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Hybrid number matrices

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Abstract. The aim of this study is to investigate some properties of hybrid number matrices. Firstly, we introduce hybrid numbers with some of their properties. Then we prove that any hybrid number has a 2×2 complex matrix representation, and we investigate hybrid number matrices using the properties of complex matrices. Also we give answers to the following basic question "If **AB** $=I$, is it true that **BA** $=I$ for hybrid number matrices?" Then we define the complex adjoint matrix and the $q-$determinant of hybrid number matrices and give some important properties. Finally, we give an explicit formula for the inverse of a hybrid number matrix by using complex matrices.

1. Introduction

Examining the matrices of different non-field number sets such as quaternions, split quaternions, complex split quaternions, and octanions is one of the topics of interest for mathematicians in recent years. Each quaternion can be represented by a complex matrix of the 2×2 type. Besides, each split quaternion can be represented by a complex matrix of type 2×2 as well as by a real matrix of type 2×2 it is. Because split quaternions are isomorphic to the ring of 2×2 real matrices. With the help of these representations, matrices whose inputs are quaternions and split quaternions can be examined, and the determinants of the matrices of these non-commutative number sets can be calculated. Especially, Zhang's study [32] is one of the most important studies on this subject. Properties and some applications of split quaternion matrices are examined in detail in the articles [2], [5], [10], [12], [15], [17], [19], [20], [21], [30] and [31]. Moreover, the analysis was made in case the inputs were complex, hyperbolic and dual numbers ([3], [4], [11], [13], [14], [18] and [26]).

The set of split quaternions is isomorphic to the set of 2x2 real matrices with

$$\begin{aligned} \varphi &: \mathbb{H} \to \mathbb{M}_{2\times 2}(\mathbb{R}), \\ \varphi(q_0 + q_1 i + q_2 j + q_3 h) &= \begin{bmatrix} q_0 + q_2 & q_3 - q_1 \\ q_3 + q_1 & q_0 - q_2 \end{bmatrix} [21]. \end{aligned}$$

Similarly, the set of hybrid numbers is isomorphic to the set of 2x2 real matrices with

$$\begin{aligned} \varphi &: \mathbb{K} \to \mathbb{M}_{2\times 2}(\mathbb{R}), \\ \varphi(q_0 + q_1 i + q_2 j + q_3 h) &= \begin{bmatrix} q_0 + q_2 & q_1 - q_2 + q_3 \\ q_2 - q_1 + q_3 & q_0 - q_2 \end{bmatrix} [22]. \end{aligned}$$

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Therefore, these two sets of numbers are isomorphic to each other. In split quaternions, the squares of the units are taken as 1 or -1, while in hybrid numbers 1, -1 and 0 are taken. This provides some advantages. As a matter of fact, there are split quaternions with a square of 0 and these are called lightlike quaternions in the literature. Particularly in Lorentzian rotations, split quaternions should be classified. This classification is lightlike, spacelike and timelike. On the other hand, vector parts are also classified separately. However, such a classification is not required for hybrid numbers. Hybrid numbers, which are a combination of dual, complex and hyperbolic numbers, show direct elliptic, parabolic and hyperbolic rotation depending on the type of number. That is, it has a more natural classification structure compared to split quaternions. For example, a hyperbolic hybrid number corresponds to a hyperbolic rotation, and an elliptical hybrid number corresponds to an elliptical rotation [25]. With the help of both split and hybrid numbers, it is possible to talk about the polar representation of a 2x2 matrix. Thus, calculating the n-th degree roots of a 2x2 matrix can also be easily done with the properties of these two sets of numbers [23]. Because these two sets of numbers are isomorphic, the properties provided for split quaternions and matrices can be applied to hybrid numbers and matrices. However, due to the ease of classification of hybrid numbers, we believe that it will be important to adapt the properties known for split quaternions for the hybrid number set.

The set of hybrid numbers is non-commutative quaternary numbers that combined and generalized the dual, complex, and hyperbolic numbers. Also, it is isomorphic to the ring of 2×2 matrices, like the split quaternions [22]. With the help of this isomorphism, some applications can be given by classifying the matrices. For example, explicit formulas giving the n-th roots of any 2×2 matrix with the help of hybrid numbers are expressed by Özdemir [23]. In addition, De Moivre's and Euler's formulas for 4×4 matrices of hybrid numbers were given by M. Akbıyık et al. using trigonometric identities [1]. Since hybrid numbers combine three well-known number systems, numerous applications can be mentioned in many fields such as number theory, linear algebra, kinematics, geometry and physics ([6], [7], [8], [9], [16], [27], [28] and [29]).

In this paper, we aim to examine some properties of matrices of hybrid numbers that bring together the complex, dual and hyperbolic number system, which has attracted great interest recently. First, we present an introduction of hybrid numbers to provide the necessary background. We define the 2×2 complex matrix representation of the hybrid number by proving that any hybrid number is isomorphic to a particular subset of 2×2 complex matrices. We also show that the determinant of this 2×2 complex matrix is consistent with the determinant of the 2×2 real matrix. Then we give some properties of matrix representations of hybrid numbers with their proofs. In the last part, we define hybrid number matrices and give some of their properties. Next, we define the complex adjoint matrix of a hybrid number matrix and give a method of finding the inverse of a hybrid number matrix using the complex adjoint matrix. This is also a method of finding the inverse of a hybrid number. We show that the determinant of the complex adjoint matrix of an $n \times n$ hybrid number matrix is also consistent with the determinant of the $2n \times 2n$ real matrix representation.

2. Hybrid Numbers

The set of hybrid numbers can be represented as

$$\mathbb{K} = \left\{ p_1 + p_2\mathbf{i} + p_3\,\varepsilon + p_4\mathbf{h} : \mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon,\ p_1, p_2, p_3, p_4 \in \mathbb{R} \right\}.$$

The multiplication table of the units $\mathbf{i}$, ε and $\mathbf{h}$ is as follows and the product of two hybrid numbers is done with the help of this table.

$\bullet$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$\mathbf{1-h}$	$\varepsilon + \mathbf{i}$
ε	$\mathbf{h+1}$	0	$-\varepsilon$
$\mathbf{h}$	$-\varepsilon - \mathbf{i}$	ε	1

The set of hybrid numbers is a non-commutative ring, so that in general we have to expect $\mathbf{pq} \neq \mathbf{qp}$ for $\mathbf{p}, \mathbf{q} \in \mathbb{K}$. For any hybrid number $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\,\varepsilon + p_4\mathbf{h}$, we define $S_{\mathbf{p}} = p_1$, the scalar part of $\mathbf{p}$;

$V_{\mathbf{p}} = p_2\mathbf{i} + p_3\,\varepsilon + p_4\mathbf{h}$, the vector part of $\mathbf{p}$. The conjugate of a hybrid number is denoted by $\overline{\mathbf{p}}$, and it is

$$\overline{\mathbf{p}} = S_{\mathbf{p}} - V_{\mathbf{p}} = p_1 - p_2\mathbf{i} - p_3\varepsilon - p_4\mathbf{h}.$$

The real number

$$C(\mathbf{p}) = \mathbf{p}\overline{\mathbf{p}} = \mathbf{p}\,\overline{\mathbf{p}} = p_1^2 + (p_2 - p_3)^2 - p_3^2 - p_4^2$$

is called the character of the hybrid number $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h}$. Besides, the product $C(\mathbf{p}) = \mathbf{p}\overline{\mathbf{p}}$ determines the character of a hybrid number. $\mathbf{p}$ is called spacelike, timelike, or lightlike, if $C(\mathbf{p}) < 0$, $C(\mathbf{p}) > 0$ or $C(\mathbf{p}) = 0$, respectively. Also, using the product of hybrid numbers, one can show the equality

$$C(\mathbf{pq}) = C(\mathbf{p})\,C(\mathbf{q}).$$

The norm of a hybrid number is defined as

$$\|\mathbf{p}\| = \sqrt{|C(\mathbf{p})|} = \sqrt{\left|p_1^2 + (p_2 - p_3)^2 - p_3^2 - p_4^2\right|}.$$

The inverse of the hybrid number $\mathbf{p}$ is

$$\mathbf{p}^{-1} = \frac{\overline{\mathbf{p}}}{C(\mathbf{p})},\ \ \|\mathbf{p}\| \neq 0.$$

Therefore, it can be said that lightlike hybrid numbers are not inverted. Detailed information for hybrid numbers can be found in Özdemir's article [22].

As with quaternions, right and left product matrices are defined as follows. Consider the transformations

$$\begin{array}{l} \mathcal{L}_{\mathbf{p}} : \mathbb{K} \to \mathbb{K} \\ \mathbf{q} \to \mathcal{L}_{\mathbf{p}}(\mathbf{q}) = \mathbf{pq} \end{array} \quad \text{and} \quad \begin{array}{l} \mathcal{R}_{\mathbf{p}} : \mathbb{K} \to \mathbb{K} \\ \mathbf{q} \to \mathcal{R}_{\mathbf{p}}(\mathbf{q}) = \mathbf{qp} \end{array}$$

for a hybrid number $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h} \in \mathbb{K}$. With these transformations, 4×4 right and left real product matrices of a hybrid number are obtained as follows, respectively:

$$\mathcal{R}_{\mathbf{p}} = \begin{bmatrix} p_1 & p_3 - p_2 & p_2 & p_4 \\ p_2 & p_1 + p_4 & 0 & -p_2 \\ p_3 & p_4 & p_1 - p_4 & p_3 - p_2 \\ p_4 & -p_3 & p_2 & p_1 \end{bmatrix},$$

$$\mathcal{L}_{\mathbf{p}} = \begin{bmatrix} p_1 & -p_2 + p_3 & p_2 & p_4 \\ p_2 & p_1 - p_4 & 0 & p_2 \\ p_3 & -p_4 & p_1 + p_4 & p_2 - p_3 \\ p_4 & p_3 & -p_2 & p_1 \end{bmatrix}.$$

Also, we have $\det \mathcal{L}_{\mathbf{p}} = \det \mathcal{R}_{\mathbf{p}} = \|\mathbf{p}\|^4$ [24].

A hybrid number $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h}$ can be written as $\mathbf{p} = x + y\mathbf{h}$ or $\mathbf{p} = x + \mathbf{h}\overline{y}$ as the sum of two complex numbers, where $x = a + (b - c)\,\mathbf{i} \in \mathbb{C}$, $y = d + c\mathbf{i} \in \mathbb{C}$. Therefore, for $\mathbf{p} = x + y\mathbf{h}$ and $\mathbf{q} = u + v\mathbf{h}$, the multiplication of $\mathbf{p}$ and $\mathbf{q}$ can be given by

$$\begin{aligned} \mathbf{pq} &= (x + y\mathbf{h})(u + v\mathbf{h}) \\ &= xu + y\mathbf{h}v\mathbf{h} + y\mathbf{h}u + xv\mathbf{h} \\ &= xu + y\overline{v} + (y\overline{u} + xv)\,\mathbf{h}. \end{aligned}$$

The following properties can be easily shown.

Theorem 2.1. *For any* $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{K}$, *the following properties are satisfied.*

1. $\mathbf{h}x_1 = \overline{x_1}\mathbf{h}$ *for* $x_1 \in \mathbb{C}$
2. $\mathbf{h}(x_1 + x_2\mathbf{h}) = \overline{x_2} + \overline{x_1}\mathbf{h}$ *for* $x_1, x_2 \in \mathbb{C}$
3. $\overline{\mathbf{ph}} = -\mathbf{h}\overline{\mathbf{p}}$
4. $\mathbf{p}^2 = S_{\mathbf{p}}^2 - \left\|V_{\mathbf{p}}\right\| + 2S_{\mathbf{p}}V_{\mathbf{p}}$
5. $\overline{\mathbf{pq}} = \overline{\mathbf{q}}\ \overline{\mathbf{p}}$
6. $(\mathbf{pq})\mathbf{r} = \mathbf{p}(\mathbf{qr})$
7. $\mathbf{pq} \neq \mathbf{qp}$ *in general*
8. $\mathbf{p} = \overline{\mathbf{p}} \Leftrightarrow \mathbf{p} \in \mathbb{R}$
9. *If* $p_1^2 + (p_2 - p_3)^2 \neq p_3^2 + p_4^2$ *then* $\mathbf{p}^{-1} = \dfrac{\overline{\mathbf{p}}}{\left\|\mathbf{p}\right\|^2}$
10. $\forall \mathbf{p} \in \mathbb{K}$ *there exists a unique representation of the form* $\mathbf{p} = x_1 + x_2\mathbf{h} \in \mathbb{K}$ *such that* $x_1, x_2 \in \mathbb{C}$.

2.1. The Matrix Representation of Hybrid Numbers

The matrix representation of hybrid numbers is particularly important to facilitate multiplication of hybrid numbers. By defining an isomorphism between 2×2 real matrices and hybrid numbers, and between a special subset of 2×2 complex matrices and hybrid numbers, we can easily operate between hybrid numbers and prove many of their properties. In [22], by defining an isomorphism between 2×2 real matrices and hybrid numbers, the representation of hybrid numbers with 2×2 real matrices is given by Özdemir as

$$\aleph_{\mathbf{p}} = \begin{bmatrix} p_1 + p_3 & p_2 - p_3 + p_4 \\ p_3 - p_2 + p_4 & p_1 - p_3 \end{bmatrix}$$

for $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h} \in \mathbb{K}$.

We can give some properties of this matrix as follows. Proofs of these properties will not be given here, as they can be seen quite easily.

Theorem 2.2. *Let* $\mathbf{p}, \mathbf{q} \in \mathbb{K}$ *and* $\lambda \in \mathbb{R}$. *Then*

1. $\aleph_{\mathbf{p}} = \aleph_{\mathbf{q}} \Longleftrightarrow \mathbf{p} = \mathbf{q}$
2. $\aleph_{\mathbf{p}+\mathbf{q}} = \aleph_{\mathbf{p}} + \aleph_{\mathbf{q}}$
3. $\aleph_{\mathbf{pq}} = \aleph_{\mathbf{p}}\aleph_{\mathbf{q}}$
4. $\aleph_{\lambda\mathbf{p}} = \aleph_{\mathbf{p}\lambda} = \lambda\aleph_{\mathbf{p}}$
5. $\aleph_{\mathbf{1}} = I_2$
6. $\aleph_{\mathbf{p}} + \aleph_{\overline{\mathbf{p}}} = 2p_1 I_2$
7. $\aleph_{\mathbf{p}^{-1}} = \aleph_{\mathbf{p}}^{-1}$, *where* $\left\|\mathbf{p}\right\| \neq 0$
8. $\left(\aleph_{\mathbf{pq}}\right)^T = \aleph_{\mathbf{q}}^T\aleph_{\mathbf{p}}^T$
9. $\left(\aleph_{\mathbf{pq}}\right)^{-1} = \aleph_{\mathbf{q}}^{-1}\aleph_{\mathbf{p}}^{-1}$
10. $\left(\aleph_{\mathbf{p}}^T\right)^{-1} = \left(\aleph_{\mathbf{p}}^{-1}\right)^T$

Also, some properties of right and left product matrices for Hybridian product are as follows.

Theorem 2.3. *[24]Let* $\mathbf{p}, \mathbf{q} \in \mathbb{K}$ *and* $\lambda \in \mathbb{R}$. *Then*

1. $\mathcal{L}_{\mathbf{p}} = \mathcal{L}_{\mathbf{q}} \Longleftrightarrow \mathbf{p} = \mathbf{q} \Longleftrightarrow \mathcal{R}_{\mathbf{p}} = \mathcal{R}_{\mathbf{q}}$
2. $\mathcal{L}_{\mathbf{p}+\mathbf{q}} = \mathcal{L}_{\mathbf{p}} + \mathcal{L}_{\mathbf{q}}, \quad \mathcal{R}_{\mathbf{p}+\mathbf{q}} = \mathcal{R}_{\mathbf{p}} + \mathcal{R}_{\mathbf{q}}$
3. $\mathcal{L}_{\mathbf{pq}} = \mathcal{L}_{\mathbf{p}}\mathcal{L}_{\mathbf{q}}, \quad \mathcal{R}_{\mathbf{pq}} = \mathcal{R}_{\mathbf{p}}\mathcal{R}_{\mathbf{q}}$
4. $\mathcal{L}_{\mathbf{p}}\mathcal{R}_{\mathbf{q}} = \mathcal{R}_{\mathbf{q}}\mathcal{L}_{\mathbf{p}}$
5. $\mathcal{L}_{\lambda\mathbf{p}} = \mathcal{L}_{\mathbf{p}\lambda} = \lambda\mathcal{L}_{\mathbf{p}}, \quad \mathcal{R}_{\lambda\mathbf{p}} = \mathcal{R}_{\mathbf{p}\lambda} = \lambda\mathcal{R}_{\mathbf{p}}$

6. $\mathcal{L}_{\mathbf{1}} = \mathcal{R}_{\mathbf{1}} = \mathrm{I}_4$
7. $\mathcal{L}_{\mathbf{p}} + \mathcal{L}_{\overline{\mathbf{p}}} = 2p_1\mathrm{I}_4, \quad \mathcal{R}_{\mathbf{p}} + \mathcal{R}_{\overline{\mathbf{p}}} = 2p_1\mathrm{I}_4$
8. $\mathcal{L}_{\mathbf{p}^{-1}} = \mathcal{L}_{\mathbf{p}}^{-1}, \quad \mathcal{R}_{\mathbf{p}^{-1}} = \mathcal{R}_{\mathbf{p}}^{-1}$ *where* $\|\mathbf{p}\| \neq 0$
9. $\mu\mathcal{R}_{\overline{\mathbf{p}}} = \mathcal{L}_{\mathbf{p}}\mu$, $\mu = \begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{I}_3 \end{bmatrix}$

In this section, we will examine the representation of hybrid numbers with 2×2 complex matrices together with their properties.

Theorem 2.4. *Every hybrid number can be represented by a* 2×2 *complex matrix.*

Proof. Let $\mathbf{p} = p_1 + p_2\mathbf{i} + p_3\varepsilon + p_4\mathbf{h} \in \mathbb{K}$, then there exist complex numbers $x_1 = p_1 + (p_2 - p_3)\,\mathbf{i}$ and $x_2 = p_4 + p_3\mathbf{i}$ such that $\mathbf{p} = x_1 + x_2\mathbf{h}$. The linear map $f_{\mathbf{p}} : \mathbb{K} \to \mathbb{K}$ is defined by $f_{\mathbf{p}}(\mathbf{q}) = \mathbf{qp}$ for all $\mathbf{q} \in \mathbb{K}$. Then, we have

$$\begin{aligned} f_{\mathbf{p}}(1) &= 1\,(x_1 + x_2\mathbf{h}) = x_1 + x_2\mathbf{h} \\ f_{\mathbf{p}}(\mathbf{h}) &= \mathbf{h}\,(x_1 + x_2\mathbf{h}) = \mathbf{h}x_1 + \mathbf{h}x_2\mathbf{h} \\ &= \overline{x_1}\mathbf{h} + \overline{x_2}\mathbf{hh} \\ &= \overline{x_2} + \overline{x_1}\mathbf{h}. \end{aligned}$$

Using this map, we may define an isomorphism between $\mathbb{K}$ and algebra of the 2×2 complex matrices:

$$\left\{ \begin{bmatrix} x_1 & x_2 \\ \overline{x_2} & \overline{x_1} \end{bmatrix} : x_1, x_2 \in \mathbb{C} \right\}$$

We denote above corresponding 2×2 complex matrix for any hybrid number $\mathbf{p}$ by

$$\Upsilon(\mathbf{p}) = \begin{bmatrix} x_1 & x_2 \\ \overline{x_2} & \overline{x_1} \end{bmatrix} = \begin{bmatrix} p_1 + (p_2 - p_3)\,\mathbf{i} & p_4 + p_3\mathbf{i} \\ p_4 - p_3\mathbf{i} & p_1 - (p_2 - p_3)\,\mathbf{i} \end{bmatrix}$$

and it is called the complex matrix representation of hybrid number $\mathbf{p}$. Also

$$\left|\det \Upsilon(\mathbf{p})\right| = \left|\det \aleph_{\mathbf{p}}\right| = \|\mathbf{p}\|^2 .$$

□

Example 2.5. *Let* $\mathbf{p} = 1 - 2\varepsilon + 3\mathbf{h}$ *and* $\mathbf{q} = -2 + \mathbf{i} + \varepsilon - \mathbf{h}$. *Then*

$\mathbf{p} = 1 + 2\mathbf{i} + (3 - 2\mathbf{i})\,\mathbf{h}$ *and* $\mathbf{q} = -2 + (-1 + \mathbf{i})\,\mathbf{h}$

$$\begin{aligned} \mathbf{pq} &= \begin{bmatrix} 1 + 2\mathbf{i} & 3 - 2\mathbf{i} \\ 3 + 2\mathbf{i} & 1 - 2\mathbf{i} \end{bmatrix} \begin{bmatrix} -2 & -1 + \mathbf{i} \\ -1 - \mathbf{i} & -2 \end{bmatrix} = \begin{bmatrix} -7 - 5\mathbf{i} & -9 + 3\mathbf{i} \\ -9 - 3\mathbf{i} & -7 + 5\mathbf{i} \end{bmatrix} \\ &= -7 - 5\mathbf{i} + (-9 + 3\mathbf{i})\,\mathbf{h} \end{aligned}$$

$$\begin{aligned} \mathbf{qp} &= \begin{bmatrix} -2 & -1 + \mathbf{i} \\ -1 - \mathbf{i} & -2 \end{bmatrix} \begin{bmatrix} 1 + 2\mathbf{i} & 3 - 2\mathbf{i} \\ 3 + 2\mathbf{i} & 1 - 2\mathbf{i} \end{bmatrix} = \begin{bmatrix} -7 - 3\mathbf{i} & -5 + 7\mathbf{i} \\ -5 - 7\mathbf{i} & -7 + 3\mathbf{i} \end{bmatrix} \\ &= -7 - 3\mathbf{i} + (-5 + 7\mathbf{i})\,\mathbf{h}. \end{aligned}$$

3. Hybrid Number Matrices

In this section we will deal with matrices whose inputs are hybrid numbers. The set of $m \times n$ matrices with hybrid number entries, which is denoted by $\mathbb{M}_{m\times n}(\mathbb{K})$, with ordinary matrix addition and multiplication is a ring with unity. If $m = n$, then the set of hybrid matrices is denoted $\mathbb{M}_n(\mathbb{K})$. The set of hybrid number matrices can be represented as:

$$\mathbb{M}_{m\times n}(\mathbb{K}) = \{\mathbf{A} = (\mathbf{a}_{st}) : \mathbf{a}_{st} \in \mathbb{K}\},$$

$$\mathbb{M}_{m\times n}(\mathbb{K}) = \{\mathbf{A} = A_1 + A_2\mathbf{i} + A_3\varepsilon + A_4\mathbf{h} : A_1, A_2, A_3, A_4 \in \mathbb{M}_{m\times n}(\mathbb{R})\}$$

or

$$\mathbb{M}_{m\times n}(\mathbb{K}) = \left\{\mathbf{A} = C_1 + C_2\mathbf{h} : \mathbf{h}^2 = 1,\ C_1, C_2 \in \mathbb{M}_{m\times n}(\mathbb{C})\right\}.$$

For $\mathbf{A} = (\mathbf{a}_{st}) \in \mathbb{M}_{m\times n}(\mathbb{K})$ and $\mathbf{p} \in \mathbb{K}$, right and left scalar multiplication are defined as

$$\mathbf{Ap} = (\mathbf{a}_{st}\mathbf{p}) \text{ and } \mathbf{pA} = \left(\mathbf{pa}_{st}\right),$$

respectively. It is easy to see that for $\mathbf{A} \in \mathbb{M}_{m\times n}(\mathbb{K})$, $\mathbf{B} \in \mathbb{M}_{n\times k}(\mathbb{K})$ and $\mathbf{p}, \mathbf{q} \in \mathbb{K}$

$$\begin{aligned}(\mathbf{pA})\mathbf{B} &= \mathbf{p}(\mathbf{AB}),\\ (\mathbf{Aq})\mathbf{B} &= \mathbf{A}(\mathbf{qB}),\\ (\mathbf{pq})\mathbf{A} &= \mathbf{p}(\mathbf{qA}).\end{aligned}$$

Moreover $\mathbb{M}_{m\times n}(\mathbb{K})$ is a module over the ring $\mathbb{K}$.

Just as for complex matrices, we associate $\mathbf{A} = (\mathbf{a}_{st}) \in \mathbb{M}_{m\times n}(\mathbb{K})$ with $\overline{\mathbf{A}} = (\overline{\mathbf{a}_{st}}) \in \mathbb{M}_{m\times n}(\mathbb{K})$, the conjugate of $\mathbf{A}$; $\mathbf{A}^T = (\mathbf{a}_{ts}) \in \mathbb{M}_{n\times m}(\mathbb{K})$, the transpose of $\mathbf{A}$; and $\mathbf{A}^* = \left(\overline{\mathbf{A}}\right)^T \in \mathbb{M}_{n\times m}(\mathbb{K})$ the conjugate transpose of $\mathbf{A}$. For a square hybrid number matrix $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$; if $\mathbf{AA}^* = \mathbf{A}^*\mathbf{A}$, then $\mathbf{A}$ is called normal hybrid matrix; if $\mathbf{A} = \mathbf{A}^*$, then $\mathbf{A}$ is called Hermitian hybrid matrix; if $\mathbf{AA}^* = I$, then $\mathbf{A}$ is called unitary hybrid matrix. For $\mathbf{B} \in \mathbb{M}_n(\mathbb{K})$, if $\mathbf{AB} = \mathbf{BA} = I$, then $\mathbf{A}$ is called invertible matrix and $\mathbf{B}$ is called the inverse of $\mathbf{A}$.

Example 3.1. *Let* $\mathbf{A} = \begin{bmatrix}\mathbf{i}+\mathbf{h} & 2\\ \mathbf{i}+\varepsilon & \varepsilon-\mathbf{h}\end{bmatrix}$, $\mathbf{B} = \begin{bmatrix}\mathbf{i} & \mathbf{h}\\ 1 & \varepsilon\end{bmatrix} \in \mathbb{M}_2(\mathbb{K})$. *By long computations, we find*

1. $\mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix}-2+\mathbf{i}+\mathbf{h} & 2\mathbf{i}-2\mathbf{h}-4\varepsilon\\ 2+\mathbf{i}+\mathbf{h} & 2\mathbf{i}+2\mathbf{h}\end{bmatrix}$ *and* $\overline{\mathbf{A}} = \begin{bmatrix}-\mathbf{i}-\mathbf{h} & 2\\ -\mathbf{i}-\varepsilon & -\varepsilon+\mathbf{h}\end{bmatrix}$
2. $\left(\overline{\mathbf{A}}\right)^{-1} = \frac{1}{4}\begin{bmatrix}-2+3\mathbf{i}+3\mathbf{h} & -6\mathbf{i}-4\varepsilon-2\mathbf{h}\\ 2-\mathbf{i}-\mathbf{h} & 2\mathbf{i}+2\mathbf{h}\end{bmatrix}$
   $\neq \frac{1}{4}\begin{bmatrix}-2-\mathbf{i}-\mathbf{h} & -2\mathbf{i}+2\mathbf{h}+4\varepsilon\\ 2-\mathbf{i}-\mathbf{h} & -2\mathbf{i}-2\mathbf{h}\end{bmatrix} = \overline{(\mathbf{A}^{-1})}$
3. $\left(\mathbf{A}^T\right)^{-1} = \frac{1}{4}\begin{bmatrix}-2-3\mathbf{i}-3\mathbf{h} & 2+\mathbf{i}+\mathbf{h}\\ 6\mathbf{i}+4\varepsilon+2\mathbf{h} & -2\mathbf{i}-2\mathbf{h}\end{bmatrix}$
   $\neq \frac{1}{4}\begin{bmatrix}-2+\mathbf{i}+\mathbf{h} & 2+\mathbf{i}+\mathbf{h}\\ 2\mathbf{i}-4\varepsilon-2\mathbf{h} & 2\mathbf{i}+2\mathbf{h}\end{bmatrix} = \left(\mathbf{A}^{-1}\right)^T$
4. $\overline{\mathbf{AB}} = \begin{bmatrix}1+\mathbf{i}+\varepsilon & 1-\mathbf{i}-3\varepsilon\\ -\varepsilon & -\mathbf{i}+\varepsilon\end{bmatrix} \neq \begin{bmatrix}1-\mathbf{i}-\varepsilon & 1+\mathbf{i}-\varepsilon\\ -\varepsilon+2\mathbf{h} & \mathbf{i}+\varepsilon\end{bmatrix} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}$
5. $(\mathbf{AB})^T = \begin{bmatrix}1-\mathbf{i}-\varepsilon & \varepsilon\\ 1+\mathbf{i}+3\varepsilon & \mathbf{i}-\varepsilon\end{bmatrix} \neq \begin{bmatrix}1+\mathbf{i}+\varepsilon & \varepsilon-2\mathbf{h}\\ 1-\mathbf{i}+\varepsilon & -\mathbf{i}+\varepsilon\end{bmatrix} = \mathbf{B}^T\mathbf{A}^T.$

Based on these examples, we can express the following Results for hybrid number matrices.

Corollary 3.2. *For any* $\mathbf{A} \in \mathbb{M}_{m\times n}(\mathbb{K})$, $\mathbf{B} \in \mathbb{M}_{n\times k}(\mathbb{K})$, $\mathbf{C} \in \mathbb{M}_n(\mathbb{K})$ *the following properties generally hold.*

1. $\left(\overline{\mathbf{C}}\right)^{-1} \neq \overline{(\mathbf{C}^{-1})}$,
2. $\left(\mathbf{C}^T\right)^{-1} \neq \left(\mathbf{C}^{-1}\right)^T$,
3. $\left(\overline{\mathbf{AB}}\right) \neq \overline{\mathbf{A}}\,\overline{\mathbf{B}}$,
4. $(\mathbf{AB})^T \neq \mathbf{B}^T\mathbf{A}^T$

Theorem 3.3. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(\mathbb{K})$ *and* $\mathbf{p} \in \mathbb{K}$. *Then the followings are satisfied:*

1. $\left(\overline{\mathbf{A}}\right)^T = \overline{(\mathbf{A}^T)}$,
2. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ *if* $\mathbf{A}$ *and* $\mathbf{B}$ *are invertible,*
3. $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$,
4. $(\mathbf{A}^*)^{-1} = \left(\mathbf{A}^{-1}\right)^*$ *if* $\mathbf{A}$ *is invertible,*

Proof. Let $\mathbf{A} = (\mathbf{a}_{st})$, $\mathbf{B} = (\mathbf{b}_{st}) \in \mathbb{M}_n(\mathbb{K})$. Then

1. $\left(\overline{\mathbf{A}}\right)^T = (\overline{\mathbf{a}_{st}})^T = (\overline{\mathbf{a}_{ts}}) = \overline{(\mathbf{A}^T)}$,
2. If $\mathbf{A}$ and $\mathbf{B}$ are invertible then

$$\begin{aligned}(\mathbf{AB})\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) &= \mathbf{A}\left(\mathbf{BB}^{-1}\right)\mathbf{A}^{-1} = I \text{ and}\\ \left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)(\mathbf{AB}) &= \mathbf{B}^{-1}\left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{B} = I.\end{aligned}$$

   So $\mathbf{AB}$ is also invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
3. Let $\mathbf{A} = C_1 + C_2\mathbf{h}$ and $\mathbf{B} = D_1 + D_2\mathbf{h}$, where C_1, C_2, D_1 and D_2 are complex matrices. Then

$$\begin{aligned}(\mathbf{AB})^* &= \left[C_1D_1 + C_2\overline{D_2} + \left(C_2\overline{D_1} + C_1D_2\right)\mathbf{h}\right]^*\\ &= \left[\overline{C_1D_1} + \overline{C_2}D_2 - \left(C_2\overline{D_1} + C_1D_2\right)\mathbf{h}\right]^T\\ &= \left(\overline{C_1D_1} + \overline{C_2}D_2\right)^T - \left(C_2\overline{D_1} + C_1D_2\right)^T\mathbf{h}\\ &= \left(\overline{D_1}\right)^T\left(\overline{C_1}\right)^T + (D_2)^T\left(\overline{C_2}\right)^T - \left(\overline{D_1}\right)^T(C_2)^T\mathbf{h} - (D_2)^T(C_1)^T\mathbf{h}.\end{aligned}$$

   On the other hand, for $\mathbf{A}^* = \left(\overline{C_1}\right)^T - (C_2)^T\mathbf{h}$ and $\mathbf{B}^* = \left(\overline{D_1}\right)^T - (D_2)^T\mathbf{h}$ we can obtain

$$\begin{aligned}\mathbf{B}^*\mathbf{A}^* &= \left(\left(\overline{D_1}\right)^T - (D_2)^T\mathbf{h}\right)\left(\left(\overline{C_1}\right)^T - (C_2)^T\mathbf{h}\right)\\ &= \left(\overline{D_1}\right)^T\left(\overline{C_1}\right)^T + (D_2)^T\left(\overline{C_2}\right)^T - \left(\overline{D_1}\right)^T(C_2)^T\mathbf{h} - (D_2)^T(C_1)^T\mathbf{h}.\end{aligned}$$

   Thus we get

$$(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*.$$

4. Suppose that $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \mathbf{B}$. In that case

$$(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$$

$$\begin{aligned}&\Rightarrow \left(\mathbf{AA}^{-1}\right)^* = \left(\mathbf{A}^{-1}\right)^*\mathbf{A}^*\\ &\Rightarrow \left(\mathbf{A}^{-1}\right)^*\mathbf{A}^* = I\\ &\Rightarrow \left(\mathbf{A}^{-1}\right)^* = (\mathbf{A}^*)^{-1}.\end{aligned}$$

□

Proposition 3.4. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(\mathbb{K})$*. If* $\mathbf{AB} = I_n$*, then* $\mathbf{BA} = I_n$*.*

Proof. Let $\mathbf{A} = C_1 + C_2\mathbf{h}$, $\mathbf{B} = D_1 + D_2\mathbf{h}$, where C_1, C_2, D_1, D_2 are $n \times n$ complex matrices. Then

$$\begin{aligned}\mathbf{AB} &= (C_1 + C_2\mathbf{h})(D_1 + D_2\ \mathbf{h}) \\ &= C_1D_1 + C_2\overline{D_2} + \left(C_2\overline{D_1} + C_1D_2\right)\mathbf{h} = I_n\end{aligned}$$

$$\begin{aligned}&\Rightarrow \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} D_1 & D_2 \\ \overline{D_2} & \overline{D_1} \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} C_1 & C_2 \\ \overline{C_2} & \overline{C_1} \end{bmatrix}\begin{bmatrix} D_1 & D_2 \\ \overline{D_2} & \overline{D_1} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} D_1 & D_2 \\ \overline{D_2} & \overline{D_1} \end{bmatrix}\begin{bmatrix} C_1 & C_2 \\ \overline{C_2} & \overline{C_1} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \\ &\Rightarrow D_1C_1 + D_2\overline{C_2} = I_n, \quad D_1C_2 + D_2\overline{C_1} = 0 \\ &\Rightarrow D_1C_1 + D_2\overline{C_2} + \left(D_1C_2 + D_2\overline{C_1}\right)\mathbf{h} = I_n \\ &\Rightarrow \mathbf{BA} = I_n.\end{aligned}$$

□

3.1. The Complex Adjoint Matrix of a Hybrid Number Matrix

In this section, we will define the complex adjoint matrix of a hybrid number matrix. After that we will give a way to find the inverse of a hybrid number matrix by using its complex adjoint matrix. Moreover, we will give some relations between hybrid number matrices and their complex adjoint matrices.

Definition 3.5. *For* $\mathbf{A} = C_1 + C_2\mathbf{h} \in \mathbb{M}_n(\mathbb{K})$*, we shall call the* $2n \times 2n$ *complex matrix*

$$\begin{bmatrix} C_1 & C_2 \\ \overline{C_2} & \overline{C_1} \end{bmatrix}$$

uniquely determined by $\mathbf{A}$*, the complex adjoint matrix or adjoint of the hybrid number matrix* $\mathbf{A}$*, symbolized* $\chi_{\mathbf{A}}$*.*

Theorem 3.6. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(\mathbb{K})$*, then the following properties are satisfied:*

1. $\chi_{I_n} = I_{2n}$,
2. $\chi_{\mathbf{A}+\mathbf{B}} = \chi_{\mathbf{A}} + \chi_{\mathbf{B}}$,
3. $\chi_{\mathbf{AB}} = \chi_{\mathbf{A}}\chi_{\mathbf{B}}$,
4. $\chi_{\mathbf{A}^{-1}} = (\chi_{\mathbf{A}})^{-1}$ *if* $\mathbf{A}^{-1}$ *exist,*
5. $\chi_{\mathbf{A}^*} \neq (\chi_{\mathbf{A}})^*$ *in general.*

Proof. Let $\mathbf{A} = C_1 + C_2\mathbf{h}$, $\mathbf{B} = D_1 + D_2\mathbf{h} \in \mathbb{M}_n(\mathbb{K})$ where $C_1, C_2, D_1, D_2 \in \mathbb{M}_n(\mathbb{C})$. Let $I_n = I$ be $n \times n$ identity matrix and 0 be $n \times n$ zero matrix.

1. By definition of complex adjoint matrix, we have

$$\chi_{I_n} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I_{2n}.$$

2. We may write $\mathbf{A}+\mathbf{B}=(C_1+D_1)+(C_2+D_2)\,\mathbf{h}$. Using the definition of complex adjoint matrix, we have

$$\begin{aligned}\chi_{\mathbf{A}+\mathbf{B}} &= \begin{bmatrix} C_1+D_1 & C_2+D_2 \\ \overline{C_2+D_2} & \overline{C_1+D_1}\end{bmatrix} = \begin{bmatrix} C_1+D_1 & C_2+D_2 \\ \overline{C_2}+\overline{D_2} & \overline{C_1}+\overline{D_1}\end{bmatrix} \\ &= \begin{bmatrix} C_1 & C_2 \\ \overline{C_2} & \overline{C_1}\end{bmatrix} + \begin{bmatrix} D_1 & D_2 \\ \overline{D_2} & \overline{D_1}\end{bmatrix} \\ &= \chi_{\mathbf{A}} + \chi_{\mathbf{B}}.\end{aligned}$$

3. Using multiplication of hybrid number, we have

$$\begin{aligned}\mathbf{AB} &= (C_1+C_2\mathbf{h})\,(D_1+D_2\ \mathbf{h}) \\ &= C_1D_1+C_2\overline{D_2}+\left(C_2\overline{D_1}+C_1D_2\right)\mathbf{h}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\chi_{\mathbf{AB}} &= \begin{bmatrix} C_1D_1+C_2\overline{D_2} & C_2\overline{D_1}+C_1D_2 \\ \overline{C_2\overline{D_1}+C_1D_2} & \overline{C_1D_1+C_2\overline{D_2}}\end{bmatrix} = \begin{bmatrix} C_1D_1+C_2\overline{D_2} & C_2\overline{D_1}+C_1D_2 \\ \overline{C_2\overline{D_1}}+\overline{C_1D_2} & \overline{C_1D_1}+\overline{C_2\overline{D_2}}\end{bmatrix} \\ &= \begin{bmatrix} C_1D_1+C_2\overline{D_2} & C_2\overline{D_1}+C_1D_2 \\ \overline{C_2}D_1+\overline{C_1}\overline{D_2} & \overline{C_1}\overline{D_1}+\overline{C_2}D_2\end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ \overline{C_2} & \overline{C_1}\end{bmatrix}\begin{bmatrix} D_1 & D_2 \\ \overline{D_2} & \overline{D_1}\end{bmatrix} \\ &= \chi_{\mathbf{A}}\chi_{\mathbf{B}}.\end{aligned}$$

4. If $\mathbf{A}$ is invertible then $\mathbf{AA}^{-1}=\mathbf{A}^{-1}\mathbf{A}=I$. By properties 1 and 3, we may write

$$I_{2n}=\chi_{I_n}=\chi_{\mathbf{AA}^{-1}}=\chi_{\mathbf{A}}\chi_{\mathbf{A}^{-1}}.$$

Thus, we get

$$(\chi_{\mathbf{A}})^{-1}=\chi_{\mathbf{A}^{-1}}.$$

5. For this property, we give the following example.

□

Example 3.7. *Let* $\mathbf{A}=\begin{bmatrix}\mathbf{i}+\mathbf{h} & \mathbf{i}+\varepsilon-3\mathbf{h} \\ 2-\varepsilon & 3+\mathbf{i}-2\varepsilon-3\mathbf{h}\end{bmatrix}\in\mathbb{M}_2(\mathbb{K})$. *Then the complex adjoint matrix of* $\mathbf{A}$ *is*

$$\chi_{\mathbf{A}}=\begin{bmatrix}\mathbf{i} & 0 & 1 & -3+\mathbf{i} \\ 2+\mathbf{i} & 3+3\mathbf{i} & -\mathbf{i} & -3+\mathbf{i} \\ 1 & -3-\mathbf{i} & -\mathbf{i} & 0 \\ \mathbf{i} & -3-\mathbf{i} & 2-\mathbf{i} & 3-3\mathbf{i}\end{bmatrix}.$$

Furthermore, $\mathbf{A}^*=\begin{bmatrix}-\mathbf{i}-\mathbf{h} & 2+\varepsilon \\ -\mathbf{i}-\varepsilon+3\mathbf{h} & 3-\mathbf{i}+2\varepsilon+3\mathbf{h}\end{bmatrix}$ *and the complex adjoint matrix of* $\mathbf{A}^*$ *is*

$$\chi_{\mathbf{A}^*}=\begin{bmatrix}-\mathbf{i} & 2-\mathbf{i} & -1 & \mathbf{i} \\ 0 & 3-3\mathbf{i} & 3-\mathbf{i} & 3+2\mathbf{i} \\ -1 & -\mathbf{i} & \mathbf{i} & 2+\mathbf{i} \\ 3+\mathbf{i} & 3-2\mathbf{i} & 0 & 3+3\mathbf{i}\end{bmatrix}.$$

But

$$(\chi_{\mathbf{A}})^*=\begin{bmatrix}-\mathbf{i} & 2-\mathbf{i} & 1 & -\mathbf{i} \\ 0 & 3-3\mathbf{i} & -3+\mathbf{i} & -3+\mathbf{i} \\ 1 & \mathbf{i} & \mathbf{i} & 2+\mathbf{i} \\ -3-\mathbf{i} & -3-\mathbf{i} & 0 & 3+3\mathbf{i}\end{bmatrix}$$

Thus, we get, $\chi_{\mathbf{A}^*}\neq(\chi_{\mathbf{A}})^*$.

Theorem 3.8. *Let* $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$. *If* $\chi_{\mathbf{A}}$ *is invertible then* $\mathbf{A}$ *is also invertible.*

Proof. Let $\mathbf{A} = C_1 + C_2\mathbf{h} \in \mathbb{M}_n(\mathbb{K})$, where $C_1, C_2 \in \mathbb{M}_n(\mathbb{C})$. Then the complex adjoint matrix of $\mathbf{A}$ is

$$\chi_{\mathbf{A}} = \begin{bmatrix} C_1 & C_2 \\ \overline{C_2} & \overline{C_1} \end{bmatrix}.$$

Also, suppose that $\chi_{\mathbf{A}}$ is invertible. Then there exists $\chi_{\mathbf{B}} \in \mathbb{M}_{2n\times 2n}(\mathbb{C})$ such that $\chi_{\mathbf{A}}\chi_{\mathbf{B}} = \chi_{\mathbf{B}}\chi_{\mathbf{A}} = I$. We can write the matrix $\chi_{\mathbf{B}}$ in the following form:

$$\chi_{\mathbf{B}} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where $B_{ij} \in \mathbb{M}_n(\mathbb{C})$ for $i, j = 1, 2$.

If we use that $\chi_{\mathbf{A}}\chi_{\mathbf{B}} = I$, we get

$$\begin{bmatrix} C_1 & C_2 \\ \overline{C_2} & \overline{C_1} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

From here we get the following equations:

$$\begin{aligned} C_1B_{11} + C_2B_{21} &= I \\ \overline{C_2}B_{11} + \overline{C_1}B_{21} &= 0 \Rightarrow C_1\overline{B_{21}} + C_2\overline{B_{11}} = 0 \end{aligned}$$

Using these equations, we have

$$I = C_1B_{11} + C_2B_{21} + \left(C_1\overline{B_{21}} + C_2\overline{B_{11}}\right)\mathbf{h}.$$

If we take $\mathbf{B} = B_{11} + \overline{B_{21}}\mathbf{h}$, the equation obtained above is equivalent to

$$\begin{aligned} \mathbf{AB} &= (C_1 + C_2\mathbf{h})\left(B_{11} + \overline{B_{21}}\mathbf{h}\right) \\ &= C_1B_{11} + C_2\mathbf{h}\overline{B_{21}}\mathbf{h} + C_2\mathbf{h}B_{11} + C_1\overline{B_{21}}\mathbf{h} \\ &= C_1B_{11} + C_2B_{21} + \left(C_2\overline{B_{11}} + C_1\overline{B_{21}}\right)\mathbf{h} = I. \end{aligned}$$

By proposition 3.4, we also have $\mathbf{BA} = I$. So, we get that $\mathbf{A}$ is invertible and $\mathbf{B}$ is the inverse of $\mathbf{A}$. □

Corollary 3.9. *Let* $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$. $\mathbf{A}$ *is invertible if and only if* $\chi_{\mathbf{A}}$ *is invertible.*

Corollary 3.10. *Let* $\mathbf{A} = A_1 + A_2\mathbf{i} + A_3\varepsilon + A_4\mathbf{h}$ *and* $A_1, A_2, A_3, A_4 \in \mathbb{M}_n(\mathbb{R})$. *Then, since the hybrid number matrix* $\mathbf{A}$ *can be written as*

$$\mathbf{A} = A_1 + (A_2 - A_3)\,\mathbf{i} + (A_4 + A_3\mathbf{i})\,\mathbf{h},$$

we can write the $2n \times 2n$ *complex adjoint matrix and the* $2n \times 2n$ *real matrix representation of the hybrid number matrix*

$$\chi_{\mathbf{A}} = \begin{bmatrix} A_1 + (A_2 - A_3)\,\mathbf{i} & A_4 + A_3\mathbf{i} \\ A_4 - A_3\mathbf{i} & A_1 - (A_2 - A_3)\,\mathbf{i} \end{bmatrix}$$

and

$$\aleph_{\mathbf{A}} = \begin{bmatrix} A_1 + A_3 & A_2 - A_3 + A_4 \\ A_3 - A_2 + A_4 & A_1 - A_3 \end{bmatrix},$$

respectively. Therefore, it is seen that

$$\det \chi_{\mathbf{A}} = \det \aleph_{\mathbf{A}} = A_1^2 + A_2^2 - 2A_3A_2 - A_4^2.$$

Corollary 3.11. *Let* $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$ *be an invertible matrix. If the inverse of the* $2n \times 2n$ *complex matrix* $\chi_{\mathbf{A}}$ *is*

$$\chi_{\mathbf{A}}^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where $\mathbf{B}_{ij} \in \mathbb{M}_n(\mathbb{C})$ *for* $i, j = 1, 2$ *then*

$$\mathbf{A}^{-1} = B_{11} + \overline{B_{21}}\mathbf{h}.$$

Now, with the help of this corollary, let's give an example of how to find the inverse of a hybrid number matrix.

Example 3.12. *Let*

$$\mathbf{A} = \begin{bmatrix} 3-\mathbf{i}+2\mathbf{h} & \mathbf{i}+2\varepsilon-\mathbf{h} \\ 1-2\mathbf{h} & 2-2\varepsilon \end{bmatrix} = \begin{bmatrix} 3-\mathbf{i} & -\mathbf{i} \\ 1 & 2+2\mathbf{i} \end{bmatrix} + \begin{bmatrix} 2 & -1+2\mathbf{i} \\ -2 & -2\mathbf{i} \end{bmatrix}\mathbf{h}.$$

Then, since the hybrid number matrix $\mathbf{A}$ *can be written as*

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 3-\mathbf{i}+2\mathbf{h} & \mathbf{i}+2\varepsilon-\mathbf{h} \\ 1-2\mathbf{h} & 2-2\varepsilon \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{i} + \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}\varepsilon + \begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix}\mathbf{h} \end{aligned}$$

and

$$\mathbf{A} = \begin{bmatrix} 3-\mathbf{i}+2\mathbf{h} & \mathbf{i}+2\varepsilon-\mathbf{h} \\ 1-2\mathbf{h} & 2-2\varepsilon \end{bmatrix} = \begin{bmatrix} 3-\mathbf{i} & -\mathbf{i} \\ 1 & 2+2\mathbf{i} \end{bmatrix} + \begin{bmatrix} 2 & -1+2\mathbf{i} \\ -2 & -2\mathbf{i} \end{bmatrix}\mathbf{h}$$

we can write the 4×4 *complex adjoint matrix and the* 4×4 *real matrix representation of the hybrid number matrix*

$$\chi_{\mathbf{A}} = \begin{bmatrix} 3-\mathbf{i} & -\mathbf{i} & 2 & -1+2\mathbf{i} \\ 1 & 2+2\mathbf{i} & -2 & -2\mathbf{i} \\ 2 & -1-2\mathbf{i} & 3+\mathbf{i} & \mathbf{i} \\ -2 & 2\mathbf{i} & 1 & 2-2\mathbf{i} \end{bmatrix}$$

and

$$\aleph_{\mathbf{A}} = \begin{bmatrix} 3 & 2 & 1 & -2 \\ 1 & 0 & -2 & 2 \\ 3 & 0 & 3 & -2 \\ -2 & -2 & 1 & 4 \end{bmatrix},$$

respectively. From here, the inverse of the complex adjoint matrix and real matrix representation of $\mathbf{A}$ *can be found as*

$$\chi_{\mathbf{A}}^{-1} = \begin{bmatrix} -\frac{1}{32} & \frac{9+3\mathbf{i}}{32} & \frac{8-\mathbf{i}}{32} & \frac{-5+\mathbf{i}}{32} \\ \frac{20-25\mathbf{i}}{64} & \frac{1+5\mathbf{i}}{64} & \frac{-7+28\mathbf{i}}{64} & \frac{11-17\mathbf{i}}{64} \\ \frac{8+\mathbf{i}}{32} & \frac{-5-\mathbf{i}}{32} & -\frac{1}{32} & \frac{9-3\mathbf{i}}{32} \\ \frac{-7-28\mathbf{i}}{64} & \frac{11+17\mathbf{i}}{64} & \frac{20+25\ \mathbf{i}}{64} & \frac{1-5\mathbf{i}}{64} \end{bmatrix}$$

and

$$\aleph_{\mathbf{A}}^{-1} = \begin{bmatrix} -\frac{1}{16} & \frac{5}{16} & \frac{1}{4} & -\frac{1}{16} \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} \\ \frac{9}{32} & \frac{3}{32} & -\frac{1}{8} & \frac{9}{32} \end{bmatrix},$$

respectively. So, according to the equality $\chi_{\mathbf{A}}^{-1} = \left[B_{ij}\right]$*, the* 2×2 *complex matrices are found as follows;*

$$\begin{aligned} B_{11} &= \begin{bmatrix} -\frac{1}{32} & \frac{9+3\mathbf{i}}{32} \\ \frac{20-25\mathbf{i}}{64} & \frac{1+5\mathbf{i}}{64} \end{bmatrix}, \quad B_{12} = \begin{bmatrix} \frac{8-\mathbf{i}}{32} & \frac{-5+\mathbf{i}}{32} \\ \frac{-7+28\mathbf{i}}{64} & \frac{11-17\mathbf{i}}{64} \end{bmatrix}, \\ B_{21} &= \begin{bmatrix} \frac{8+\mathbf{i}}{32} & \frac{-5-\mathbf{i}}{32} \\ \frac{-7-28\mathbf{i}}{64} & \frac{11+17\mathbf{i}}{64} \end{bmatrix}, \quad B_{22} = \begin{bmatrix} -\frac{1}{32} & \frac{9-3\mathbf{i}}{32} \\ \frac{20+25\mathbf{i}}{64} & \frac{1-5\mathbf{i}}{64} \end{bmatrix}. \end{aligned}$$

Thus, using the Corollary 3.11, we find the inverse of the hybrid number matrix $\mathbf{A}$ *as*

$$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{32} & \frac{9+3\mathbf{i}}{32} \\ \frac{20-25\mathbf{i}}{64} & \frac{1+5\mathbf{i}}{64} \end{bmatrix} + \begin{bmatrix} \frac{8-\mathbf{i}}{32} & \frac{-5+\mathbf{i}}{32} \\ \frac{-7+28\mathbf{i}}{64} & \frac{11-17\mathbf{i}}{64} \end{bmatrix}\mathbf{h}$$

It can also be observed that

$$\det \chi_{\mathbf{A}} = \det \aleph_{\mathbf{A}} = -64.$$

Definition 3.13. *Let* $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$ *and* $\chi_{\mathbf{A}}$ *be the complex adjoint matrix of* $\mathbf{A}$*. We define the q -determinant of* $\mathbf{A}$ *by* $|\mathbf{A}|_q = |\chi_{\mathbf{A}}|$*. Here* $|\chi_{\mathbf{A}}|$ *is the usual determinant of* $\chi_{\mathbf{A}}$*.*

Theorem 3.14. *Let* $\mathbf{A} \in \mathbb{M}_n(\mathbb{K})$*, then*

1. $\mathbf{A}$ *is invertible* $\Leftrightarrow |\mathbf{A}|_q = |\chi_{\mathbf{A}}| \neq 0$.
2. $|\mathbf{AB}|_q = |\mathbf{A}|_q |\mathbf{B}|_q$ *consequently* $\left|\mathbf{A}^{-1}\right|_q = |\mathbf{A}|_q^{-1}$ *if* $\mathbf{A}^{-1}$ *is exist.*
3. $|\mathbf{PAQ}|_q = |\mathbf{A}|_q$ *for any elementary matrices* $\mathbf{P}$ *and* $\mathbf{Q}$.

Proof.
1. It is clear from Corollary 3.9.
2. We know from Theorem 3.6 that $\chi_{\mathbf{AB}} = \chi_{\mathbf{A}}\chi_{\mathbf{B}}$. And so

$$|\mathbf{AB}|_q = |\chi_{\mathbf{AB}}| = |\chi_{\mathbf{A}}\chi_{\mathbf{B}}| = |\chi_{\mathbf{A}}|\,|\chi_{\mathbf{B}}| = |\mathbf{A}|_q\,|\mathbf{B}|_q\,.$$

Here, if we let $\mathbf{B} = \mathbf{A}^{-1}$ then we can find easily

$$\left|\mathbf{A}^{-1}\right|_q = |\mathbf{A}|_q^{-1}\,.$$

3. It is sufficient to observe that $|\chi_{\mathbf{P}}| = 1$ where $\mathbf{P}$ is an elementary hybrid number matrix. □

4. Conclusion

In this study, we have given hybrid number matrices. Since hybrid number matrices have complex matrix representation, the study of this matrix algebra has provided some advantages. We have defined the $2n \times 2n$ complex adjoint matrix and the $2n \times 2n$ real matrix representation of $n \times n$ type hybrid number matrices and gave some important properties. These properties are very important for the application. We have also defined the q-determinant of hybrid number matrices and showed that this determinant is consistent with the $2n \times 2n$ real matrix representation. In addition, we have also given an explicit formula for the inverse of hybrid number matrices using the complex adjoint matrix.

References

[1] Akbıyık, M., Akbıyık, S. Y., Karaca, E., & Yılmaz, F. (2021) De Moivre's and Euler Formulas for Matrices of Hybrid Numbers. Axioms, 10(3), 213.

[2] Alagöz, Y., Oral, K. H., & Yüce, S. (2012). Split quaternion matrices. Miskolc Mathematical Notes, 13(2), 223-232.

[3] Alagoz, Y., & Ozyurt, G. (2019). Some properties of complex quaternion and complex split quaternion matrices. Miskolc Mathematical Notes, 20(1), 45-58.

[4] Alagöz, Y., & Özyurt, G. (2019). Real and hyperbolic matrices of split semi quaternions. Advances in Applied Clifford Algebras, 29(3), 1-20.

[5] Ali, I. (2018). A note on quaternion matrices and split quaternion matrix pencils. Journal of Applied Mathematics and Computing, 58(1), 323-334.

[6] Alp, Y., & Kocer, E. G. (2021). Hybrid Leonardo numbers. Chaos, Solitons & Fractals, 150, 111128.

[7] Catarino, P. (2019). On k-Pell hybrid numbers. Journal of Discrete Mathematical Sciences and Cryptography, 22(1), 83-89.

[8] Dattoli, G., Licciardi, S., Pidatella, R. M., & Sabia, E. (2018). Hybrid complex numbers: the matrix version. Advances in Applied Clifford Algebras, 28(3), 1-17.

[9] Dattoli, G., Di Palma, E., Gielis, J., & Licciardi, S. (2020). Parabolic Trigonometry. International Journal of Applied and Computational Mathematics, 6(2), 1-10.

[10] Erdoğdu, M., & Özdemir, M. (2013). On eigenvalues of split quaternion matrices. Advances in Applied Clifford Algebras, 23(3), 615-623.

[11] Erdoğdu, M., & Özdemir, M. (2013). On complex split quaternion matrices. Advances in Applied Clifford Algebras, 23(3), 625-638.

[12] Erdoğdu, M., & Özdemir, M. (2017). On exponential of split quaternionic matrices. Applied Mathematics and Computation, 315, 468-476.

[13] Erdoğdu, M., & Özdemir, M. (2016). Matrices over hyperbolic split quaternions. Filomat, 30(4), 913-920.

[14] Erdoğdu, M., & Özdemir, M. (2015). Split quaternion matrix representation of dual split quaternions and their matrices. Advances in Applied Clifford Algebras, 25(4), 787-798.

[15] Jiang, T., Zhang, Z., & Jiang, Z. (2018). Algebraic techniques for eigenvalues and eigenvectors of a split quaternion matrix in split quaternionic mechanics. Computer Physics Communications, 229, 1-7.

[16] Kızılateş, C. (2020). A new generalization of Fibonacci hybrid and Lucas hybrid numbers. Chaos, Solitons & Fractals, 130, 109449.

[17] Kösal, H. H., Akyigit, M., & Tosun, M. (2016). On the consimilarity of split quaternions and split quaternion matrices. An. St. Univ. Ovidius Constanta, Ser. Mat, 24(3), 189-207.

[18] Kyrchei, I. (2017). Cramer's rules for some Hermitian coquaternionic matrix equations. Advances in Applied Clifford Algebras, 27(3), 2509-2529.

[19] Kyrchei, I. (2015). The column and row immanants over a split quaternion algebra. Advances in Applied Clifford Algebras, 25(3), 611-619.

[20] Liu, X., & He, Z. H. (2020). On the split quaternion matrix equation AX=B. Banach Journal of Mathematical Analysis, 14(1), 228-248.

[21] Ni, Q. Y., Ding, J. K., Cheng, X. H., & Jiao, Y. N. (2019). 2×2 Matrix Representation Forms and Inner Relationships of Split Quaternions. Advances in Applied Clifford Algebras, 29(2), 1-12.

[22] Özdemir, M. (2018). Introduction to hybrid numbers. Advances in Applied Clifford Algebras, 28(1), 1-32.

[23] Özdemir, M. (2019). Finding n-th Roots of a 2×2 Real Matrix Using De Moivre's Formula. Advances in Applied Clifford Algebras, 29(1), 1-25.

[24] Öztürk, İ., & Özdemir, M. (2020). Similarity of hybrid numbers. Mathematical Methods in the Applied Sciences, 43(15), 8867-8881.

[25] Öztürk, İ., & Özdemir, M. (2022). Elliptical rotations with hybrid numbers. Indian Journal of Pure and Applied Mathematics, 1-17.

[26] Özyurt, G., & Alagöz, Y. (2018). On hyperbolic split quaternions and hyperbolic split quaternion matrices. Advances in Applied Clifford Algebras, 28(5), 1-11.

[27] Szynal-Liana, A., & Włoch, I. (2019). On Jacobsthal and Jacobsthal–Lucas hybrid numbers. In Annales Mathematicae Silesianae (Vol. 33, No. 1, pp. 276-283).

[28] Szynal-Liana, A., & Włoch, I. (2020). Introduction to Fibonacci and Lucas hybrinomials. Complex Variables and Elliptic Equations, 65(10), 1736-1747.

[29] Taşcı, D., & Sevgi, E. (2021). Some Properties between Mersenne, Jacobsthal and Jacobsthal-Lucas Hybrid Numbers. Chaos, Solitons & Fractals, 146, 110862.

[30] Wang, M., Yue, L., & Liu, Q. (2020). Elementary Transformation and its Applications for Split Quaternion Matrices. Advances in Applied Clifford Algebras, 30(1), 1-10.

[31] Yuan, S. F., Wang, Q. W., Yu, Y. B., & Tian, Y. (2017). On Hermitian Solutions of the Split Quaternion Matrix Equation AXB+CXD=E. Advances in Applied Clifford Algebras, 27(4), 3235-3252.

[32] Zhang, F. (1997). Quaternions and matrices of quaternions. Linear algebra and its applications, 251, 21-57.