{\epsilon}, & y^p \le t \le b^p. \end{cases} \tag{6}$$

$$h_{1,p}(t) := \begin{cases} t-a^p, & a^p \le t < x^p; \\ (1-2\alpha)t + \frac{3a^p\alpha+\alpha b^p-2a^p}{2}, & x^p \le t < \frac{a^p+b^p}{2}; \\ (2\alpha-1)t + \frac{2b^p-3b^p\alpha-\alpha a^p}{2}, & \frac{a^p+b^p}{2} \le t < y^p; \\ b^p-t, & y^p \le t \le b^p. \end{cases}$$

$$h_{0,p}(t) := \begin{cases} t-a^p, & a^p \le t < x^p; \\ t-\frac{\alpha b^p-\alpha a^p+2a^p}{2}, & x^p \le t < \frac{a^p+b^p}{2}; \\ t-\frac{\alpha a^p-\alpha b^p+2b^p}{2}, & \frac{a^p+b^p}{2} \le t < y^p; \\ t-b^p, & y^p \le t \le b^p. \end{cases}$$

$$h_{2,p}(t) := \begin{cases} t - \frac{\alpha b^p - \alpha a^p + 2a^p}{2}, & a^p \le t < \frac{a^p+b^p}{2}; \\ t - \frac{\alpha a^p - \alpha b^p + 2b^p}{2}, & \frac{a^p+b^p}{2} \le t \le b^p. \end{cases}$$

$$\begin{aligned}
&\mathfrak{I}_1(x,y,a,b) \\
&:= \frac{1}{(b^p-a^p)^3}\int_{a^p}^{b^p} h_{1,p}(t)(b^p-t)t^{\frac{2-p}{p}}\,dt \\
&= \frac{1}{(b^p-a^p)^3}\Bigg\{\frac{p}{2p+2}\Bigg[a^{2+2p}-x^{2+2p}+b^{2+2p}-y^{2+2p}+(2\alpha-1) \\
&\times\left(2\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^{2+2p}-x^{2+2p}-y^{2+2p}\right)\Bigg]+\frac{p}{2}b^p\Bigg[a^{2+p}-a^px^2+b^{2+p}-b^py^2 \\
&+\frac{3a^p\alpha+\alpha b^p-2a^p}{2}\left(\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^2-x^2\right)+\frac{2b^p-3b^p\alpha-\alpha a^p}{2} \\
&\times\left(y^2-\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^2\right)\Bigg]+\frac{p}{p+2}\Bigg[(b^p+a^p)(x^{2+p}-a^{2+p})-2b^{2+2p}+2b^py^{2+p} \\
&+\left(b^p(1-2\alpha)-\frac{3a^p\alpha+\alpha b^p-2a^p}{2}\right)\left(\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^{2+p}-x^{2+p}\right) \\
&+\left(b^p(2\alpha-1)-\frac{2b^p-3b^p\alpha-\alpha a^p}{2}\right)\left(y^{2+p}-\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^{2+p}\right)\Bigg]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{I}_2(x,y,a,b) \\
&:= \frac{1}{(b^p-a^p)^3}\int_{a^p}^{b^p} h_{1,p}(t)(t-a^p)t^{\frac{2-p}{p}}\,dt \\
&= \frac{1}{(b^p-a^p)^3}\Bigg\{\frac{p}{2p+2}\Bigg[x^{2+2p}-a^{2+2p}-b^{2+2p}+y^{2+2p}+(1-2\alpha) \\
&\times\left(2\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^{2+2p}-x^{2+2p}-y^{2+2p}\right)\Bigg]+\frac{p}{2}a^p[a^px^2-a^{2+p}-b^{2+p}+b^py^2 \\
&-\frac{3a^p\alpha+\alpha b^p-2a^p}{2}\left(\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^2-x^2\right)-\frac{2b^p-3b^p\alpha-\alpha a^p}{2} \\
&\times\left(y^2-\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^2\right)\Bigg]+\frac{p}{p+2}[2a^{2+2p}-2a^px^{2+p}+(a^p+b^p)(b^{2+p}-y^{2+p}) \\
&+\left(\frac{3a^p\alpha+\alpha b^p-2a^p}{2}-a^p(1-2\alpha)\right)\left(\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^{2+p}-x^{2+p}\right) \\
&+\left(\frac{2b^p-3b^p\alpha-\alpha a^p}{2}-a^p(2\alpha-1)\right)\left(y^{2+p}-\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^{2+p}\right)\Bigg]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{A}(x,y,a,b) \\
&:= \frac{1}{(b^p-a^p)^3}\int_{a^p}^{b^p} h_{1,p}(t)(b^p-t)t^{\frac{1-p}{p}}\,dt
\end{aligned}$$

$$= \frac{1}{(b^p - a^p)^3} \left\{ \frac{p}{2p+1} \left[a^{1+2p} - x^{1+2p} + b^{1+2p} - y^{1+2p} - (1-2\alpha) \right. \right.$$

$$\left. \times \left(2 \left(\sqrt[p]{\frac{a^p + b^p}{2}} \right)^{1+2p} - x^{1+2p} - y^{1+2p} \right) \right] + pb^p \left[a^{1+p} - a^p x + b^{1+p} - b^p y \right.$$

$$+ \frac{3a^p\alpha + \alpha b^p - 2a^p}{2} \left(\sqrt[p]{\frac{a^p + b^p}{2}} - x \right) + \frac{2b^p - 3b^p\alpha - \alpha a^p}{2}$$

$$\left. \left(y - \sqrt[p]{\frac{a^p + b^p}{2}} \right) \right] + \frac{p}{p+1} \left[(b^p + a^p)(y^{1+p} - a^{1+p}) - 2b^{1+2p} + 2b^p y^{1+p} \right.$$

$$+ \left(b^p(1-2\alpha) - \frac{3a^p\alpha + \alpha b^p - 2a^p}{2} \right) \left(\left(\sqrt[p]{\frac{a^p + b^p}{2}} \right)^{1+p} - x^{1+p} \right)$$

$$\left. \left. + \left(b^p(2\alpha - 1) - \frac{2b^p - 3b^p\alpha - \alpha a^p}{2} \right) \left(y^{1+p} - \left(\sqrt[p]{\frac{a^p + b^p}{2}} \right)^{1+p} \right) \right] \right\}$$

$$\mathfrak{B}(x, y, a, b)$$

$$:= \frac{1}{(b^p - a^p)^3} \int_{a^p}^{b^p} h_{1,p}(t)(t - a^p) t^{\frac{1-p}{p}} dt$$

$$= \frac{1}{(b^p - a^p)^3} \left\{ \frac{p}{2p+1} \left[x^{1+2p} - a^{1+2p} - b^{1+2p} + y^{1+2p} + (1-2\alpha) \right. \right.$$

$$\left. \times \left(2 \left(\sqrt[p]{\frac{a^p + b^p}{2}} \right)^{1+2p} - x^{1+2p} - y^{1+2p} \right) \right] + pa^p \left[a^p x - a^{1+p} - b^{1+p} + b^p y \right.$$

$$\left. - \frac{3a^p\alpha + \alpha b^p - 2a^p}{2} \left(\sqrt[p]{\frac{a^p + b^p}{2}} - x \right) - \frac{2b^p - 3b^p\alpha - \alpha a^p}{2} \left(y - \sqrt[p]{\frac{a^p + b^p}{2}} \right) \right]$$

$$+ \frac{p}{p+1} \left[2a^{1+2p} - 2a^p x^{1+p} + (a^p + b^p)(b^{1+p} - y^{1+p}) \right.$$

$$+ \left(\frac{3a^p\alpha + \alpha b^p - 2a^p}{2} - a^p(1-2\alpha) \right) \left(\left(\sqrt[p]{\frac{a^p + b^p}{2}} \right)^{1+p} - x^{1+p} \right)$$

$$\left. \left. + \left(\frac{2b^p - 3b^p\alpha - \alpha a^p}{2} - a^p(2\alpha - 1) \right) \left(y^{1+p} - \left(\sqrt[p]{\frac{a^p + b^p}{2}} \right)^{1+p} \right) \right] \right\}$$

$$\mathfrak{A}(x, y, a, b) + \mathfrak{B}(x, y, a, b)$$

$$= \frac{1}{(b^p - a^p)^2} \int_{a^p}^{b^p} h_{1,p}(t) t^{\frac{1-p}{p}} dt$$

$$= \frac{1}{(b^p - a^p)^2} \left\{ \frac{p}{p+1} \left[x^{1+p} + y^{1+p} - a^{1+p} - b^{1+p} + (1-2\alpha) \right. \right.$$

$$\left. \times \left(2 \left(\sqrt[p]{\frac{a^p + b^p}{2}} \right)^{1+p} - x^{1+p} - y^{1+p} \right) \right] + p[b^{1+p} + a^{1+p} - b^p y - a^p x$$

$$+ \left(\sqrt[p]{\frac{a^p + b^p}{2}} - x \right) \left[\alpha \frac{a^p + b^p}{2} - (1-\alpha) a^p \right] + \left(y - \sqrt[p]{\frac{a^p + b^p}{2}} \right)$$

$$\times\left[(1-\alpha)b^p-\alpha\frac{a^p+b^p}{2}\right]\Bigg]\Bigg\}$$

$$\begin{aligned}
&\mathfrak{I}_1(x,y,a,b)+\mathfrak{I}_2(x,y,a,b)\\
&:=\frac{1}{(b^p-a^p)^2}\int_{a^p}^{b^p}h_{1,p}(t)t^{\frac{2-p}{p}}dt\\
&=\frac{1}{(b^p-a^p)^2}\left\{\frac{p}{p+2}\left[x^{2+p}+y^{2+p}-a^{2+p}-b^{2+p}+(1-2\alpha)\right.\right.\\
&\times\left.\left(2\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^{2+p}-x^{2+p}-y^{2+p}\right)\right]\\
&+\frac{p}{2}\left[b^{2+p}+a^{2+p}-b^py^2-a^px^2+\frac{3a^p\alpha+\alpha b^p-2a^p}{2}\left(\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^2-x^2\right)\right.\\
&\left.\left.+\frac{2b^p-3b^p\alpha-\alpha a^p}{2}\left(y^2-\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)^2\right)\right]\right\}
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{C}(x,y,a,b):=\\
&\frac{2\alpha-1}{12}\int_\phi\frac{(\psi(x)+\chi(x))^3-4\chi^3(x)-4\psi^3(x)}{(\chi(x)-\psi(x))^2}\left|f'\left(\frac{\psi(x)}{\chi(x)}\right)\right|d\mu(x)\\
&+\frac{1-\alpha}{2}\int_\phi\chi(x)\left|f'\left(\frac{\psi(x)}{\chi(x)}\right)\right|d\mu(x)+\int_\phi\left|f'\left(\frac{\psi(x)}{\chi(x)}\right)\right|\\
&\times\frac{(2\psi(x)-5\alpha\chi(x)-3\alpha\psi(x)+2\chi(x))(\chi^2(x)+2\psi(x)\chi(x)-3\psi^2(x))}{16(\chi(x)-\psi(x))^2}d\mu(x)\\
&+\int_\phi\frac{(7\alpha\chi(x)-4\chi(x)+\alpha\psi(x))(3\chi^2(x)-2\psi(x)\chi(x)-\psi^2(x))}{16(\chi(x)-\psi(x))^2}\left|f'\left(\frac{\psi(x)}{\chi(x)}\right)\right|d\mu(x) \qquad (7)
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{D}(x,y,a,b):=\\
&\frac{2\alpha-1}{12}\int_\phi\frac{|(\psi(x)+\chi(x))^3-4\chi^3(x)-4\psi^3(x)|}{|\psi(x)-\chi(x)|^2}d\mu(x)+\frac{1-\alpha}{2}\int_\phi\psi(x)d\mu(x)\\
&+\int_\phi\frac{(2\psi(x)-5\alpha\psi(x)-3\alpha\chi(x)+2\chi(x))(3\chi^2(x)-\psi^2(x)-2\psi(x)\chi(x))}{16(\psi(x)-\chi(x))^2}d\mu(x)\\
&+\int_\phi\frac{(7\alpha\psi(x)-4\psi(x)+\alpha\chi(x))(\chi^2(x)-3\psi^2(x)+2\psi(x)\chi(x))}{16(\psi(x)-\chi(x))^2}d\mu(x) \qquad (8)
\end{aligned}$$

3. Results

Lemma 3.1. *Let $f: I\subseteq\mathbf{R}^+\to\mathbf{R}$ be a differentiable function on I°, interior of I, $a,b\in I^\circ$ with $a<b$; $\rho,\beta>0$; $0\le\alpha\le1$ and $w\in\mathbf{R}$; let $g(\xi)=\sqrt[p]{\xi}$, $\xi>0$, then*

$$\begin{aligned}
\Omega(x,y,a,b)&:=p\left\{(x^p-a^p)^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}[w(x^p-a^p)^{2\rho}]-\left[\frac{2(x^p-a^p)-\alpha(b^p-a^p)}{2}\right]^\beta\right.\\
&\left.\times\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{a^p+b^p-2x^p}{2}\right)^\rho\left(\frac{2(x^p-a^p)-\alpha(b^p-a^p)}{2}\right)^\rho\right]\right\}f(x)
\end{aligned}$$

$$+p\left\{(b^p-y^p)^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}[w(b^p-y^p)^{2\rho}]-\left[\frac{2(b^p-y^p)-\alpha(b^p-a^p)}{2}\right]^\beta\right.$$

$$\left.\times\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{2y^p-a^p-b^p}{2}\right)^\rho\left(\frac{2(b^p-y^p)-\alpha(b^p-a^p)}{2}\right)^\rho\right]\right\}f(y)$$

$$+p\left\{\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{a^p+b^p-2x^p}{2}\right)^\rho\left((1-\alpha)\frac{b^p-a^p}{2}\right)^\rho\right]\right.$$

$$\left.+\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{2y^p-a^p-b^p}{2}\right)^\rho\left((1-\alpha)\frac{b^p-a^p}{2}\right)^\rho\right]\right\}\left[(1-\alpha)\frac{b^p-a^p}{2}\right]^\beta$$

$$\times f\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)-p\left\{\left(\mathfrak{J}^\sigma_{\rho,\beta,x^p-;w(x^p-a^p)^\rho}f\circ g\right)(a^p)\right.$$

$$+\left(\mathfrak{J}^\sigma_{\rho,\beta,\frac{a^p+b^p}{2}-;w\left(\frac{a^p+b^p-2x^p}{2}\right)^\rho}f\circ g\right)\left(\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)$$

$$-\left(\mathfrak{J}^\sigma_{\rho,\beta,x^p-;w\left(\frac{a^p+b^p-2x^p}{2}\right)^\rho}f\circ g\right)\left(\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)$$

$$+\left(\mathfrak{J}^\sigma_{\rho,\beta,\frac{a^p+b^p}{2}+;w\left(\frac{2y^p-a^p-b^p}{2}\right)^\rho}f\circ g\right)\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}\right)$$

$$+\left(\mathfrak{J}^\sigma_{\rho,\beta,y^p-;w\left(\frac{2y^p-a^p-b^p}{2}\right)^\rho}f\circ g\right)\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}\right)$$

$$\left.+\left(\mathfrak{J}^\sigma_{\rho,\beta,y^p+;w(b^p-y^p)^\rho}f\circ g\right)(b^p)\right\}=\int_{a^p}^{b^p}\kappa_0(t)t^{\frac{1-p}{p}}f'\left(\sqrt[p]{t}\right)dt \tag{9}$$

provided that $\kappa_0(t)$ is defined by (6).

Proof. *Integrating by parts and change of variable technique*

$$I_1:=\int_{a^p}^{x^p}(t-a^p)^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(x^p-a^p\right)^\rho\left(t-a^p\right)^\rho\right]t^{\frac{1-p}{p}}f'\left(\sqrt[p]{t}\right)dt$$

$$=\int_{a^p}^{x^p}(t-a^p)^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(x^p-a^p\right)^\rho\left(t-a^p\right)^\rho\right]\left[\frac{x^p-t}{x^p-a^p}a^p+\frac{t-a^p}{x^p-a^p}x^p\right]^{\frac{1-p}{p}}$$
$$\times f'\left(\sqrt[p]{\frac{x^p-t}{x^p-a^p}a^p+\frac{t-a^p}{x^p-a^p}x^p}\right)dt$$

$$=p(t-a^p)^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}[w(x^p-a^p)^\rho(t-a^p)^\rho]f\left(\sqrt[p]{\frac{x^p-t}{x^p-a^p}a^p+\frac{t-a^p}{x^p-a^p}x^p}\right)\Bigg|_{a^p}^{x^p}$$

$$-p\int_{a^p}^{x^p}(t-a^p)^{\beta-1}\mathfrak{F}^{\sigma}_{\rho,\beta}[w(x^p-a^p)^{\rho}(t-a^p)^{\rho}]f\left(\sqrt[p]{\frac{x^p-t}{x^p-a^p}a^p+\frac{t-a^p}{x^p-a^p}x^p}\right)dt$$

$$=p(x^p-a^p)^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(x^p-a^p)^{2\rho}]f(x)$$

$$-p\int_{a^p}^{x^p}(u-a^p)^{\beta-1}\mathfrak{F}^{\sigma}_{\rho,\beta}[w(x^p-a^p)^{\rho}(u-a^p)^{\rho}](f\circ g)(u)du$$

$$=p(x^p-a^p)^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}[w(x^p-a^p)^{2\rho}]f(x)-p(\mathfrak{J}^{\sigma}_{\rho,\beta,x^p-;w(x^p-a^p)^{\rho}}f\circ g)(a^p) \tag{10}$$

$$I_2:=\int_{x^p}^{\frac{a^p+b^p}{2}}\left(t-\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)^{\beta}t^{\frac{1-p}{p}}f'\left(\sqrt[p]{t}\right)$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{a^p+b^p-2x^p}{2}\right)^{\rho}\left(t-\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)^{\rho}\right]dt$$

$$=\int_{x^p}^{\frac{a^p+b^p}{2}}\left(t-\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)^{\beta}\left[\frac{a^p+b^p-2t}{a^p+b^p-2x^p}x^p+\frac{(t-x^p)(a^p+b^p)}{a^p+b^p-2x^p}\right]^{\frac{1-p}{p}}$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{a^p+b^p-2x^p}{2}\right)^{\rho}\left(t-\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)^{\rho}\right]$$

$$\times f'\left(\sqrt[p]{\frac{a^p+b^p-2t}{a^p+b^p-2x^p}x^p+\frac{(t-x^p)(a^p+b^p)}{a^p+b^p-2x^p}}\right)dt$$

$$=p\left(t-\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{a^p+b^p-2x^p}{2}\right)^{\rho}\left(t-\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)^{\rho}\right]$$

$$\times f\left(\sqrt[p]{\frac{a^p+b^p-2t}{a^p+b^p-2x^p}x^p+\frac{(t-x^p)(a^p+b^p)}{a^p+b^p-2x^p}}\right)\Bigg|_{x^p}^{\frac{a^p+b^p}{2}}$$

$$-p\int_{x^p}^{\frac{a^p+b^p}{2}}\left(t-\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)^{\beta-1}\mathfrak{F}^{\sigma}_{\rho,\beta}\left[w\left(\frac{a^p+b^p-2x^p}{2}\right)^{\rho}\left(t-\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)^{\rho}\right]$$

$$\times f\left(\sqrt[p]{\frac{a^p+b^p-2t}{a^p+b^p-2x^p}x^p+\frac{(t-x^p)(a^p+b^p)}{a^p+b^p-2x^p}}\right)dt$$

$$=p\left\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{a^p+b^p-2x^p}{2}\right)^{\rho}((1-\alpha)\frac{b^p-a^p}{2})^{\rho}\right]\right.$$

$$\times\left[(1-\alpha)\frac{b^p-a^p}{2}\right]^{\beta}f\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)-\left[\frac{2(x^p-a^p)-\alpha(b^p-a^p)}{2}\right]^{\beta}$$

$$\left.\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{a^p+b^p-2x^p}{2}\right)^{\rho}\left(\frac{2(x^p-a^p)-\alpha(b^p-a^p)}{2}\right)^{\rho}\right]f(x)\right\}$$

$$-p\int_{x^p}^{\frac{a^p+b^p}{2}}\left(u-\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)^{\beta-1}$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta}\left[w\left(\frac{a^p+b^p-2x^p}{2}\right)^{\rho}\left(u-\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)^{\rho}\right](f\circ g)(u)du.$$

$$\Rightarrow I_2 \quad = \quad p\left\{\left((1-\alpha)\frac{b^p-a^p}{2}\right)^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{a^p+b^p-2x^p}{2}\right)^{\rho}\left((1-\alpha)\frac{b^p-a^p}{2}\right)^{\rho}\right]\right.$$

$$\times f\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)-\left[\frac{2(x^p-a^p)-\alpha(b^p-a^p)}{2}\right]^{\beta}$$

$$\left.\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{a^p+b^p-2x^p}{2}\right)^{\rho}\left(\frac{2(x^p-a^p)-\alpha(b^p-a^p)}{2}\right)^{\rho}\right]f(x)\right\}$$

$$-p\left(\mathfrak{J}^{\sigma}_{\rho,\beta,\frac{a^p+b^p}{2}-;w\left(\frac{a^p+b^p-2x^p}{2}\right)^{\rho}}f\circ g\right)\left(\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right)$$

$$+p\left(\mathfrak{J}^{\sigma}_{\rho,\beta,x^p-;w\left(\frac{a^p+b^p-2x^p}{2}\right)^{\rho}}f\circ g\right)\left(\frac{\alpha b^p-\alpha a^p+2a^p}{2}\right) \tag{11}$$

$$I_3 \quad := \quad -\int_{\frac{a^p+b^p}{2}}^{y^p}\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-t\right)^{\beta}t^{\frac{1-p}{p}}f'\left(\sqrt[p]{t}\right)$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{2y^p-a^p-b^p}{2}\right)^{\rho}\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-t\right)^{\rho}\right]dt$$

$$=-\int_{\frac{a^p+b^p}{2}}^{y^p}\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-t\right)^{\beta}$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{2y^p-a^p-b^p}{2}\right)^{\rho}\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-t\right)^{\rho}\right]$$

$$\times\left[\frac{2t-a^p-b^p}{2y^p-a^p-b^p}y^p+\frac{(y^p-t)(a^p+b^p)}{2y^p-a^p-b^p}\right]^{\frac{1-p}{p}}$$

$$\times f'\left(\sqrt[p]{\frac{2t-a^p-b^p}{2y^p-a^p-b^p}y^p+\frac{(y^p-t)(a^p+b^p)}{2y^p-a^p-b^p}}\right)dt$$

$$=-p\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-t\right)^{\beta}$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w(\frac{2y^p-a^p-b^p}{2})^{\rho}(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-t)^{\rho}\right]$$

$$\times f\left(\sqrt[p]{\frac{2t-a^p-b^p}{2y^p-a^p-b^p}y^p+\frac{(y^p-t)(a^p+b^p)}{2y^p-a^p-b^p}}\right)\Bigg|_{\frac{a^p+b^p}{2}}^{y^p}$$

$$-p\int_{\frac{a^p+b^p}{2}}^{y^p}\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-t\right)^{\beta-1}$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta}\left[w\left(\frac{2y^p-a^p-b^p}{2}\right)^{\rho}\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-t\right)^{\rho}\right]$$

$$\times f\left(\sqrt[p]{\frac{2t-a^p-b^p}{2y^p-a^p-b^p}y^p+\frac{(y^p-t)(a^p+b^p)}{2y^p-a^p-b^p}}\right)dt$$

$$=p\left\{-\left[\frac{2(b^p-y^p)-\alpha(b^p-a^p)}{2}\right]^{\beta}\right.$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{2y^p-a^p-b^p}{2}\right)^{\rho}\left(\frac{2(b^p-y^p)-\alpha(b^p-a^p)}{2}\right)^{\rho}\right]f(y)$$

$$+\left[(1-\alpha)\frac{b^p-a^p}{2}\right]^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{2y^p-a^p-b^p}{2}\right)^{\rho}\left((1-\alpha)\frac{b^p-a^p}{2}\right)^{\rho}\right]$$

$$\left.\times f\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)\right\}-p\int_{\frac{a^p+b^p}{2}}^{y^p}\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-u\right)^{\beta-1}$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta}\left[w\left(\frac{2y^p-a^p-b^p}{2}\right)^{\rho}\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}-u\right)^{\rho}\right](f\circ g)(u)du$$

$$=p\left\{-\left[\frac{2(b^p-y^p)-\alpha(b^p-a^p)}{2}\right]^{\beta}\right.$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{2y^p-a^p-b^p}{2}\right)^{\rho}\left(\frac{2(b^p-y^p)-\alpha(b^p-a^p)}{2}\right)^{\rho}\right]f(y)$$

$$+\left[(1-\alpha)\frac{b^p-a^p}{2}\right]^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{2y^p-a^p-b^p}{2}\right)^{\rho}\left((1-\alpha)\frac{b^p-a^p}{2}\right)^{\rho}\right]$$

$$\left.\times f\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)\right\}-p\left(\mathfrak{I}^{\sigma}_{\rho,\beta,\frac{a^p+b^p}{2}+;w(\frac{2y^p-a^p-b^p}{2})^{\rho}}f\circ g\right)\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}\right)$$

$$-p\left(\mathfrak{I}^{\sigma}_{\rho,\beta,y^p-;w(\frac{2y^p-a^p-b^p}{2})^{\rho}}f\circ g\right)\left(\frac{\alpha a^p-\alpha b^p+2b^p}{2}\right)\tag{12}$$

Integrating by parts and change of variable technique

$$I_4 := -\int_{y^p}^{b^p} (b^p - t)^{\beta} \mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\,(b^p - y^p)^{\rho}\,(b^p - t)^{\rho}\right] t^{\frac{1-p}{p}} f'\left(\sqrt[p]{t}\right) dt$$

$$= -\int_{y^p}^{b^p} (b^p - t)^{\beta} \mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\,(b^p - y^p)^{\rho}\,(b^p - t)^{\rho}\right] \left[\frac{b^p - t}{b^p - y^p} y^p + \frac{t - y^p}{b^p - y^p} b^p\right]^{\frac{1-p}{p}}$$

$$\times f'\left(\sqrt[p]{\frac{b^p - t}{b^p - y^p} y^p + \frac{t - y^p}{b^p - y^p} b^p}\right) dt$$

$$= -p(b^p - t)^{\beta} \mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w(b^p - y^p)^{\rho}(b^p - t)^{\rho}\right] f\left(\sqrt[p]{\frac{b^p - t}{b^p - y^p} y^p + \frac{t - y^p}{b^p - y^p} b^p}\right)\Bigg|_{y^p}^{b^p}$$

$$-p\int_{y^p}^{b^p} (b^p - t)^{\beta-1} \mathfrak{F}^{\sigma}_{\rho,\beta}\left[w(b^p - y^p)^{\rho}(b^p - t)^{\rho}\right] f\left(\sqrt[p]{\frac{b^p - t}{b^p - y^p} y^p + \frac{t - y^p}{b^p - y^p} b^p}\right) dt$$

$$= p(b^p - y^p)^{\beta} \mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w(b^p - y^p)^{2\rho}\right] f(y)$$

$$-p\int_{y^p}^{b^p} (b^p - u)^{\beta-1} \mathfrak{F}^{\sigma}_{\rho,\beta}\left[w(b^p - y^p)^{\rho}(b^p - u)^{\rho}\right] (f \circ g)(u) du$$

$$= p(b^p - y^p)^{\beta} \mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w(b^p - y^p)^{2\rho}\right] f(y) - p\left(\mathfrak{J}^{\sigma}_{\rho,\beta,y^p+;w(b^p-y^p)^{\rho}} f \circ g\right)(b^p) \tag{13}$$

Addition of (10)-(13) *yields the desired identity* (9). □

Note 3.2. *It may be noted that for* $\beta, \sigma(0) = 1$, $w = 0$, *identity* (9) *reduces to*

$$\begin{aligned} \Upsilon(x, y, a, b) &:= \alpha p \frac{f(x) + f(y)}{2} + p(1-\alpha) f\left(\sqrt[p]{\frac{a^p + b^p}{2}}\right) - \frac{p}{b^p - a^p}\int_{a^p}^{b^p} (f \circ g)(u) du \\ &= \frac{1}{b^p - a^p}\int_{a^p}^{b^p} h_{0,p}(t) t^{\frac{1-p}{p}} f'\left(\sqrt[p]{t}\right) dt, \end{aligned} \tag{14}$$

and hence for $p = 1$, *it reduces to identity* (2.4) *in [11]. For* $x = a$ *and* $y = b$, *it reduces to*

$$\begin{aligned} \mathfrak{Z}(a, b; p) &:= \alpha p \tfrac{f(a)+f(b)}{2} + p(1-\alpha) f\left(\sqrt[p]{\tfrac{a^p+b^p}{2}}\right) - \tfrac{p}{b^p - a^p}\int_{a^p}^{b^p} (f \circ g)(u) du \\ &= \frac{1}{b^p - a^p}\int_{a^p}^{b^p} h_{2,p}(t) t^{\frac{1-p}{p}} f'\left(\sqrt[p]{t}\right) dt. \end{aligned}$$

Theorem 3.3. *Let* $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ *be a differentiable function on* I°, *interior of* I, $a, b \in I^{\circ}$ *with* $a < b$ *such that* $|f'|$ *is* $p-$*convex and* $\rho, \beta > 0$; *let* $g(\xi) = \sqrt[p]{\xi}$, $\xi > 0$; $0 \le \alpha \le 1$, *then*

$$|\Omega(x, y, a, b)| \le \mathfrak{L}_{1,0}|f'(a)| + \mathfrak{L}_{2,0}|f'(b)|$$

Proof. By $p-$convexity of $|f'|$ and properties of modulus to (9)

$$\begin{aligned}|\Omega(x,y,a,b)| &= \left|\int_{a^p}^{b^p} \kappa_0(t) t^{\frac{1-p}{p}} f'\left(\sqrt[p]{t}\right) dt\right| \\ &\leq \int_{a^p}^{b^p} |\kappa_0(t)| \left|f'\left(\sqrt[p]{t}\right)\right| t^{\frac{1-p}{p}} dt \\ &= \int_{a^p}^{b^p} |\kappa_0(t)| \left|f'\left(\sqrt[p]{\frac{b^p-t}{b^p-a^p}a^p + \frac{t-a^p}{b^p-a^p}b^p}\right)\right| t^{\frac{1-p}{p}} dt \\ &\leq \int_{a^p}^{b^p} |\kappa_0(t)| \left\{\frac{b^p-t}{b^p-a^p}\left|f'(a)\right| + \frac{t-a^p}{b^p-a^p}\left|f'(b)\right|\right\} t^{\frac{1-p}{p}} dt.\end{aligned}$$

This completes the proof. □

Corollary 3.4. *Let $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, $a, b \in I^\circ$ such that $|f'|$ is $p-$convex; let $g(\xi) = \sqrt[p]{\xi}$ for $\xi > 0$ and $0 \leq \alpha \leq 1$, then*

$$|\Upsilon(x,y,a,b)| \leq (b^p - a^p)\left[\mathfrak{A}(x,y,a,b)|f'(a)| + \mathfrak{B}(x,y,a,b)|f'(b)|\right] \tag{15}$$

Proof. The proof directly follows from Theorem 3.3 for $\beta, \sigma(0) = 1$, $w = 0$, that is, by properties of modulus, $p-$convexity of $|f'|$ and identity (14), we have

$$\begin{aligned}|\Upsilon(x,y,a,b)| &= \frac{1}{b^p-a^p}\left|\int_{a^p}^{b^p} h_{0,p}(t) t^{\frac{1-p}{p}} f'\left(\sqrt[p]{t}\right) dt\right| \\ &\leq \frac{1}{b^p-a^p}\int_{a^p}^{b^p} |h_{0,p}(t)| \left|f'\left(\sqrt[p]{t}\right)\right| t^{\frac{1-p}{p}} dt \\ &= \frac{1}{b^p-a^p}\int_{a^p}^{b^p} |h_{0,p}(t)| \left|f'\left(\sqrt[p]{\frac{b^p-t}{b^p-a^p}a^p + \frac{t-a^p}{b^p-a^p}b^p}\right)\right| t^{\frac{1-p}{p}} dt \\ &\leq \frac{1}{b^p-a^p}\int_{a^p}^{b^p} h_{1,p}(t) \left\{\frac{b^p-t}{b^p-a^p}|f'(a)| + \frac{t-a^p}{b^p-a^p}|f'(b)|\right\} t^{\frac{1-p}{p}} dt.\end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, interior of I, $a, b \in I^\circ$ with $a < b$ such that $|f'|^s$ is $p-$convex and $s > 1$ such that $r = \frac{s}{s-1}$; let $g(\xi) = \sqrt[p]{\xi}$, $\xi > 0$; $0 \leq \alpha \leq 1$ and $\rho, \beta > 0$, then*

$$|\Omega(x,y,a,b)| \leq \sqrt[s]{\frac{(b^p-a^p)(|f'(a)|^s + |f'(b)|^s)}{2}}\sqrt[r]{\int_{a^p}^{b^p} |\kappa_0(t)|^r t^{\frac{r(1-p)}{p}} dt}$$

Proof. By Hölder inequality, $p-$convexity of $|f'|^s$ properties of modulus to (14)

$$\begin{aligned}&|\Omega(x,y,a,b)| \\ &= \left|\int_{a^p}^{b^p} \kappa_0(t) t^{\frac{1-p}{p}} f'\left(\sqrt[p]{t}\right) dt\right| \\ &\leq \int_{a^p}^{b^p} |\kappa_0(t)| \left|f'\left(\sqrt[p]{t}\right)\right| t^{\frac{1-p}{p}} dt \\ &\leq \sqrt[r]{\int_{a^p}^{b^p} |\kappa_0(t)|^r t^{\frac{r(1-p)}{p}} dt}\sqrt[s]{\int_{a^p}^{b^p} \left|f'\left(\sqrt[p]{\frac{b^p-t}{b^p-a^p}a^p + \frac{t-a^p}{b^p-a^p}b^p}\right)\right|^s dt}\end{aligned}$$

$$\leq \sqrt[r]{\int_{a^p}^{b^p} |\kappa_0(t)|^r t^{\frac{r(1-p)}{p}} dt} \sqrt[s]{\int_{a^p}^{b^p} \left\{ \frac{b^p - t}{b^p - a^p} |f'(a)|^s + \frac{t - a^p}{b^p - a^p} |f'(b)|^s \right\} dt}$$

This completes the proof. □

Theorem 3.6. *Let $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, interior of I, $a, b \in I^\circ$ with $a < b$ such that $|f'|^s$ is $p-$convex and $s \geq 1$; let $g(\xi) = \sqrt[p]{\xi}$, $\xi > 0$; $0 \leq \alpha \leq 1$ and $\rho, \beta > 0$, then*

$$\begin{aligned} |\Omega(x, y, a, b)| &\leq (\mathfrak{L}_{1,0}(x, y, a, b) + \mathfrak{L}_{2,0}(x, y, a, b)) \\ &\times \sqrt[s]{\frac{\mathfrak{L}_{1,0}(x, y, a, b)|f'(a)|^s + \mathfrak{L}_{2,0}(x, y, a, b)|f'(b)|^s}{\mathfrak{L}_{1,0}(x, y, a, b) + \mathfrak{L}_{2,0}(x, y, a, b)}} \end{aligned}$$

Proof. By power-mean inequality, $p-$convexity of $|f'|^s$ and properties of modulus to (9)

$$\begin{aligned} &|\Omega(x, y, a, b)| \\ &= \left| \int_{a^p}^{b^p} \kappa_0(t) t^{\frac{1-p}{p}} f'\left(\sqrt[p]{t}\right) dt \right| \\ &\leq \int_{a^p}^{b^p} |\kappa_0(t)| \left| f'\left(\sqrt[p]{t}\right) \right| t^{\frac{1-p}{p}(1-\frac{1}{s}+\frac{1}{s})} dt \\ &\leq \left[\int_{a^p}^{b^p} |\kappa_0(t)| t^{\frac{1-p}{p}} dt \right]^{\frac{s-1}{s}} \sqrt[s]{\int_{a^p}^{b^p} |\kappa_0(t)| \left| f'\left(\sqrt[p]{\frac{b^p - t}{b^p - a^p} a^p + \frac{t - a^p}{b^p - a^p} b^p} \right) \right|^s t^{\frac{1-p}{p}} dt} \\ &\leq \left[\int_{a^p}^{b^p} |\kappa_0(t)| t^{\frac{1-p}{p}} dt \right]^{\frac{s-1}{s}} \sqrt[s]{\int_{a^p}^{b^p} |\kappa_0(t)| \left\{ \frac{b^p - t}{b^p - a^p} |f'(a)|^s + \frac{t - a^p}{b^p - a^p} |f'(b)|^s \right\} t^{\frac{1-p}{p}} dt} \\ &= (\mathfrak{L}_{1,0}(x, y, a, b) + \mathfrak{L}_{2,0}(x, y, a, b))^{\frac{s-1}{s}} \\ &\times \sqrt[s]{\mathfrak{L}_{1,0}(x, y, a, b)|f'(a)|^s + \mathfrak{L}_{2,0}(x, y, a, b)|f'(b)|^s}. \end{aligned}$$

This completes the proof. □

Corollary 3.7. *Let $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, interior of I, $a, b \in I^\circ$ with $a < b$ such that $|f'|^s$ is $p-$convex and $s \geq 1$; let $g(\xi) = \sqrt[p]{\xi}$, $\xi > 0$; $0 \leq \alpha \leq 1$, then*

$$\begin{aligned} |\Upsilon(x, y, a, b)| &\leq (b^p - a^p)(\mathfrak{A}(x, y, a, b) + \mathfrak{B}(x, y, a, b))^{\frac{s-1}{s}} \\ &\times \sqrt[s]{\mathfrak{A}(x, y, a, b)|f'(a)|^s + \mathfrak{B}(x, y, a, b)|f'(b)|^s}. \end{aligned} \tag{16}$$

Proof. The proof directly follows from Theorem 3.6 for $\beta, \sigma(0) = 1$, $w = 0$, that is, by power-mean inequality, $p-$convexity of $|f'|^s$ and properties of modulus to identity (14), we have

$$\begin{aligned} &|\Upsilon(x, y, a, b)| \\ &= \frac{1}{b^p - a^p} \left| \int_{a^p}^{b^p} h_{0,p}(t) t^{\frac{1-p}{p}} f'\left(\sqrt[p]{t}\right) dt \right| \\ &\leq \frac{1}{b^p - a^p} \int_{a^p}^{b^p} |h_{0,p}(t)| \left| f'\left(\sqrt[p]{t}\right) \right| t^{\frac{1-p}{p}(1-\frac{1}{s}+\frac{1}{s})} dt \\ &\leq \frac{1}{b^p - a^p} \left[\int_{a^p}^{b^p} h_{1,p}(t) t^{\frac{1-p}{p}} dt \right]^{\frac{s-1}{s}} \\ &\times \sqrt[s]{\int_{a^p}^{b^p} h_{1,p}(t) \left| f'\left(\sqrt[p]{\frac{b^p - t}{b^p - a^p} a^p + \frac{t - a^p}{b^p - a^p} b^p} \right) \right|^s t^{\frac{1-p}{p}} dt} \end{aligned}$$

$$\leq \frac{1}{b^p - a^p}\left[\int_{a^p}^{b^p} h_{1,p}(t) t^{\frac{1-p}{p}} dt\right]^{\frac{s-1}{s}}$$
$$\times \sqrt[s]{\int_{a^p}^{b^p} h_{1,p}(t)\left\{\frac{b^p - t}{b^p - a^p}|f'(a)|^s + \frac{t - a^p}{b^p - a^p}|f'(b)|^s\right\} t^{\frac{1-p}{p}} dt}.$$

This completes the proof. □

Theorem 3.8. *Let $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, interior of I, $a, b \in I^\circ$ with $a < b$ such that $|f'|$ is $p-$concave for $0 < p$; let $g(\xi) = \sqrt[p]{\xi}$, $\xi > 0$; $0 \leq \alpha \leq 1$ and $\rho, \beta > 0$, then*

$$\begin{aligned} |\Omega(x,y,a,b)| &\leq [\mathfrak{L}_{1,0}(x,y,a,b) + \mathfrak{L}_{2,0}(x,y,a,b)] \\ &\quad \times \left| f'\left(\frac{\mathfrak{L}_{1,\frac{1}{p}}(x,y,a,b) + \mathfrak{L}_{2,\frac{1}{p}}(x,y,a,b)}{\mathfrak{L}_{1,0}(x,y,a,b) + \mathfrak{L}_{2,0}(x,y,a,b)}\right)\right| \end{aligned}$$

Proof. By $p-$concavity of $|f'|$, properties of modulus to (9) and Jensen's integral inequality [30]:

$$\begin{aligned} |\Omega(x,y,a,b)| &= \left|\int_{a^p}^{b^p} \kappa_0(t) t^{\frac{1-p}{p}} f'\left(\sqrt[p]{t}\right) dt\right| \\ &\leq \int_{a^p}^{b^p} |\kappa_0(t)| |f'\left(\sqrt[p]{t}\right)| t^{\frac{1-p}{p}} dt \\ &\leq \left[\int_{a^p}^{b^p} t^{\frac{1-p}{p}} |\kappa_0(t)| dt\right] \left| f'\left(\frac{\int_{a^p}^{b^p} t^{\frac{1-p}{p}} |\kappa_{\frac{1}{p}}(t)| dt}{\int_{a^p}^{b^p} t^{\frac{1-p}{p}} |\kappa_0(t)| dt}\right)\right|. \end{aligned}$$

This completes the proof. □

Corollary 3.9. *Let $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, interior of I, $a, b \in I^\circ$ with $a < b$ such that $|f'|$ is $p-$concave for $0 < p$; let $g(\xi) = \sqrt[p]{\xi}$, $\xi > 0$; $0 \leq \alpha \leq 1$, then*

$$\begin{aligned} |\Upsilon(x,y,a,b)| &\leq (b^p - a^p)[\mathfrak{A}(x,y,a,b) + \mathfrak{B}(x,y,a,b)] \\ &\quad \times \left| f'\left(\frac{\mathfrak{J}_1(x,y,a,b) + \mathfrak{J}_2(x,y,a,b)}{\mathfrak{A}(x,y,a,b) + \mathfrak{B}(x,y,a,b)}\right)\right|. \end{aligned} \tag{17}$$

Proof. The proof directly follows from Theorem 3.8 for $\beta, \sigma(0) = 1$, $w = 0$, that is, by Jensen's integral inequality, $p-$concavity of $|f'|$ and properties of modulus to (14)

$$\begin{aligned} |\Upsilon(x,y,a,b)| &= \frac{1}{b^p - a^p}\left|\int_{a^p}^{b^p} h_{0,p}(t) t^{\frac{1-p}{p}} f'\left(\sqrt[p]{t}\right) dt\right| \\ &\leq \frac{1}{b^p - a^p}\int_{a^p}^{b^p} |h_{0,p}(t)| \left|f'\left(\sqrt[p]{t}\right)\right| t^{\frac{1-p}{p}} dt \\ &\leq \frac{1}{b^p - a^p}\left[\int_{a^p}^{b^p} t^{\frac{1-p}{p}} h_{1,p}(t) dt\right] \left| f'\left(\frac{\int_{a^p}^{b^p} t^{\frac{2-p}{p}} h_{1,p}(t) dt}{\int_{a^p}^{b^p} t^{\frac{1-p}{p}} h_{1,p}(t) dt}\right)\right|. \end{aligned}$$

This completes the proof. □

Note 3.10. 1. *It may be noted that for $p = 1$, $x = a$, $y = b$ Corollary 3.7 coincides with*

- *[11, Corollary 2.5].*
- *[11, Remarks 2.6, 2.7] for $\alpha = \frac{1}{3}, \frac{1}{2}$.*

- *[30, Theorems 1, 2] for* $\alpha = 1, 0$.

2. *For* $p = 1$*, Corollaries 3.7 and 3.9 coincide with [11, Theorems 2.3, 2.12].*
3. *For* $p = 1$*,* $x = a$*,* $y = b$ *Corollary 3.9 coincides with*
   - *[11, Remarks 2.15, 2.16] for* $\alpha = \frac{1}{3}, \frac{1}{2}$.
   - *trapezoid, midpoint inequalities in [30, Theorems 3] for* $\alpha = 1, 0$.
4. *For* $\alpha = 1$*,* $x = \frac{3a+b}{4}$ *and* $y = \frac{a+3b}{4}$ *Corollary 3.9 coincides with the second inequality in [12, Theorem 1.1]*

4. Applications

Here, in this section, we provide some applications on means, $f-$divergence measure, probability density function and some quadrature rules by using the results proved in Section 3.

4.1. Application for special means

Here, We shall consider the following special means:

- The arithmetic mean: $A(a,b) := \frac{a+b}{2}$, $a, b \geq 0$.
- The $p-$power mean:

$$M_p(a,b) := \sqrt[p]{\frac{a^p + b^p}{2}},\ a, b > 0.$$

- The $p-$logarithmic mean:

$$L_p(a,b) := \begin{cases} a, & a = b; \\ \sqrt[p]{\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}}, & a \neq b; a, b > 0; \mathbf{R} \ni p \neq -1. \end{cases}$$

Proposition 4.1. *Let* $f(t) = \frac{t^{1-p}}{1-p}; p > 1$ *so that* $|f'(t)|$ *is* $p-$*concave and* $g \equiv J$*, identity function, then all the conditions of Corollary 3.9 are satisfied and hence the following inequality holds:*

$$\left|\alpha A(a^{1-p}, b^{1-p}) + (1-\alpha) M_p^{1-p}(a,b) - L_{1-p}^{1-p}(a^p, b^p)\right|$$

$$\leq \frac{1-p}{b^p - a^p} \left\{ \frac{(M_p^{p+1}(a,b) - A(a^{p+1}, b^{p+1}))(2-4\alpha)}{p+1} \right.$$

$$\left. + (M_p(a,b) - a)(\alpha A(a^p, b^p) - (1-\alpha)a^p) + (M_p(a,b) - b)(\alpha A(a^p, b^p) - (1-\alpha)b^p) \right\}^{p+1}$$

$$\times \left\{ \frac{(M_p^{p+2}(a,b) - A(a^{p+2}, b^{p+2}))(2-4\alpha)}{p+2} \right.$$

$$\left. + \frac{(3\alpha b^p + \alpha a^p - 2b^p)(M_p^2(a,b) - b^2) + (3\alpha a^p + \alpha b^p - 2a^p)(M_p^2(a,b) - a^2)}{4} \right\}^{-p}$$

4.2. Applications on quadrature rules

Let I_n be a partition of the interval $[a^p, b^p]$ such that: $(a^p =) r_0^p < r_1^p < \dots < r_n^p (= b^p)$ and $d_i^p = r_{i+1}^p - r_i^p$, $\xi_i^p \in \left[r_i^p, \frac{r_i^p + r_{i+1}^p}{2}\right]$, $\zeta_i^p \in \left[\frac{r_i^p + r_{i+1}^p}{2}, r_{i+1}^p\right]$, $0 \leq i \leq n-1$. Let $S_\alpha(f, I_n, \xi, \zeta)$ be the extended fractional Simpson quadrature formula and $R_\alpha(f, I_n, \xi, \zeta)$ be the associated error of $I_\alpha(f \circ g, I_n, \xi, \zeta)$ by $S_\alpha(f, I_n, \xi, \zeta)$ for $\alpha \in [0, 1]$, then

$$I_\alpha(f \circ g, I_n, \xi, \zeta) = S_\alpha(f, I_n, \xi, \zeta) + R_\alpha(f, I_n, \xi, \zeta), \tag{18}$$

provided that:

$$\begin{aligned} S_\alpha(f, I_n, \xi, \zeta) := \sum_{i=0}^{n-1} \Bigg[&\Bigg\{ \left(\xi_i^p - r_i^p\right)^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\xi_i^p - r_i^p\right)^{2\rho}\right] - \left[\frac{2(\xi_i^p - r_i^p) - \alpha d_i^p}{2}\right]^\beta \\ &\times \mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{r_i^p + r_{i+1}^p - 2\xi_i^p}{2}\right)^\rho \left(\frac{2(\xi_i^p - r_i^p) - \alpha d_i^p}{2}\right)^\rho\right]\Bigg\} f(\xi_i) \\ &+ \Bigg\{ (r_{i+1}^p - \zeta_i^p)^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(r_{i+1}^p - \zeta_i^p\right)^{2\rho}\right] - \left[\frac{2(r_{i+1}^p - \zeta_i^p) - \alpha d_i^p}{2}\right]^\beta \\ &\times \mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{2\zeta_i^p - r_i^p - r_{i+1}^p}{2}\right)^\rho \left(\frac{2(r_{i+1}^p - \zeta_i^p) - \alpha d_i^p}{2}\right)^\rho\right]\Bigg\} f(\zeta_i) \\ &+ \Bigg\{ \mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{r_i^p + r_{i+1}^p - 2\xi_i^p}{2}\right)^\rho ((1-\alpha)\frac{d_i^p}{2})^\rho\right] \\ &+ \mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{2\zeta_i^p - r_i^p - r_{i+1}^p}{2}\right)^\rho \left(\frac{(1-\alpha) d_i^p}{2}\right)^\rho\right]\Bigg\} \left[\frac{(1-\alpha) d_i^p}{2}\right]^\beta f\left(\sqrt[p]{\frac{r_i^p + r_{i+1}^p}{2}}\right)\Bigg], \end{aligned} \tag{19}$$

$$\begin{aligned} I_\alpha(f \circ g, I_n, \xi, \zeta) := \sum_{i=0}^{n-1} d_i^p \Bigg\{ &\left(\mathfrak{J}^\sigma_{\rho,\beta,\xi_i^p-;w(\xi_i^p - r_i^p)^\rho} f \circ g\right)(r_i^p) \\ &+ \left(\mathfrak{J}^\sigma_{\rho,\beta,\frac{r_i^p + r_{i+1}^p}{2}-;w\left(\frac{r_i^p + r_{i+1}^p - 2\xi_i^p}{2}\right)^\rho} f \circ g\right)\left(\frac{\alpha r_{i+1}^p - \alpha r_i^p + 2r_i^p}{2}\right) \\ &- \left(\mathfrak{J}^\sigma_{\rho,\beta,\xi_i^p-;w\left(\frac{r_i^p + r_{i+1}^p - 2\xi_i^p}{2}\right)^\rho} f \circ g\right)\left(\frac{\alpha r_{i+1}^p - \alpha r_i^p + 2r_i^p}{2}\right) \\ &+ \left(\mathfrak{J}^\sigma_{\rho,\beta,\frac{r_i^p + r_{i+1}^p}{2}+;w\left(\frac{2\zeta_i^p - r_i^p - r_{i+1}^p}{2}\right)^\rho} f \circ g\right)\left(\frac{\alpha r_i^p - \alpha r_{i+1}^p + 2r_{i+1}^p}{2}\right) \end{aligned}$$

$$+\left(\mathfrak{I}^{\sigma}_{\rho,\beta,\zeta_i^p-;w\left(\frac{2\zeta_i^p-r_i^p-r_{i+1}^p}{2}\right)^{\rho}}f\circ g\right)\left(\frac{\alpha r_i^p-\alpha r_{i+1}^p+2r_{i+1}^p}{2}\right)$$

$$+\left(\mathfrak{I}^{\sigma}_{\rho,\beta,\zeta_i^p+;w\left(r_{i+1}^p-\zeta_i^p\right)^{\rho}}f\circ g\right)\left(r_{i+1}^p\right)\Bigg\} \tag{20}$$

$$R_\alpha(f,I_n,\xi,\zeta) \;:=\; \sum_{i=0}^{n-1}\frac{d_i^p}{p}\int_{r_i^p}^{r_{i+1}^p}\kappa_0(t)t^{\frac{1-p}{p}}f'\left(\sqrt[p]{t}\right)dt. \tag{21}$$

For $\alpha=0,\frac{1}{3},1$ in identity (18) we have the following special formulae.

- The generalized fractional mid point formula

$$S_0(f,I_n,\xi,\zeta):=\sum_{i=0}^{n-1}\Bigg[\Bigg\{\left(\xi_i^p-r_i^p\right)^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\xi_i^p-r_i^p\right)^{2\rho}\right]-(\xi_i^p-r_i^p)^{\beta}$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{(r_i^p+r_{i+1}^p-2\xi_i^p)(\xi_i^p-r_i^p)}{2}\right)^{\rho}\right]\Bigg\}f(\xi_i)$$

$$+\Bigg\{(r_{i+1}^p-\zeta_i^p)^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(r_{i+1}^p-\zeta_i^p\right)^{2\rho}\right]-(r_{i+1}^p-\zeta_i^p)^{\beta}$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{(2\zeta_i^p-r_i^p-r_{i+1}^p)(r_{i+1}^p-\zeta_i^p)}{2}\right)^{\rho}\right]\Bigg\}f\left(\zeta_i\right)+\Bigg\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{r_i^p+r_{i+1}^p-2\xi_i^p}{4}d_i^p\right)^{\rho}\right]$$

$$+\;\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{2\zeta_i^p-r_i^p-r_{i+1}^p}{4}d_i^p\right)^{\rho}\right]\Bigg\}\left[\frac{d_i^p}{2}\right]^{\beta}f\left(\sqrt[p]{\frac{r_i^p+r_{i+1}^p}{2}}\right)\Bigg],$$

- The generalized fractional trapezoid formula

$$S_1(f,I_n,\xi,\zeta):=\sum_{i=0}^{n-1}\Bigg[\Bigg\{\left(\xi_i^p-r_i^p\right)^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\xi_i^p-r_i^p\right)^{2\rho}\right]-\left[\frac{2\xi_i^p-2r_i^p-d_i^p}{2}\right]^{\beta}$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{(r_i^p+r_{i+1}^p-2\xi_i^p)(2\xi_i^p-2r_i^p-d_i^p)}{4}\right)^{\rho}\right]\Bigg\}f(\xi_i)$$

$$+\Bigg\{(r_{i+1}^p-\zeta_i^p)^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(r_{i+1}^p-\zeta_i^p\right)^{2\rho}\right]-\left[\frac{2r_{i+1}^p-2\zeta_i^p-d_i^p}{2}\right]^{\beta}$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{(2\zeta_i^p-r_i^p-r_{i+1}^p)(2r_{i+1}^p-2\zeta_i^p-d_i^p)}{4}\right)^{\rho}\right]\Bigg\}f\left(\zeta_i\right)\Bigg]$$

- The generalized fractional Simpson formula

$$S_{\frac{1}{3}}(f,I_n,\xi,\zeta) := \sum_{i=0}^{n-1}\left[\left\{\left(\xi_i^p - r_i^p\right)^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\xi_i^p - r_i^p\right)^{2\rho}\right] - \left[\frac{6\xi_i^p - 6r_i^p - d_i^p}{6}\right]^{\beta}\right.\right.$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{(r_i^p + r_{i+1}^p - 2\xi_i^p)(6\xi_i^p - 6r_i^p - d_i^p)}{12}\right)^{\rho}\right]\Bigg\} f(\xi_i)$$

$$+\left\{(r_{i+1}^p - \zeta_i^p)^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(r_{i+1}^p - \zeta_i^p\right)^{2\rho}\right] - \left[\frac{6r_{i+1}^p - 6\zeta_i^p - d_i^p}{6}\right]^{\beta}\right.$$

$$\times\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{(2\zeta_i^p - r_i^p - r_{i+1}^p)(6r_{i+1}^p - 6\zeta_i^p - d_i^p)}{12}\right)^{\rho}\right]\Bigg\} f(\zeta_i)$$

$$+\left\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{r_i^p + r_{i+1}^p - 2\xi_i^p}{6}d_i^p\right)^{\rho}\right]\right.$$

$$\left.\left.+\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{2\zeta_i^p - r_i^p - r_{i+1}^p}{6}d_i^p\right)^{\rho}\right]\right\}\left[\frac{d_i^p}{3}\right]^{\beta} f\left(\sqrt[p]{\frac{\xi_i^p + \zeta_i^p}{2}}\right)\right],$$

Theorem 4.2. *Let the condition of Theorem* 3.6 *be satisfied. Let* $S_\alpha(f,I_n,\xi,\zeta)$, $I_\alpha(f,I_n,\xi,\zeta)$ *and* $R_\alpha(f,I_n,\xi,\zeta)$ *be defined by* (19), (20) *and* (21) *respectively, then*

$$|R_\alpha(f,I_n,\xi,\zeta)| \leq \max\{|f'(a)|,|f'(b)|\}\sum_{i=0}^{n-1} d_i^p\frac{\mathfrak{L}_{1,0}(\xi_i,\zeta_i,r_i,r_{i+1}) + \mathfrak{L}_{2,0}(\xi_i,\zeta_i,r_i,r_{i+1})}{p}.$$

Proof. Application of Theorem 3.6 to the subinterval $[r_i^p, r_{i+1}^p]$, yields the following:

$$\begin{aligned}
&\left|R_\alpha(f,I_n,\xi,\zeta)\right| \\
&= \left|\sum_{i=0}^{n-1}\frac{d_i^p}{p}\int_{r_i^p}^{r_{i+1}^p}\kappa_0(t)t^{\frac{1-p}{p}}f'(\sqrt[p]{t})dt\right| \\
&\leq \sum_{i=0}^{n-1}\frac{d_i^p}{p}\sqrt[s]{\int_{r_i^p}^{r_{i+1}^p}|\kappa_0(t)|t^{\frac{1-p}{p}}\left|f'\left(\sqrt[p]{\frac{r_{i+1}^p - t}{r_{i+1}^p - r_i^p}r_i^p + \frac{t - r_i^p}{r_{i+1}^p - r_i^p}r_{i+1}^p}\right)\right|^s dt} \\
&\times\left[\int_{r_i^p}^{r_{i+1}^p}|\kappa_0(t)|t^{\frac{1-p}{p}}dt\right]^{\frac{s-1}{s}} \\
&\leq \sum_{i=0}^{n-1}\frac{d_i^p}{p}\sqrt[s]{\int_{r_i^p}^{r_{i+1}^p}|\kappa_0(t)|\left\{\frac{r_{i+1}^p - t}{r_{i+1}^p - r_i^p}|f'(r_i)|^s + \frac{t - r_i^p}{r_{i+1}^p - r_i^p}|f'(r_{i+1})|^s\right\}t^{\frac{1-p}{p}}dt}
\end{aligned}$$

$$\times\left[\int_{r_i^p}^{r_{i+1}^p}|\kappa_0(t)|t^{\frac{1-p}{p}}dt\right]^{\frac{s-1}{s}}$$
$$=\sum_{i=0}^{n-1}\frac{d_i^p}{p}\left[\mathfrak{L}_{1,0}(\xi_i,\zeta_i,r_i,r_{i+1})+\mathfrak{L}_{2,0}(\xi_i,\zeta_i,r_i,r_{i+1})\right]^{\frac{s-1}{s}}$$
$$\times\sqrt[s]{\mathfrak{L}_{1,0}(\xi_i,\zeta_i,r_i,r_{i+1})|f'(r_i)|^s+\mathfrak{L}_{2,0}(\xi_i,\zeta_i,r_i,r_{i+1})|f'(r_{i+1})|^s}.$$
$$=\sum_{i=0}^{n-1}d_i^p\frac{\mathfrak{L}_{1,0}(\xi_i,\zeta_i,r_i,r_{i+1})+\mathfrak{L}_{2,0}(\xi_i,\zeta_i,r_i,r_{i+1})}{p}$$
$$\times\sqrt[s]{\frac{\mathfrak{L}_{1,0}(\xi_i,\zeta_i,r_i,r_{i+1})|f'(r_i)|^s+\mathfrak{L}_{2,0}(\xi_i,\zeta_i,r_i,r_{i+1})|f'(r_{i+1})|^s}{\mathfrak{L}_{1,0}(\xi_i,\zeta_i,r_i,r_{i+1})+\mathfrak{L}_{2,0}(\xi_i,\zeta_i,r_i,r_{i+1})}}.$$

This completes the proof. □

Theorem 4.3. *Let the condition of Theorem* 3.8 *be satisfied. Let* $S_\alpha(f,I_n,\xi,\zeta)$, $I_\alpha(f,I_n,\xi,\zeta)$ *and* $R_\alpha(f,I_n,\xi,\zeta)$ *be defined by* (19), (20) *and* (21) *respectively, then*

$$|R_\alpha(f,I_n,\xi,\zeta)| \le \sum_{i=0}^{n-1}d_i^p\frac{\mathfrak{L}_{1,0}(\xi_i,\zeta_i,r_i,r_{i+1})+\mathfrak{L}_{2,0}(\xi_i,\zeta_i,r_i,r_{i+1})}{p}\times\left|f'\left(\frac{\mathfrak{L}_{1,\frac{1}{p}}(\xi_i,\zeta_i,r_i,r_{i+1})+\mathfrak{L}_{2,\frac{1}{p}}(\xi_i,\zeta_i,r_i,r_{i+1})}{\mathfrak{L}_{1,0}(\xi_i,\zeta_i,r_i,r_{i+1})+\mathfrak{L}_{2,0}(\xi_i,\zeta_i,r_i,r_{i+1})}\right)\right|$$

Proof. Application of Theorem 3.8 to the subinterval $[r_i^p, r_{i+1}^p]$, yields the following:

$$|R_\alpha(f,I_n,\xi,\zeta)|=\left|\sum_{i=0}^{n-1}\frac{d_i^p}{p}\int_{r_i^p}^{r_{i+1}^p}\kappa_0(t)t^{\frac{1-p}{p}}f'\left(\sqrt[p]{t}\right)dt\right|$$

$$\le\sum_{i=0}^{n-1}\frac{d_i^p}{p}\int_{r_i^p}^{r_{i+1}^p}|\kappa_0(t)|\left|f'\left(\sqrt[p]{\frac{r_{i+1}^p-t}{r_{i+1}^p-r_i^p}r_i^p+\frac{t-r_i^p}{r_{i+1}^p-r_i^p}r_{i+1}^p}\right)\right|t^{\frac{1-p}{p}}dt$$

$$\le\sum_{i=0}^{n-1}\frac{d_i^p}{p}\int_{r_i^p}^{r_{i+1}^p}|\kappa_0(t)|t^{\frac{1-p}{p}}dt\left|f'\left(\frac{\int_{r_i^p}^{r_{i+1}^p}t^{\frac{1-p}{p}}|\kappa_{\frac{1}{p}}(t)|dt}{\int_{r_i^p}^{r_{i+1}^p}t^{\frac{1-p}{p}}|\kappa_0(t)|dt}\right)\right|$$

$$=\sum_{i=0}^{n-1}d_i^p\frac{\mathfrak{L}_{1,0}(\xi_i,\zeta_i,r_i,r_{i+1})+\mathfrak{L}_{2,0}(\xi_i,\zeta_i,r_i,r_{i+1})}{p}$$

$$\times\left|f'\left(\frac{\mathfrak{L}_{1,\frac{1}{p}}(\xi_i,\zeta_i,r_i,r_{i+1})+\mathfrak{L}_{2,\frac{1}{p}}(\xi_i,\zeta_i,r_i,r_{i+1})}{\mathfrak{L}_{1,0}(\xi_i,\zeta_i,r_i,r_{i+1})+\mathfrak{L}_{2,0}(\xi_i,\zeta_i,r_i,r_{i+1})}\right)\right|$$

□

Note 4.4. *For* $\alpha=0,\frac{1}{3},1$ *in Theorems 4.2, 4.3, we get some generalisations of results in [11, 30]*

4.3. $f-$divergence measures

Let the set ϕ and the σ finite measure μ be given, and let the set of all probability densities on μ to be defined on $\Omega := \{\chi | \chi : \phi \to \mathbf{R}, \chi(\varpi) > 0, \int_\phi \chi(\varpi) d\mu(\varpi) = 1\}$. Let $f : (0,\infty) \to \mathbf{R}$ be given mapping and consider $D_f(\chi,\psi)$ defined by:

$$D_f(\chi,\psi) := \int_\phi \chi(\varpi) f\left[\frac{\psi(\varpi)}{\chi(\varpi)}\right] d\mu(\varpi), \quad \chi,\psi \in \Omega. \tag{22}$$

If f is convex, then (22) is called as the Csisźar $f-$divergence. Consider the following Hermite-Hadamard (HH) divergence:

$$D^f_{HH}(\chi,\psi) := \int_\phi \chi(\varpi) \frac{\int_1^{\frac{\psi(\varpi)}{\chi(\varpi)}} f(t)dt}{\frac{\psi(\varpi)}{\chi(\varpi)} - 1} d\mu(\varpi), \quad \chi,\psi \in \Omega, \tag{23}$$

where f is convex on $(0,\infty)$ with $f(1) = 0$. Note that $D^f_{HH}(\chi,\psi) \geq 0$ with the equality holds if and only if $\chi = \psi$.

Proposition 4.5. *Let $f : I \subseteq \mathbf{R}^+ \to \mathbf{R}$ be a differentiable function on I°, $a,b \in I^\circ$ such that $|f'|$ is convex and $f(1) = 0$, then*

$$\left|\frac{\alpha}{2} D_f(\chi,\psi) + (1-\alpha)\int_\phi \chi(\varpi) f\left(\frac{\psi(\varpi)+\chi(\varpi)}{2\chi(\varpi)}\right) d\mu(\varpi) - D^f_{HH}(\chi,\psi)\right| \leq \mathfrak{C} + \mathfrak{D}|f'(1)|, \tag{24}$$

$\mathfrak{C}$ and $\mathfrak{D}$ are defined by (7) *and* (8) *respectively.*

Proof. Let $\Phi_1 := \{\varpi \in \phi : \psi(\varpi) > \chi(\varpi)\}$; $\Phi_2 := \{\varpi \in \phi : \psi(\varpi) < \chi(\varpi)\}$ and $\Phi_3 := \{\varpi \in \phi : \psi(\varpi) = \chi(\varpi)\}$. Obviously, if $\varpi \in \Phi_3$, then equality holds in (24). Now, if $\varpi \in \Phi_1$, then for $x = a = 1$, $p = 1$; $b = \frac{\psi(\varpi)}{\chi(\varpi)}$; $g \equiv I$, identity function, in Corollary 3.4, multiplying both sides to the obtained result by $\chi(\varpi)$ and integrating over Φ_1, we have

$$\left|(1-\alpha)\int_{\Phi_1} \chi(\varpi) f\left(\frac{\psi(\varpi)+\chi(\varpi)}{2\chi(\varpi)}\right) d\mu(\varpi) + \frac{\alpha}{2}\int_{\Phi_1} \chi(\varpi) f\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right) d\mu(\varpi) - \int_{\Phi_1} \chi(\varpi) \frac{\int_1^{\frac{\psi(\varpi)}{\chi(\varpi)}} f(t)dt}{\frac{\psi(\varpi)}{\chi(\varpi)} - 1} d\mu(\varpi)\right| \leq [\mathfrak{A}_1|f'(1)| + \mathfrak{B}_1], \tag{25}$$

provided that:

$$\mathfrak{A}_1 := \frac{2\alpha-1}{12}\int_{\Phi_1} \frac{(\psi(\varpi)+\chi(\varpi))^3 - 4\chi^3(\varpi) - 4\psi^3(\varpi)}{(\psi(\varpi)-\chi(\varpi))^2} d\mu(\varpi)$$

$$+\frac{1-\alpha}{2}\int_{\Phi_1} \psi(\varpi)d\mu(\varpi) + \int_{\Phi_1} \frac{(2\psi(\varpi) - 5\alpha\psi(\varpi) - 3\alpha\chi(\varpi) + 2\chi(\varpi))(\psi^2(\varpi) + 2\psi(\varpi)\chi(\varpi) - 3\chi^2(\varpi))}{16(\psi(\varpi)-\chi(\varpi))^2} d\mu(\varpi$$

$$+\int_{\Phi_1} \frac{(7\alpha\psi(\varpi) - 4\psi(\varpi) + \alpha\chi(\varpi))(3\psi^2(\varpi) - 2\psi(\varpi)\chi(\varpi) - \chi^2(\varpi))}{16(\psi(\varpi)-\chi(\varpi))^2} d\mu(\varpi)$$

$$\mathfrak{B}_1 := \frac{2\alpha-1}{12}\int_{\Phi_1} \frac{[(\psi(\varpi)+\chi(\varpi))^3 - 4\chi^3(\varpi) - 4\psi^3(\varpi)]}{(\chi(\varpi)-\psi(\varpi))^2} \left|f'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right| d\mu(\varpi)$$

$$+\frac{1-\alpha}{2}\int_{\Phi_1}\chi(\varpi)\left|f'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|d\mu(\varpi)+\int_{\Phi_1}\left|f'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|$$

$$\times\frac{(2\psi(\varpi)-5\alpha\chi(\varpi)-3\alpha\psi(\varpi)+2\chi(\varpi))(\chi^2(\varpi)+2\psi(\varpi)\chi(\varpi))-3\psi^2(\varpi)}{16(\chi(\varpi)-\psi(\varpi))^2}d\mu(\varpi)$$

$$+\int_{\Phi_1}\frac{(7\alpha\chi(\varpi)-4\chi(\varpi)+\alpha\psi(\varpi))(3\chi^2(\varpi)-\psi^2(\varpi)-2\psi(\varpi)\chi(\varpi))}{16(\chi(\varpi)-\psi(\varpi))^2}\left|f'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|d\mu(\varpi)$$

Similarly, if $\varpi\in\Phi_2$, then for $y=b=1$, $p=1$; $a=\frac{\psi(\varpi)}{\chi(\varpi)}$; $g\equiv I$, identity function, in Corollary 3.4, multiplying both sides to the obtained result by $\chi(\varpi)$ and integrating over Φ_2, we have

$$\left|(1-\alpha)\int_{\Phi_2}\chi(\varpi)f\left(\frac{\psi(\varpi)+\chi(\varpi)}{2\chi(\varpi)}\right)d\mu(\varpi)+\frac{\alpha}{2}\int_{\Phi_2}\chi(\varpi)f\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)d\mu(\varpi)\right.$$

$$\left.-\int_{\Phi_2}\chi(\varpi)\frac{\int_1^{\frac{\psi(\varpi)}{\chi(\varpi)}}f(t)dt}{\frac{\psi(\varpi)}{\chi(\varpi)}-1}d\mu(\varpi)\right|\leq[\mathfrak{A}_2+\mathfrak{B}_2|f'(1)|],\tag{26}$$

provided that:

$$\mathfrak{A}_2:=\frac{2\alpha-1}{12}\int_{\Phi_2}\frac{(\psi(\varpi)+\chi(\varpi))^3-4\chi^3(\varpi)-4\psi^3(\varpi)}{(\chi(\varpi)-\psi(\varpi))^2}\left|f'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|d\mu(\varpi)$$

$$+\frac{1-\alpha}{2}\int_{\Phi_2}\chi(\varpi)\left|f'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|d\mu(\varpi)+\int_{\Phi_2}\left|f'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|$$

$$\times\frac{(2\psi(\varpi)-5\alpha\chi(\varpi)-3\alpha\psi(\varpi)+2\chi(\varpi))(\chi^2(\varpi)+2\psi(\varpi)\chi(\varpi)-3\psi^2(\varpi))}{16(\chi(\varpi)-\psi(\varpi))^2}d\mu(\varpi)$$

$$+\int_{\Phi_2}\frac{(7\alpha\chi(\varpi)-4\chi(\varpi)+\alpha\psi(\varpi))(3\chi^2(\varpi)-2\psi(\varpi)\chi(\varpi)-\psi^2(\varpi))}{16(\chi(\varpi)-\psi(\varpi))^2}\left|f'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|d\mu(\varpi)$$

$$\mathfrak{B}_2:=\frac{1-2\alpha}{12}\int_{\Phi_2}\frac{(\psi(\varpi)+\chi(\varpi))^3-4\chi^3(\varpi)-4\psi^3(\varpi)}{\left|\psi(\varpi)-\chi(\varpi)\right|^2}d\mu(\varpi)$$

$$+\frac{1-\alpha}{2}\int_{\Phi_2}\psi(\varpi)d\mu(\varpi)$$

$$+\int_{\Phi_2}\frac{(2\psi(\varpi)-5\alpha\psi(\varpi)-3\alpha\chi(\varpi)+2\chi(\varpi))(3\chi^2(\varpi)-\psi^2(\varpi)-2\psi(\varpi)\chi(\varpi))}{16\left|\psi(\varpi)-\chi(\varpi)\right|^2}d\mu(\varpi)$$

$$+\int_{\Phi_2}\frac{(7\alpha\psi(\varpi)-4\psi(\varpi)+\alpha\chi(\varpi))(\chi^2(\varpi)-3\psi^2(\varpi)+2\psi(\varpi)\chi(\varpi))}{16\left|\psi(\varpi)-\chi(\varpi)\right|^2}d\mu(\varpi)$$

Adding inequalities (25) and (26), and utilizing triangular inequality, we get the desired result (24). □

4.4. Probability density functions

Let $g : [a, b] \to [0, 1]$ be the probability density function of a continuous random variable X with the cumulative distribution function, F, given by:

$$F(\varrho) = Pr(X \leq \varrho) = \int_a^{\varrho} g(t)dt \text{ and } E(X) = \int_a^b tdF(t) = b - \int_a^b F(t)dt. \tag{27}$$

Then, from Corollary 3.4 for $g \equiv J$, identity function, $p = 1$, $y = b$, $x = a$, we have the following result:

$$\left|\frac{\alpha}{2} + (1-\alpha)Pr\left(X \leq \frac{a+b}{2}\right) - \frac{1}{b-a}(b - E(X))\right| \leq \mathfrak{E}|g(a)| + \mathfrak{F}|g(b)|,$$

provided that:

$$\begin{aligned} \mathfrak{E} \;:=\; & \frac{(2\alpha-1)[(a+b)^3 - 4a^3 - 4b^3] + 6b(b-a)^2(1-\alpha)}{12(b-a)^2} \\ & + \frac{(2b - 5b\alpha - 3a\alpha + 2a)(b^2 + 2ab - 3a^2) + (7b\alpha - 4b + a\alpha)(3b^2 - a^2 - 2ab)}{16(b-a)^2} \end{aligned}$$

$$\begin{aligned} \mathfrak{F} \;:=\; & \frac{[(a+b)^3 - 4a^3 - 4b^3](1-2\alpha) + 6a(b-a)^2(\alpha-1)}{12(b-a)^2} \\ & + \frac{(7a\alpha + \alpha b - 4a)(b^2 + 2ab - 3a^2) + (2b - 3b\alpha - 5a\alpha + 2a)(3b^2 - a^2 - 2ab)}{16(b-a)^2} \end{aligned}$$

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