Filomat 37:29 (2023), 9791–9821
https://doi.org/10.2298/FIL2329791A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Relative boundedness in non-Archimedean Banach spaces

Aymen Ammar[a]

[a] *Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax Soukra Road Km 3.5, B. P. 1171, 3000, Sfax, Tunisia*

Abstract. This paper treats general unbounded operators, closed operators, and relative boundedness in non-Archimedean Banach spaces.

1. Introduction

The theory of valuations was begun in 1912 by J. Kürschák who defined the valuation axioms as we are used today. The main motivation was to supply a solid base for the theory of p-adic fields as formulated by K. Hensel. In 1934, A. Ostrowski published essential contributions to valuation theory. Simultaneous, W. Krull generalized the notion of an absolute value to that of a valuation. This generalization made possible applications in other mathematical fields, including algebraic geometry and functional analysis. Since the early 1940s, non-Archimedean field analysis has been attempted from different perspectives. In 1943, A. F. Monna [14] outlined the non-Archimedean normed vector spaces.

One of the successful applications of p-adic functional analysis was the use by B. Dwork of an ad hoc linear operator in his study of the rationality of the zeta function of a hypersurface over finite fields (a part of Weil conjectures) (see [6]). Immediately, J. P. Serre has given a general setting of this operator by constructing the Fredholm determinant of completely continuous operators which applies very well to Dwork's operator (see [19]).

1.1. Valuation

The valued fields of the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are important to several mathematics theories. Noting that there are two types of valuation, one is the Archimedean valuation, as in the cases of $\mathbb{C}$ and $\mathbb{R}$, and the other is the non-Archimedean valuation. The effects of replacing $\mathbb{R}$ or $\mathbb{C}$ by the more general object of a non-Archimedean valued field $(\mathbb{K}, |\cdot|)$ in these theories has long been investigated. In fact, the study of non-Archimedean analysis is much concerned with the base valued field. The adopted definition segregates the classical norm from all other the non-Archimedean norm as to avoid carrying out generalizations.

2020 *Mathematics Subject Classification*. Primary 47A10; Secondary 47A12.
Keywords. Non-Archimedean Banach spaces; Linear operator; Spectra; Relative boundedness.
Received: 04 April 2023; Accepted: 06 June 2023
Communicated by Dragan S. Djordjević
Email address: ammar_aymen84@yahoo.fr (Aymen Ammar)

1.2. Definitions and some properties

Let $\mathbb{K}$ be a field. It's unit element, we dented by $1_{\mathbb{K}}$. The symbol 1 is the unit element of $\mathbb{R}$.

We start by presentation of the notion of valuation on a field general $\mathbb{K}$. The following concept will play a key role in this paper.

Definition 1.1. *A valuation on $\mathbb{K}$ is a map $|\cdot| : \mathbb{K} \longrightarrow \mathbb{R}$ satisfying*:
(i) $|x| \geq 0$ *for any $x \in \mathbb{K}$ with equality only for $x = 0$.*
(ii) $|xy| = |x| \cdot |y|$ *for any $x, y \in \mathbb{K}$.*
(iii) *For some real number $C \in]1, +\infty[$, which will be called Artin constant and any $x \in \mathbb{K}$, if $|x| \leq 1$, then $|x + 1_{\mathbb{K}}| \leq C$.*

The following result can be easily derived from Definition 1.1.

Proposition 1.2. *Let $|\cdot|$ be a valuation on $\mathbb{K}$. Then, the following hold:*
(i) $|1_{\mathbb{K}}| = 1$.
(ii) *For $x \in \mathbb{K}$ and $n \in \mathbb{N}$, if $|x^n| = 1$, then $|x| = 1$.*
(iii) $|-1_{\mathbb{K}}| = 1$.
(iv) *For all $x \in \mathbb{K}$, we have $|-x| = |x|$.*
(v) *For $x \in \mathbb{K} \setminus \{0\}$, we have $|x^{-1}| = |x|^{-1}$.*

Proof. (i) In view of Definition 1.1 (ii), we may deduce that,

$$\begin{aligned} |1_{\mathbb{K}}| &= |1_{\mathbb{K}} \times 1_{\mathbb{K}}| \\ &= |1_{\mathbb{K}}| \times |1_{\mathbb{K}}| \\ &= |1_{\mathbb{K}}|^2. \end{aligned}$$

Then, we infer that $|1_{\mathbb{K}}|\,(1 - |1_{\mathbb{K}}|) = 0$. Moreover, by using ($i$) of Definition 1.1, we have that $|1_{\mathbb{K}}| = 1$.

(ii) Let $x \in \mathbb{K}$ and $n \in \mathbb{N}$. According to Definition 1.1 (ii), we have

$$\begin{aligned} |x^n| &= \underbrace{|x \times \ldots \times x|}_{n \text{ times}} \\ &= \underbrace{|x| \times \ldots \times |x|}_{n \text{ times}} \\ &= |x|^n. \end{aligned} \tag{1}$$

Using the fact that $|x^n| = 1$, we infer from Eq. (1) that $|x|^n = 1$. Hence, we obtain

$$|x| = \sqrt[n]{|x|^n} = 1.$$

(iii) Observe that

$$\begin{aligned} |1_{\mathbb{K}}| &= |-(-1_{\mathbb{K}})| \\ &= |-1_{\mathbb{K}} \times -1_{\mathbb{K}}| \\ &= |-1_{\mathbb{K}}|^2. \end{aligned}$$

So, from both (i) and Definition 1.1 (ii), we have $|-1_{\mathbb{K}}|^2 = 1$. Hence, by virtue of (ii), we conclude that $|-1_{\mathbb{K}}| = 1$.

(iv) Let $x \in \mathbb{K}$. Since $-x = -1_{\mathbb{K}} \times x$. then by using Definition 1.1 (ii), we see that

$$|-x| = |-1_{\mathbb{K}}| \times |x|.$$

The use of (*iii*) makes us to conclude that $|-x| = |x|$.
(*v*) It follows from both Proposition 1.2 (*i*) and Definition 1.1 (*ii*) that

$$\begin{aligned} 1 &= |1_{\mathbb{K}}| \\ &= |x \times x^{-1}| \\ &= |x|\,|x^{-1}|. \end{aligned}$$

This implies that $|x| = |x^{-1}|^{-1}$. Hence, we deduce $|x^{-1}| = |x|^{-1}$. This completes the proof. □

Remark 1.3. (*i*) *The usual absolute values of* $\mathbb{R}$ *and* $\mathbb{C}$ *are examples of valuations.*
(*ii*) *Let* $|\cdot|$ *be a valuation on* $\mathbb{K}$*. The valuation* $|\cdot|$ *is called the trivial valuation, if*

$$|x| = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

Proposition 1.4. *Let* $|\cdot|$ *be a valuation on* $\mathbb{K}$ *and* $C \in]1, +\infty[$*. Then, the following statements are equivalent:*
(*i*) *For any* $\alpha \in \mathbb{K}$*, if* $|\alpha| \leq 1$*, then* $|\alpha + 1_{\mathbb{K}}| \leq C$*.*
(*ii*) *For any* $\alpha, \alpha_1 \in \mathbb{K}$*,* $|\alpha + \alpha_1| \leq C \max\Big(|\alpha|, |\alpha_1|\Big)$*.*

Proof. First we suppose that (*i*) holds. Let α, $\alpha_1 \in \mathbb{K}$. Without loss of generality, we may suppose that $0 < |\alpha| \leq |\alpha_1|$. Hence, $\left|\dfrac{\alpha}{\alpha_1}\right| \leq 1$, and so we get

$$\left|\frac{\alpha}{\alpha_1} + 1_{\mathbb{K}}\right| \leq C. \tag{2}$$

Multiplying the inequality (2), by $|\alpha_1|$, we get

$$\begin{aligned} |\alpha + \alpha_1| &\leq C|\alpha_1| \\ &= C \max\Big(|\alpha|, |\alpha_1|\Big). \end{aligned}$$

Conversely, let $\alpha \in \mathbb{K}$ be such that $|\alpha| \leq 1$. This entails that,

$$\begin{aligned} |\alpha + 1_{\mathbb{K}}| &\leq C \max\Big(|\alpha|, |1_{\mathbb{K}}|\Big) \\ &= C \max\Big(|\alpha|, 1\Big) \\ &= C. \end{aligned}$$

This completes the proof. □

Definition 1.5. *A valuation* $|\cdot|$ *on the field* $\mathbb{K}$ *satisfies the triangle inequality if for any* $\alpha, \alpha_1 \in \mathbb{K}$*, we have*

$$|\alpha + \alpha_1| \leq |\alpha| + |\alpha_1|.$$

We start our investigation with the following lemma, which constitute a preparation for the proof of Theorem 1.8

Lemma 1.6. *Let* $|\cdot|$ *be a valuation on* $\mathbb{K}$*.*
(*i*) *Let* $p \in \mathbb{N}$ *and let* $\alpha_1, \ldots, \alpha_{2^p} \in \mathbb{K}$*. Then,*

$$|\alpha_1 + \alpha_2 + \ldots + \alpha_{2^p}| \leq 2^p \max_{1 \leq i \leq 2^p} |\alpha_i|.$$

(*ii*) *Let* $q \in \mathbb{N}$ *and let* $\alpha_1, \ldots, \alpha_q \in \mathbb{K}$*. Then,*

$$\Big|\alpha_1 + \alpha_2 + \ldots + \alpha_q\Big| \leq 2q \max_{1 \leq i \leq 2q} |\alpha_i|. \tag{3}$$

Proof. (i) By induction on p. For $p = 1$. Without loss of generality, we may suppose that $0 < |\alpha_1| \leq |\alpha_2|$. Multiplying by $|\alpha_2|^{-1}$, then $\left|\frac{\alpha_1}{\alpha_2}\right| \leq 1$, so

$$\begin{aligned} \left|\frac{\alpha_1}{\alpha_2} + 1_{\mathbb{K}}\right| &\leq C \\ &\leq 2. \end{aligned}$$

Multiplying by $|\alpha_2|$, we get

$$\begin{aligned} |\alpha_1 + \alpha_2| &\leq 2|\alpha_2| \\ &= 2\max\Big(|\alpha_1|, |\alpha_2|\Big). \end{aligned}$$

The case $|\alpha_2| \leq |\alpha_1|$ is handled similarly. Assume the result true up to $p-1$. Since

$$\begin{aligned} |\alpha_1 + \ldots + \alpha_{2^p}| &= |\alpha_1 + \ldots + \alpha_{2^{p-1}} + \alpha_{2^{p-1}+1} + \ldots + x_{2^p}| \\ &\leq 2\max\left(|\alpha_1 + \ldots + \alpha_{2^{p-1}}|, |\alpha_{2^{p-1}+1} + \ldots + x_{2^p}|\right). \end{aligned}$$

Putting

$$\begin{cases} \beta_1 = \alpha_{2^{p-1}+1} \\ \beta_2 = \alpha_{2^{p-1}+2} \\ \vdots \\ \beta_{2^{p-1}} = \alpha_{2^{p-1}+2^{p-1}} = \alpha_{2^p} \end{cases}$$

and applying the inductive hypothesis, on $\alpha_1, \ldots, \alpha_{2^{p-1}}$ and $\beta_1, \ldots, \beta_{2^{p-1}}$ we obtain

$$|\alpha_1 + \ldots + \alpha_{2^p}| \leq 2^n \max_{1 \leq i \leq 2^p} |\alpha_i|.$$

(ii) Let p be an integer such that $2^{p-1} < q \leq 2^p$. So, if we take $x_{q+1} = x_{q+2} = \ldots = x_{2^p} = 0$, we have

$$\begin{aligned} \Big|\alpha_1 + \alpha_2 + \ldots + \alpha_q\Big| &= \Big|\alpha_1 + \alpha_2 + \ldots + \alpha_q + \alpha_q + \ldots + \alpha_{2^q}\Big| \\ &\leq 2^p \max_{1 \leq i \leq 2^p} |\alpha_i| \quad \text{(as } (i)\text{)}. \\ &\leq 2q \max_{1 \leq i \leq 2^p} |\alpha_i|. \end{aligned}$$

This completes the proof. □

Remark 1.7. *Let $n \in \mathbb{N}$. Then, $|n| \leq 2n$. Indeed, if we take $x_1 = x_2 = \ldots = x_n = 1_{\mathbb{K}}$, then by using the inequality (3) we obtain $|n| \leq 2n$.*

Theorem 1.8. *Let $\mathbb{K}$ be a field and $|\cdot|$ a valuation with Artin constant C (see Definition 1.1 (iii)). Then, $|\cdot|$ satisfies the triangle inequality if, and only if, $C \leq 2$.*

Proof. Let $|\cdot|$ be a valuation satisfying the triangle inequality, and let $\alpha \in \mathbb{K}$ such that $|\alpha| \leq 1$. Then,

$$\begin{aligned} |\alpha + 1_{\mathbb{K}}| &\leq |\alpha| + |1_{\mathbb{K}}| \\ &\leq 1 + 1 \\ &= 2. \end{aligned}$$

Conversely, let $\alpha, \beta \in \mathbb{K}$. Hence, for all $n \in \mathbb{N}$, we obtain that

$$\begin{array}{rcl} |\alpha+\beta|^n & = & \left|\sum_{0\le i\le n} C_i^n \alpha^i \beta^{n-i}\right| \\ & \le & 2(n+1)\max_{0\le i\le n}\left(|C_i^n||\alpha^i||\beta^{n-i}|\right) \\ & \le & 4(n+1)\max_{0\le i\le n}\left(C_i^n|\alpha|^i|\beta|^{n-i}\right) \\ & \le & 4(n+1)\sum_{0\le i\le n}\left(C_i^n|\alpha|^i|\beta|^{n-i}\right) \\ & = & 4(n+1)(|\alpha|+|\beta|)^n. \end{array}$$

Taking n-th root and letting $n \to \infty$, we find

$$|\alpha+\beta| \le |\alpha| + |\beta|,$$

which completes the proof. □

Note that any positive power of a valuation is still a valuation.

Proposition 1.9. *Let* $(\mathbb{K}, |\cdot|)$ *be a valued field with Artin constant* C*, and let* λ *be a positive real number, then*

$$\begin{array}{rcl} |\cdot|^\lambda : \mathbb{K} & \to & \mathbb{R}_+ \\ \alpha & \mapsto & |\alpha|^\lambda \end{array}$$

is a valuation on $\mathbb{K}$ *with Artin constant* C^λ*.*

Proof. According to Definition 1.1, the statements (i) and (ii) of can be proved easily.
For the statement (iii) of Definition 1.1. The fact that $|\alpha|^\lambda \le 1$ implies that $|\alpha| \le 1$. Since $|\cdot|$ is valuation on $\mathbb{K}$ we infer that $|\alpha + 1_{\mathbb{K}}| \le C$. Consequently,

$$|\alpha + 1_{\mathbb{K}}|^\lambda \le C^\lambda.$$

This entails $|\cdot|^\lambda$ is a valuation on $\mathbb{K}$ with Artin constant C^λ. □

The following theorem essentially separates the absolute value on $\mathbb{R}$ or $\mathbb{C}$ from all other valuations enabling us to avoid carrying out mere generalizations.

Theorem 1.10. [18, Theorem 1] *Let* $(\mathbb{K}, |\cdot|)$ *be a valued field. Then,*
(i) $\mathbb{K}$ *is a subfield of* (*or isomorphic to*) $\mathbb{C}$ *and the valuation induces the restriction topology on* $\mathbb{K}$*, or*
(ii) *the valuation on* $\mathbb{K}$ *satisfies the strong triangle inequality*

$$|\alpha_1 + \alpha_2| \le \max\left(|\alpha_1|, |\alpha_2|\right)$$

for all $\alpha_1, \alpha_2 \in \mathbb{K}$*.*

Definition 1.11. *Two valuations* $|\cdot|_1$ *and* $|\cdot|_2$ *on the field* $\mathbb{K}$ *are equivalent, if there exists a positive real numbers* λ *such that* $|\cdot|_2 = |\cdot|_1^\lambda$*.*

Remark 1.12. *It follows from Theorem 1.10 that a valuation is either Archimedean as a valuation on a subfield of* (*or isomorphic to*) $\mathbb{C}$ *for example that of* $\mathbb{R}$ *and* $\mathbb{C}$*, or is non-Archimedean.*

We define the distance between two elements $x, y \in \mathbb{K}$ by

$$d(x, y) = |x - y|.$$

The function $d(\cdot,\cdot)$ is called the topological induced by the valuation. Then, $(\mathbb{K}, d)$ is topological space. Define the closed and the open balls on $\mathbb{K}$ centered at λ_0 and with radius ε, respectively:

$$\overline{\mathrm{B}}_{\mathbb{K}}(\lambda_0, \varepsilon) = \{\lambda \in \mathbb{K} : |\lambda - \lambda_0| \leq \varepsilon\}$$

and

$$\mathrm{B}_{\mathbb{K}}(\lambda_0, \varepsilon) = \{\lambda \in \mathbb{K} : |\lambda - \lambda_0| < \varepsilon\}.$$

Hence, the fundamental system of neighborhoods of each element λ_0 in $\mathbb{K}$ consists of the set of element λ such that $|\lambda - \lambda_0| < \varepsilon$.

1.3. Non-Archimedean Valuations

A fields $\mathbb{K}$ to will be considered as a different subfield of (or isomorphic to) $\mathbb{C}$, and all non-Archimedean valuations to will be considered are non-trivial.

Theorem 1.13. [16, Theorem 1.1] *Let $|\cdot|$ be a valuation on $\mathbb{K}$. The following conditions are equivalent:*
(*i*) *The valuation is non-Archimedean.*
(*ii*) $|n\, 1_{\mathbb{K}}| \leq 1$ *for every* $n \in \mathbb{N}$.
(*iii*) *For all* $x, y \in \mathbb{K}$, $|x + y| \leq \max\{|x|, |y|\}$.
(*iv*) *If* $x, y \in \mathbb{K}$ *and* $|x| < |y|$, *then* $|x - y| = |y|$.

Remark 1.14. *A valuation $|\cdot|$ is non-Archimedean, if the set $\{|n\, 1_{\mathbb{K}}| \leq 1 : n \in \mathbb{N}\}$ is bounded, otherwise it is Archimedean* (*see* [16, Chapter 1]).

In the following, we give one of the most interesting example of the non-archimedean valuation.

Example 1.15. *Let $\mathbb{Q}$ be a field and p be a prime number. We write every non-zero rational number x as:*

$$x = \frac{a}{b} p^n$$

where n, a, b are integers, and $\gcd(p, ab) = 1$ *(where* gcd *means the greatest common divisor). Put*

$$|x|_p = \begin{cases} p^{-n} & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Then, $|\cdot|_p$ is a non-Archimedean valuation on $\mathbb{Q}$. It is called the p-adic valuation.

Proof. • It follows from (4) that $|x|_p = 0$ if, and only, if $x = 0$.

• If $x = \frac{a}{b}p^n$ and $y = \frac{c}{d}p^m$, where n, m, a, b, c, d are integers, $\gcd(p, ab) = 1$ and $\gcd(p, cd) = 1$(where gcd means the greatest common divisor), then

$$xy = \frac{ac}{bd} p^{m+n},$$

and

$$\gcd(abcd, p) = 1.$$

Therefore, $|xy|_p = p^{-(n+m)} = |x|_p\, |y|_p$. Moreover, $xy = 0$, if, and only if, $x = 0$ or $y = 0$ if, and only if, $|x|_p\, |y|_p = 0$. Then,

$$|xy|_p = 0 = |x|_p\, |y|_p.$$

• We assume x and y are non-zero rational number. If $n \leq m$, then

$$x + y = p^n \left(\frac{ad + p^{m-n} cb}{bd} \right).$$

Hence, we obtain

$$|x+y|_p \le p^{-n} \le \max\left\{|x|_p, |y|_p\right\}.$$

The case $m \le n$ is handled similarly.

If $x = y = 0$, then $|x+y|_p = 0 = \max\left\{|x|_p, |y|_p\right\}$.

If $x = 0$ and $y \ne 0$, then

$$|x+y|_p = |y|_p \le \max\left\{|x|_p, |y|_p\right\}.$$

If $y = 0$ and $x \ne 0$, then

$$|x+y|_p = |x|_p \le \max\left\{|x|_p, |y|_p\right\}.$$

So, $|\cdot|_p$ is a non-Archimedean valuation, is claimed. This completes the proof □

Remark 1.16. *Let $p \ge 2$ be a prime.*

(i) *The non-Archimedean valued field $(\mathbb{Q}, |\cdot|_p)$ is not complete.*

(ii) *The completion of $(\mathbb{Q}, |\cdot|_p)$ is called the field of p-adic numbers and denoted by $(\mathbb{Q}_p, |\cdot|_p)$. Moreover, we have the following characterization of p-adic numbers: each $x \in \mathbb{Q}_p$ can be expressed as*

$$x = \sum_{i \ge n} a_i p^i,$$

where $0 \le a_i \le p-1$, and $|x|_p = p^{-n}$.

In non-Archimedean theory, there are two kinds of valuations: discrete and dense.

Definition 1.17. *Let $(\mathbb{K}, |\cdot|)$ be a non-Archimedean valued field.*
$|\mathbb{K}^*| = \{|x| : x \in \mathbb{K}, x \ne 0\}$ *is called the value group of the non-Archimedean valuation.*

Moreover, $|\mathbb{K}^*|$ is a subgroup of the multiplicative group of positive real numbers.

Definition 1.18. (i) *The non-Archimedean valuation is called discrete, if there is a real number $0 < r < 1$ such that*

$$|\mathbb{K}^*| = \{r^n : n \in \mathbb{Z}\}.$$

(ii) *The non-Archimedean valuation is called dense, if it is non discrete.*

Remark 1.19. (i) *If a non-Archimedean valuation is trivial, then it is discrete. Indeed, it follows from Remark 1.16 that for all $x \ne 0$, we have $|x| = 1$. Then, we infer that $|\mathbb{K}^*| = 1$. This implies that there is $0 < r < 1$ such that $|\mathbb{K}^*| = \{r^0 : 0 \in \mathbb{Z}\}$. Hence, the non-Archimedean trivial valuation is discrete.*

(ii) *The p-adic valuation on $\mathbb{Q}_p$ is discrete. Indeed, by referring to Example 1.15, we have for all $x \in \mathbb{Q}_p^*$,*

$$|x|_p = p^{-n}.$$

Clearly, $0 < p^{-1} < 1$. Put $r = p^{-1}$. Hence, we infer that

$$|\mathbb{Q}_p^*| = \{r^n : n \in \mathbb{Z}\}.$$

Example 1.20. *For nonzero polynomial $P \in \mathbb{R}[X]$ given by*

$$P(X) = a_0 + a_1 X + \cdots + a_{deg(P)} X^{deg(P)},$$

where $deg(P)$: is the degree of P, $a_{deg(P)} \ne 0$ and $a_0, a_1, \cdots, a_{deg(P)} \in \mathbb{R}$. By convention, the degree of the zero polynomial is $-\infty$.

Let r be any number greater than 1. *For* $P \in \mathbb{R}[X]$, *puting*

$$|P| = \begin{cases} r^{deg(P)} & if\ P \neq 0 \\ 0 & if\ P = 0. \end{cases} \tag{5}$$

We define the map $|\cdot|_*$ *on the filed of rational function* $\mathbb{R}(X)$ *by*

$$|PQ^{-1}|_* = |P|\ |Q|^{-1},$$

where $P \in \mathbb{R}[X]$ *and* $Q \in \mathbb{R}[X]\backslash\{0\}$. *Then,*
(i) $|\cdot|$ *behaves like a non-Archimedean valuation.*
(ii) $|\cdot|_*$ *is a non-Archimedean valuation.*
(iii) *The valuation* $|\cdot|_*$ *is discrete on* $\mathbb{R}(X)$.

Proof. (i) We will prove that $|\cdot|$ satisfies the axioms of a valuation.
• It is clear that $P = 0$ if, and only if, $|P| = 0$.
• Let $P, Q \in \mathbb{R}[X]$. Then,

$$\begin{aligned} |PQ| &= \begin{cases} r^{deg(PQ)} & \text{if } PQ \neq 0 \\ 0 & \text{if } PQ = 0. \end{cases} \\ &= \begin{cases} r^{deg(P)}\ r^{deg(Q)} & \text{if } PQ \neq 0 \\ 0 & \text{if } PQ = 0. \end{cases} \end{aligned} \tag{6}$$

We discuss two cases.
First case. If $PQ \neq 0$, then we obtain $P \neq 0$ and $Q \neq 0$. It follows from (6) that

$$|PQ| = r^{deg(P)}\ r^{deg(Q)} = |P|\ |Q|.$$

Second case. If $PQ = 0$, then $P = 0$ or $Q = 0$ which yields $|P|\ |Q| = 0$. Thanks to (6), we can see that $|PQ| = 0 = |P|\ |Q|$. Hence, for all $P, Q \in \mathbb{R}[X]$, we conclude that

$$|PQ| = |P|\ |Q|. \tag{7}$$

• Let $P, Q \in \mathbb{R}[X]$. Then,

$$|P + Q| = \begin{cases} r^{deg(P+Q)} & \text{if } P \neq -Q \\ 0 & \text{if } P = -Q. \end{cases} \tag{8}$$

We discuss two cases.
First case. If $P \neq -Q$, then by using (8), we have

$$\begin{aligned} |P + Q| &= r^{deg(P+Q)} \\ &\leq r^{\max\{deg(P), deg(Q)\}}. \end{aligned}$$

Second case. If $P = -Q$, then by referring to (8), we have

$$|P + Q| = 0 \leq r^{\max\{deg(P), deg(Q)\}}.$$

Hence, for all $P, Q \in \mathbb{R}[X]$, we deduce that

$$\begin{aligned} |P + Q| &\leq r^{\max\{deg(P), deg(Q)\}} \\ &\leq \max\{r^{deg(P)}, r^{deg(Q)}\} \\ &\leq \max\{|P|, |Q|\}. \end{aligned} \tag{9}$$

Thus, $|\cdot|$ is satisfy the axioms of the valuation.

(*ii*) • Let $PQ^{-1} \in \mathbb{R}(X)$ and $|PQ^{-1}|_* = |P|\,|Q|^{-1} = 0$. Then, we obtain $|P| = 0$. It follows that $P = 0$. This implies that

$$PQ^{-1} = 0.$$

Conversely, let $0 \in \mathbb{R}(X)$. Then, $0 = 0\,Q^{-1}$ for all $Q \in \mathbb{R}[X]\backslash\{0\}$. Hence, $|0|_* = |0|\,|Q|^{-1} = 0$.

• Let $P_1Q_1^{-1}, P_2Q_2^{-1} \in \mathbb{R}(X)$. Therefore,

$$\begin{aligned}|P_1Q_1^{-1}P_2Q_2^{-1}|_* &= |(P_1P_2)(Q_2Q_1)^{-1}| \\ &= |P_1P_2|\,|Q_2Q_1|^{-1}.\end{aligned}$$

It follows from (7) that

$$\begin{aligned}|P_1Q_1^{-1}P_2Q_2^{-1}|_* &= |P_1|\,|P_2|\,(|Q_2|\,|Q_1|)^{-1} \\ &= |P_1|\,|Q_1|^{-1}\,|P_2|\,|Q_2|^{-1} \\ &= |P_1Q_1^{-1}|_*\,|P_2Q_2^{-1}|_*.\end{aligned}$$

• Let $P_1Q_1^{-1}, P_2Q_2^{-1} \in \mathbb{R}(X)$. So, we have

$$\begin{aligned}|P_1Q_1^{-1} + P_2Q_2^{-1}|_* &= |(P_1Q_2 + P_2Q_1)(Q_2Q_1)^{-1}| \\ &= |P_1Q_2 + P_2Q_1|\,|Q_2Q_1|^{-1}.\end{aligned}$$

In view of (7) and (9) implies that

$$\begin{aligned}|P_1Q_1^{-1} + P_2Q_2^{-1}|_* &\leq \max\{|P_1Q_2|, |P_2Q_1|\}\ |Q_2Q_1|^{-1} \\ &\leq \max\{|P_1|\,|Q_2|, |P_2|\,|Q_1|\}\,|Q_2|^{-1}|\,|Q_1|^{-1} \\ &\leq \max\{|P_1|\,|Q_1|^{-1}|, |P_2|\,|Q_2|^{-1}\} \\ &\leq \max\{|P_1Q_1^{-1}|_*, |P_2Q_2^{-1}|_*\}.\end{aligned}$$

Hence, we conclude that $|\cdot|_*$ is a non-Archimedean valuation.

(*iii*) For all $PQ^{-1} \in \mathbb{R}(X)^*$, we have

$$\begin{aligned}|PQ^{-1}|_* &= r^{deg(P)}\, r^{-deg(Q)} \\ &= \left(r^{-1}\right)^{deg(Q)-deg(P)}.\end{aligned}$$

Put $u = r^{-1}$. Clearly, $0 < u < 1$. Hence, we deduce that

$$|\mathbb{R}(X)^*| = \{u^n : n \in \mathbb{Z}\}.$$

This completes the proof. □

Remark 1.21. *$|\cdot|$ satisfies the axioms of a valuation but it is not a valuation, because $\mathbb{R}[X]$ is not a field.*

Example 1.22. *Let Γ be a field. Consider the formal series by*

$$f = \sum_{i\in\mathbb{N}} a_i t^{\alpha_i},$$

where $(a_i)_i \subset \Gamma$ and $(\alpha_i)_i$ is an increasing sequence of rational number such that $\alpha_i \longrightarrow +\infty$ as $i \to +\infty$. It is know that the set of formal series is a field. We denote it by $\mathbb{K}$. Let $f \in \mathbb{K}$ and $a_1 \neq 0$. We define a non-Archimedean valuation on $\mathbb{K}$ by

$$|f| = \begin{cases} 2^{-\alpha_1} & \text{if } f \neq 0 \\ 0 & \text{if } f = 0. \end{cases}$$

Then, $|\cdot|$ is discrete on $\mathbb{K}$. Indeed, for $f \in \mathbb{K}^*$, we have $|f| = 2^{-a_1}$. Then,

$$|\mathbb{K}^*| = \{2^{-a_1}, a_1 \in \mathbb{Q}\}.$$

This is equivalent to saying that $|\cdot|$ is not dense which yields is discrete.

Remark 1.23. *Let $(\mathbb{K}, |\cdot|)$ be a non-Archimedean valued field.*

(*i*) *The non-Archimedean valuation on $\mathbb{K}$ can be either dense or discrete. This nature of the valuation is closely related to the nature of the metric that it induces on $\mathbb{K}$.*

(*iii*) *The valuation on $\mathbb{K}$ is discrete if, and only if, the metric that it induces on $\mathbb{K}$ is discrete (i.e., if for any sequence $(x_n, y_n)_n$ in $\mathbb{K}^2$ such that the sequence of real numbers $(d(x_n, y_n))_n$ is strictly decreasing and $d(x_n, y_n) \longrightarrow 0$ as $n \to +\infty$).*

(*iii*) *The metric d on $\mathbb{K}$ is dense if, and only, if the valuation inducing it is dense (i.e., the value group $|\mathbb{K}^*|$ is dense in $\mathbb{R}_+^*$).*

Proposition 1.24. [18, Proposition 2] *Let $(\mathbb{K}, |\cdot|)$ be a non-archimedean valued field.*

(*i*) *Every ball is both open and closed.*

(*ii*) *Two balls are either disjoint or one is a subset of the other.*

(*iii*) *Each point of any ball is a center.*

Remark 1.25. *In general, the closed ball is not compact on non-archimedean valued field. But if $\mathbb{K}$ is a locally compact field (i.e. every point has a neighborhood which is a compact set) with a non trivial non-archimedean valuation, then every closed ball of $\mathbb{K}$ is compact.*

Proposition 1.26. [16, Exercice 1. B] *Let $\mathbb{K}$ be a field with a non trivial non-archimedean valuation. The following conditions are equivalent*

(*i*) *$\mathbb{K}$ is locally compact.*

(*ii*) *$\overline{\mathrm{B}}_{\mathbb{K}}(0, 1)$ is compact.*

(*iii*) *Every closed bounded subset of $\mathbb{K}$ is compact.*

Notice that the classification of locally compact fields is well known see for instance A. Weil in [20].

Definition 1.27. *Let $(\mathbb{K}, |\cdot|)$ be a non-Archimedean valued field.*

(*i*) *A sequence $(x_n)_n$ of $\mathbb{K}$ is said to converge to $x \in \mathbb{K}$ if, the sequence of real numbers $(|x_n - x|)$ converges to 0.*

(*ii*) *A sequence $(x_n)_n$ of $\mathbb{K}$ is called a Cauchy sequence if, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ whenever $n \geq m \geq n_0$.*

(*iii*) *A metric space $(\mathbb{K}, d)$ is said to be complete if every Cauchy sequence of elements in $\mathbb{K}$ converges.*

Definition 1.28. *X is called spherically complete if each nested sequence of balls $B_1 \supset B_2 \supset \ldots$ has a non-empty intersection.*

Now, let us assume that $(\mathbb{K}, |\cdot|)$ be a complete non-Archimedean filed.

Definition 1.29. *Let X be a vector space over non-Archimedean field $\mathbb{K}$. A non-Archimedean norm on X is a map $\|\cdot\| : X \longrightarrow \mathbb{R}_+$ satisfying*

(*i*) *$\|x\| = 0$ if, and only if, $x = 0$.*

(*ii*) *$\|\lambda x\| = |\lambda| \, \|x\|$ for any $x \in X$ and any $\lambda \in \mathbb{K}$.*

(*iii*) *$\|x + y\| \leq \max\{\|x\|, \|y\|\}$, for any $x, y \in X$.*

Property (*iii*) of Definition 1.29 is referred to as the ultrametric or strong triangle inequality. The pair $(X, \|\cdot\|)$ is called a non-Archimedean normed space, where X is a vector space over non-Archimedean filed $\mathbb{K}$ and $\|\cdot\|$ is a non-Archimedean norm on X.

Remark 1.30. (*i*) *The non-Archimedean valuation on* $\mathbb{K}$ *itself is a non-Archimedean norm.*

(*ii*) *Let* $(X, \|.\|)$ *be a non-Archimedean normed space. If* $x, y \in X$ *such that* $\|x\| \neq \|y\|$, *then we have*

$$\|x + y\| = \max\left\{\|x\|, \|y\|\right\}.$$

(*iii*) *It is clear that the strong triangle inequality implies the triangle inequality. Then, a non-Archimedean normed space are normed according to the standard definition.*

A normed vector space $(X, \|.\|)$ will be considered as a metric space with respect to the metric

$$d(x, y) = \|x - y\| \text{ for any } x, y \in X.$$

Then, $d(\cdot,\cdot)$ induces a topology on X. Define the closed and the open balls on X centered x_0 and with radius ε, respectively :

$$\overline{\mathrm{B}}_{\mathbb{X}}(x_0, \varepsilon) = \{x \in X : \|x - x_0\| \leq \varepsilon\}$$

and

$$\mathrm{B}_{\mathbb{X}}(x_0, \varepsilon) = \{x \in X : \|x - x_0\| < \varepsilon\}.$$

Definition 1.31. *Let* $(X, \|.\|)$ *be a non-Archimedean normed space and E be a nonempty subset of X.*

(*i*) *The set E is said to be bounded, if the set of real numbers* $\{\|x\| : x \in E\}$ *is bounded.*

(*ii*) *The set E is said to be absolutely convex, if* $\alpha x + \beta y \in E$, *for all* $x, y \in E$ *and* $\alpha, \beta \in \overline{\mathrm{B}}_{\mathbb{K}}(0,1)$, *i.e., if E is* $\overline{\mathrm{B}}_{\mathbb{K}}(0,1)$*-module.*

(*iii*) *The set E is said to be compactoid, if for every* $r > 0$ *there exists a finite set* $F \subset X$ *such that*

$$E \subset \overline{\mathrm{B}}_X(0, r) + \overline{Co}F,$$

where $\overline{Co}F$ *is a intersection for all closed absolutely convex subsets of X containing F.* ◊

Lemma 1.32. [16, 4.S Exercise] *Let* $(X, \|.\|)$ *be a non-Archimedean normed space over a field* $\mathbb{K}$.

(*i*) *Every compactoid is bounded.*

(*ii*) *If E, F are compactoid of X, then so E + F.*

(*iii*) *If* $\mathbb{K}$ *is locally compact, then the compactoids of X are just the precompact set.*

Definition 1.33. *Let* $(X, \|.\|)$ *be a non-Archimedean normed space. A sequence* $(x_n)_n \subset X$ *converges to* $x \in X$, *if the sequence of real numbers* $(\|x_n - x\|)_n$ *converges to* 0.

Lemma 1.34. [3, Proposition 2.13] *Let* $(X, \|.\|)$ *be a non-Archimedean normed space. If the sequence* $(x_n)_n$ *converges in X, then it is bounded.* ◊

Definition 1.35. *A non-Archimedean Banach space* $(X, \|.\|)$ *is a non-Archimedean normed space, which is complete with respect to the natural metric induced by the norm.* ◊

Remark 1.36. (*i*) *A closed subspace of a non-Archimedean Banach space is a non-Archimedean Banach space.*

(*ii*) *The space* $(\mathbb{K}, |\cdot|)$ *is a non-Archimedean Banach spaces. Indeed, the fact that* $(\mathbb{K}, |\cdot|)$ *is complete follows from Remark 1.30* (*i*) *that it is non-Archimedean Banach spaces.*

(*iii*) *The space* $(\mathbb{K}^n, \|\cdot\|)$ *is equipped with the norm defined by*

$$\|\lambda\| = \max\left\{|\lambda_i| : 1 \leq i \leq n\right\}, \textit{ for all } \lambda = (\lambda_i)_i \subset \mathbb{K}^n,$$

is a non-Archimedean Banach space. Indeed, it follows from (*ii*) *and* (*iii*) *that* $(\mathbb{K}^n, \|\cdot\|)$ *is a non-Archimedean Banach space.*

Example 1.37. *Let X be a non-Archimedean Banach space and F be a closed subspace of X. Let $P : X \to X/_F$ be the quotient map. Define the norm on the quotient space $X/_F$ by*

$$\|Px\| = d(x, F), x \in X$$

where $d(x, F) = \inf_{y \in F} \|x - y\|$. Then, this norm is a non-Archimedean norm on $X/_F$. Indeed, we have $Px = Py$, if and only if, $x - y \in F$. This implies that the norm is well defined.

- *Let $Px \in X/_F$ such that $\|Px\| = 0$. Then, $d(x, F) = 0$ which yields $x \in F$. Hence, we infer that $Px = 0$. Conversely, we have $\|0\| = \|P0\|$. The fact that F is a subspace of X implies that $d(0, F) = 0$. Consequently, $\|0\| = 0$.*
- *Let $\lambda \in \mathbb{K}^*$ and $Px \in X/_F$. Since F is a vector space, then we have*

$$\begin{aligned} \|\lambda Px\| &= \|P(\lambda x)\| \\ &= \inf_{y \in F} \|\lambda x - y\| \\ &= \inf_{\lambda^{-1} y \in F} \left\| \lambda \left(x - \lambda^{-1} y \right) \right\| \\ &= |\lambda| \inf_{\lambda^{-1} y \in F} \left\| x - \lambda^{-1} y \right\| \\ &= |\lambda| \, \|Px\|. \end{aligned}$$

- *Let $Px \in X/_F$ and $Py \in X/_F$. Then, we have*

$$\|Px + Py\| = \inf_{z \in F} \|x + y - z\|. \tag{10}$$

Let $z_1 \in F$. Then, we have $z = z - z_1 + z_1$. Putting $z_2 = z - z_1$. Since F is a vector space, then $z_2 \in F$. It follows from (10) *that*

$$\begin{aligned} \|Px + Py\| &= \inf_{z_1 + z_2 \in F} \|(x - z_2) + (y - z_1)\| \\ &\leq \inf_{z_1 + z_2 \in F} \max\{\|x - z_2\|, \|y - z_1\|\} \\ &\leq \max\{\inf_{z_1 \in F} \|x - z_2\|, \inf_{z_2 \in F} \|y - z_1\|\} \\ &\leq \max\{\|Px\|, \|Py\|\}. \end{aligned}$$

It is shown that $\| \cdot \|$ is a non-archimedean norm.

Example 1.38. [2, Example 7] *Let $\omega = (\omega_i)_i$ be a sequence of non-zero elements in $\mathbb{K}$. We define $\ell^\infty(\mathbb{N}, \mathbb{K})$ by*

$$\ell^\infty(\mathbb{N}, \mathbb{K}) = \left\{ x = (x_i)_i : x_i \in \mathbb{K}, \forall\, i \in \mathbb{N} \text{ and } \|x\| = \sup_{i \in \mathbb{N}} |\omega_i|^{1/2} \, |x_i| < +\infty \right\},$$

where $(|\omega_i|^{1/2})_i \subset \mathbb{R}_+^*$, and it is equipped with the norm

$$x = (x_i)_i \subset \mathbb{E}_\omega, \ \|x\| = \sup_{i \in \mathbb{N}} \left(|\omega_i|^{1/2} |x_i| \right).$$

The space $(\ell^\infty(\mathbb{N}, \mathbb{K}), \| \cdot \|)$ is a non-Archimedean Banach space.

Now, let us recall the space $\mathbb{E}_\omega$ which plays a very important role in the sequel. The reader interested in this space may also refer to T. Diagana and F. Ramaroson [3].

Definition 1.39. *Let $\omega = (\omega_i)_i$ be a sequence of non-zero elements in $\mathbb{K}$. We define $\mathbb{E}_\omega$ by*

$$\mathbb{E}_\omega = \left\{ x = (x_i)_i : x_i \in \mathbb{K}, \forall\, i \in \mathbb{N} \text{ and } \lim_{i \to +\infty} |\omega_i|^{1/2} |x_i| = 0 \right\},$$

and it is equipped with the norm

$$x = (x_i)_i \subset \mathbb{E}_\omega, \ \|x\| = \sup_{i \in \mathbb{N}} \left(|\omega_i|^{1/2} |x_i| \right). \qquad \diamond$$

We recall the following results due to S. Ludkovsky and B. Diarra [13].

Proposition 1.40. *The space* $(\mathbb{E}_\omega, \|\cdot\|)$ *is a non-Archimedean Banach space.*

Proof. In order to show that $(\mathbb{E}_\omega, \|\cdot\|)$ is a non-Archimedean Banach space, we will prove that $\mathbb{E}_\omega$ is a closed subspace of $\ell^\infty(\mathbb{N}, \mathbb{K})$. Let us assume that $x \in \mathbb{E}_\omega$. Then, $\lim_{i\to+\infty} |\omega_i|^{1/2}\,|x_i| = 0$. This implies that the sequence of real numbers $(|\omega_i|^{1/2}|x_i|)_i$ is bounded. Hence, we infer that $\sup_{i\in\mathbb{N}} |x_i||\omega_i|^{1/2} < +\infty$. It is show that $\mathbb{E}_\omega \subset \ell^\infty(\mathbb{N}, \mathbb{K})$. It remains to prove that $\mathbb{E}_\omega$ is a closed. Let $(x(n))_n \subset \mathbb{E}_\omega$ such that $x(n) \longrightarrow x$ as $n \to +\infty$. Then, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\|x(n) - x\| = \sup_{i\in\mathbb{N}} |x_i(n) - x_i||\omega_i|^{1/2} < \varepsilon.$$

This implies that for all $i \in \mathbb{N}$ and $n \geq n_0$,

$$|x_i(n) - x_i||\omega_i|^{1/2} < \varepsilon. \tag{11}$$

Since the valuation is non-Archimedean, then we have

$$\begin{array}{rcl} |x_i||\omega_i|^{1/2} & = & |x_i(n_0) - x_i(n_0) + x_i||\omega_i|^{1/2} \\ & \leq & \max\left\{|x_i(n_0) - x_i||\omega_i|^{1/2}, |x_i(n_0)||\omega_i|^{1/2}\right\}. \end{array}$$

It follows from (11) that

$$|x_i||\omega_i|^{1/2} \leq \max\left\{\varepsilon, |x_i(n_0)||\omega_i|^{1/2}\right\}.$$

Consequently,

$$\limsup_{i\in\mathbb{N}} |x_i||\omega_i|^{1/2} \leq \max\left\{\varepsilon, \limsup_{i\in\mathbb{N}} |x_i(n_0)||\omega_i|^{1/2}\right\}.$$

The fact that $\limsup_{i\in\mathbb{N}} |x_i(n_0)||\omega_i|^{1/2} = \lim_{i\to+\infty} |x_i(n_0)||\omega_i|^{1/2} = 0$ implies that

$$\limsup_{i\in\mathbb{N}} |x_i||\omega_i|^{1/2} \leq \varepsilon.$$

By the arbitrariness of ε, we infer that $\lim_{i\to+\infty} |x_i||\omega_i|^{1/2} = 0$. This enables us to conclude that $x \in \mathbb{E}_\omega$. Finally, the use of Remark 1.36 (*i*) and Example 1.38 allows us to conclude that $(\mathbb{E}_\omega, \|\cdot\|)$ is a non-Archimedean Banach space. □

Remark 1.41. (*i*) *For* $x = (x_i)_i$, $y = (y_i)_i$, *the inner product is defined by*

$$\begin{array}{rcl} \langle\cdot,\cdot\rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega & \longrightarrow & \mathbb{K} \\ (x, y) & \longmapsto & \displaystyle\sum_{i=0}^{+\infty} x_i y_i w_i. \end{array}$$

The space $\left(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle\right)$ *is called a non-Archimedean (or p-adic) Hilbert space (see* [3, Section 2.4] *and Example 1.38).*

(*iii*) $\mathbb{E}_\omega$ *has a canonical orthogonal base, namely,* $\{e_i : i = 0, 1, 2, \dots\}$ *where* $e_i = (0, \dots, 0, 1, 0, \dots)$ *with* 1 *at the i th place. For each* $i \in \mathbb{N}$, *we have* $\|e_i\| = |\omega_i|^{\frac{1}{2}}$ *(see* [2, Subsection 2.3.2]).

1.4. Gap between subspaces of non-Archimedean Banach spaces

Definition 1.42. *Let X be a non-Archimedean Banach space such that $\|X\| \subseteq |\mathbb{K}|$ (where $\|X\| = \{\|x\| : x \in X\}$ and $|\mathbb{K}| := \{|\lambda| : \lambda \in \mathbb{K}\}$) and M, N be two linear subspaces of X. Let us define*

$$\delta(M,N) = \begin{cases} \sup\limits_{\substack{x\in M\\ \|x\|=1}} d(x,N) & \text{if } M \neq \{0\} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\widehat{\delta}(M,N) = \max\Big\{\delta(M,N), \delta(N,M)\Big\},$$

where

$$d(x,N) = \inf\Big\{d(x,z) : z \in N\Big\} = \inf\Big\{\|x-z\| : z \in N\Big\}.$$

$\widehat{\delta}(M,N)$ is called the gap between the subspaces M and N.

Remark 1.43. (i) *One of the most difference between the Archimedean and the non-Archimedean theories is that if X is a normed vector space over $\mathbb{K}$, the set $\|X\|$ may not be the same as $|\mathbb{K}|$. As a consequence, non zero element of X may be fail to have a scalar multiple of norm 1, in fact, the unit sphere $\{x \in X : \|x\| = 1\}$ may well very be empty. Indeed, let us assume that $\|X\| \neq \mathbb{R}_+$. Consider for $r \in \mathbb{R}_+\backslash\|X\|$ the norm $\|\cdot\|_r$ on X by*

$$\|x\|_r = r^{-1}\|x\|.$$

This implies that

$$\Big\{x \in X : \|x\|_r = 1\Big\} = \emptyset.$$

(ii) *Accordingly, we added the condition $\|X\| \subseteq |\mathbb{K}|$ in Definition 1.42. This hypothesis leads to the unit sphere is not empty, and therefore the existence of δ and thus that of $\widehat{\delta}(.,.)$ are ensured.*

The following properties of the gap follow directly from the Definition 1.42.

Proposition 1.44. *Let M, N be two linear subspaces of a non-Archimedean Banach space X such that $\|X\| \subseteq |\mathbb{K}|$. Then, the following properties are hold:*

(i) $\delta(M,N) = \delta(\overline{M},\overline{N})$ *and* $\widehat{\delta}(M,N) = \widehat{\delta}(\overline{M},\overline{N})$.

(ii) $\delta(M,N) = 0$ *if, and only if,* $\overline{M} \subset \overline{N}$.

(iii) $\widehat{\delta}(M,N) = 0$ *if, and only if,* $\overline{M} = \overline{N}$.

(iv) $0 \leq \delta(M,N) \leq 1$ *and* $0 \leq \widehat{\delta}(M,N) \leq 1$.

Proof. (i) The fact that $d(x,N) = d(x,\overline{N})$ implies that

$$\begin{aligned} \delta(M,N) &= \sup_{\substack{x\in M\\ \|x\|=1}} d(x,N) \\ &= \sup_{\substack{x\in \overline{M}\\ \|x\|=1}} d(x,N) \\ &= \sup_{\substack{x\in \overline{M}\\ \|x\|=1}} d(x,\overline{N}) \\ &= \delta(\overline{M},\overline{N}). \end{aligned}$$

It follows that

$$\begin{aligned}\widehat{\delta}(M,N) &= \max\{\delta(M,N);\delta(N,M)\}\\ &= \max\left\{\delta(\overline{M},\overline{N});\delta(\overline{N},\overline{M})\right\}\\ &= \widehat{\delta}(\overline{M},\overline{N}).\end{aligned}$$

(*ii*) Assume that $\delta(M,N)=\delta(\overline{M},\overline{N})=0$. Then, we have

$$\sup_{\substack{x\in\overline{M}\\ \|x\|=1}} d(x,\overline{N})=0.$$

Hence, for all $x\in\overline{M}$ we get $d(x,\overline{N})=0$. This implies that $x\in\overline{N}$. Therefore, $\overline{M}\subset\overline{N}$. Conversely, let us assume that $\overline{M}\subset\overline{N}$. Then, for all $x\in\overline{M}$ we have $d(x,\overline{N})=0$. Consequently, $\delta(M,N)=\delta(\overline{M},\overline{N})=0$.

(*iii*) Let $\widehat{\delta}(M,N)=0$. This is equivalent to saying that $\delta(M,N)=0$ and $\delta(N,M)=0$. Then, we can conclude from (*ii*) that

$$\widehat{\delta}(M,N)=0 \text{ if, and only if, } \overline{M}\subset\overline{N} \text{ and } \overline{N}\subset\overline{M}.$$

As a result $\widehat{\delta}(M,N)=0$ if, and only if, $\overline{N}=\overline{M}$, as desired.

(*iv*) On the one hand, since the distance is always positive, then it is easy to see that $\delta(M,N)\geq 0$. On the other hand, we have

$$\begin{aligned}\delta(M,N) &= \sup_{\substack{x\in M\\ \|x\|=1}}\left[\inf_{y\in N}\|x-y\|\right]\\ &\leq \sup_{\substack{x\in M\\ \|x\|=1}}\left[\inf_{y\in N}\left(\max\{\|x\|,\|y\|\}\right)\right].\end{aligned}$$

If $\max\{\|x\|,\|y\|\}=\|x\|=1$, then we can conclude that

$$0\leq\delta(M,N)\leq 1. \tag{12}$$

If $\max\{\|x\|,\|y\|\}=\|y\|$, then we have $\inf_{y\in N}\|y\|=0$. This implies that

$$0\leq\delta(M,N)\leq 0. \tag{13}$$

Therefore, by combining (12) and (13), we deduce that $0\leq\delta(M,N)\leq 1$. As a result, $0\leq\widehat{\delta}(M,N)\leq 1$. □

Remark 1.45. (*i*) *It follows from the Definition 1.42 that* $\delta(M,0)=1$, *if* $M\neq\{0\}$.

(*ii*) *Let us assume that* $\|X\|\subseteq|\mathbb{K}|$. *Keeping in mind that* $d(cx,N)=|c|d(x,N)$ *for any non-zero element* c *in* $\mathbb{K}$, *we infer that*

$$\sup_{\substack{x\in M\\ \|x\|=|c|}} d(x,N)=|c|\delta(M,N). \tag{14}$$

2. Linear operators on a non-Archimedean Banach space

In [2], T. Diagana applied the theory of linear operators on non-Archimedean Banach spaces. He developed this theory and studied its basic properties. It is worth mentioning that the proofs which are valid for real or complex spaces cannot be given in the same way for spaces over a non-Archimedean valued field $\mathbb{K}$. Among the works in this direction, we can state, for example ([1, 4, 5]). Furthermore in [3], T. Diagana and F. Ramaroson developed this theory and used it to study the spectral theory on non-Archimedean Hilbert spaces.The analysis in and over non-archimedean valued fields is known as ultrametric (non-archimedean, p-adic) analysis.

Definition 2.1. *Let X and Y be two non-Archimedean Banach spaces and $T : \mathcal{D}(T) \subset X \longrightarrow Y$. T is called linear, if $\mathcal{D}(T)$, which designates its domain, is a vector subspace of X, and if*

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty,$$

for all $\alpha, \beta \in \mathbb{K}$ and $x, y \in \mathcal{D}(T)$.

The symbols $R(T)$ and $N(T)$ stand respectively for the range and the null space of the operator T, which are defined by

$$R(T) = \Big\{Tx : x \in \mathcal{D}(T)\Big\}, \text{ and } N(T) = \Big\{x \in \mathcal{D}(T) : Tx = 0\Big\}.$$

2.1. Bounded linear operators

Definition 2.2. *Let X and Y be two non-Archimedean Banach spaces. A linear operator $T : X \longrightarrow Y$ is called bounded, if there exists $M \geq 0$ such that*

$$\|Tx\| \leq M\|x\|, \textit{ for all } x \in X.$$

Denoted by $\mathcal{L}(X, Y)$, the set of all bounded linear operators to X from Y. If $X = Y$, then $\mathcal{L}(X, X) = \mathcal{L}(X)$. Note that $\mathcal{L}(X)$ is a unitary Banach algebra.

Remark 2.3. (i) *Let X and Y be two non-Archimedean Banach spaces. For $T \in \mathcal{L}(X, Y)$, we have*

$$\|T\| = \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|}{\|x\|} = \inf\{M \geq 0 : \|Tx\| \leq M\|x\|, \textit{ for all } x \in X\} < +\infty.$$

(ii) *T is continuous if, and only if, $\|T\| < +\infty$.*
(iii) *The null and identity operators on X will be denoted respectively by O_X and I_X, which are defined by*

$$O_X(x) = 0 \textit{ and } I_X(x) = x, \textit{ for all } x \in X.$$

Moreover, the operators O_X and I_X are bounded.

Lemma 2.4. *Let X be a non-Archimedean Banach space.*
(i) *If $T, S \in \mathcal{L}(X)$ and $\lambda \in \mathbb{K}$, then $T + S$, λT, TS and ST belong to $\mathcal{L}(X)$.*
(ii) *The space $(\mathcal{L}(X), \|\cdot\|)$ of bounded linear operators on X, is non-Archimedean.* ◊

Proof. (i) Since $T, S \in \mathcal{L}(X)$, then $\mathcal{D}(T + S) = X$, $\mathcal{D}(\lambda T) = X$, $\mathcal{D}(TS) = X$ and $\mathcal{D}(TS) = X$.
• Let $x \in X$, then we have

$$\begin{aligned} \|(T + S)x\| &= \|Tx + Sx\| \\ &\leq \max\{\|Tx\|, \|Sx\|\}. \end{aligned}$$

Based on the assumption $T, S \in \mathcal{L}(X)$, we infer that

$$\|(T + S)x\| \leq \max\{\|T\|, \|S\|\} \, \|x\|.$$

• Let $x \in X$, then by the boundedness of the operator T, we infer that

$$\begin{aligned} \|(\lambda T)x\| &= |\lambda| \, \|Tx\| \\ &\leq |\lambda| \, \|T\| \, \|x\|. \end{aligned}$$

• Let $x \in X$, then by the boundedness of the operators T and S, we have

$$\begin{aligned} \|(T S)x\| &\leq \|T\| \, \|Sx\| \\ &\leq \|T\| \, \|S\| \, \|x\|. \end{aligned}$$

• Let $x \in X$, then we have

$$\|(S\,T)x\| \leq \|S\|\,\|T\|\,\|x\|.$$

(*ii*) Let $(T_n)_n$ be a Cauchy sequence in $\mathcal{L}(X)$. This implies that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|T_n - T_m\| < \varepsilon, \text{ for all } n, m \geq n_0.$$

Then, for all $x \in X \backslash \{0\}$, we have,

$$\|T_n x - T_m x\| < \varepsilon\,\|x\|, \text{ for all } n, m \geq n_0. \tag{15}$$

This means that $(T_n x)_n$ is a Cauchy sequence in X. The fact that X is a non-Archimedean Banach space implies that there exists $y \in X$ such that $\|T_n x - y\| \longrightarrow 0$ as $n \to +\infty$. Setting $Tx = y$, where $T : X \longrightarrow X$ is a linear operator. Letting $m \to +\infty$ in (15), we infer that

$$\|T_n x - Tx\| \leq \varepsilon\,\|x\|, \text{ for all } n \geq n_0. \tag{16}$$

Now, we have

$$\begin{aligned} \|Tx\| &= \|Tx - T_n x + T_n x\| \\ &\leq \max\{\|Tx - T_n x\|, \|T_n x\|\}. \end{aligned}$$

Using the fact that $T_n \in \mathcal{L}(X)$ for all $n \geq n_0$, we deduce from (16) that

$$\begin{aligned} \|Tx\| &\leq \max\{\varepsilon\,\|x\|, \|T_n\|\,\|x\|\} \\ &\leq \max\{\varepsilon, \|T_n\|\}\,\|x\|. \end{aligned}$$

This shows that T is bounded. Moreover, $\|T_n - T\| < \varepsilon$, for all $n \geq n_0$. This is equivalent to saying that $\|T_n - T\| \longrightarrow 0$ as $n \to +\infty$. □

Remark 2.5. *Let X be a non-Archimedean Banach space. If $T \in \mathcal{L}(X)$, we defined a descent norm by*

$$\|T\|_0 = \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\|.$$

(*i*) *The descent norm $\|\cdot\|_0$ is equivalent to $\|\cdot\|$ but need not be identical with it* (*see* [16, Chapter 3]).

(*ii*) *If the valuation of $\mathbb{K}$ is dense, then these norms are always equivalent and equal* (*see* [19, Section 2]).

We illustrate Remark 2.5 with the following example:

Example 2.6. *Let $X = \mathbb{Q}_5$. Consider the linear operator*

$$I_{\mathbb{Q}_5} : (\mathbb{Q}_5, \|\cdot\|_5) \longrightarrow (\mathbb{Q}_5, |\cdot|_5)$$

where $\|x\|_5 := 2|x|_5$, for all $x \in \mathbb{Q}_5$. Then, the norms $\|\cdot\|_5$ and $|\cdot|_5$ are equivalent but not equal. Indeed, it follows from Remark 1.19 (*ii*) *that $|\cdot|_5$ is discrete. Moreover, we have*

$$\|I_{\mathbb{Q}_5}\| = \sup_{x \in \mathbb{Q}_5 \backslash \{0\}} \frac{|x|_5}{\|x\|_5} = \sup_{x \in \mathbb{Q}_5 \backslash \{0\}} \frac{|x|_5}{2|x|_5} = \frac{1}{2}.$$

Hence,

$$\|I_{\mathbb{Q}_5}\|_0 = \sup_{\substack{x \in \mathbb{Q}_5 \\ \|x\|_5 \leq 1}} |x|_5 = \sup_{\substack{x \in \mathbb{Q}_5 \\ 2|x|_5 \leq 1}} |x|_5 = \frac{1}{5}.$$

Hence, we infer that

$$\|I_{\mathbb{Q}_5}\|_0 \leq \|I_{\mathbb{Q}_5}\| \leq \alpha \|I_{\mathbb{Q}_5}\|_0, \textit{ for all } \alpha \geq \frac{5}{2}.$$

Thus, the norms $\|\cdot\|_5$ and $|\cdot|_5$ are equivalent but not equal.

Remark 2.7. *Let X be a non-Archimedean Banach space over $\mathbb{K}$.*
(i) *In the non-Archimedean theory, the set $\|X\| := \{\|x\| : x \in X\}$ may not be the same as $|\mathbb{K}| := \{|\alpha| : \alpha \in \mathbb{K}\}$ i.e., we can find $x \in X\backslash\{0\}$ and $\alpha \in \mathbb{K}\backslash\{0\}$ such that $\|x\| \neq |\alpha|$. As a consequence,*

$$\|\alpha^{-1}\, x\| = |\alpha^{-1}|\, \|x\| = |\alpha|^{-1}\, \|x\| \neq 1,$$

in fact, the set $\{x \in X : \|x\| = 1\}$ may well very be empty.
(ii) *Assume that $\|X\| \subseteq |\mathbb{K}|$. Let $T \in \mathcal{L}(X)$. Then, the operator norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent and equal. Indeed, let $x \in X\backslash\{0\}$. The fact that $\|X\| \subseteq |\mathbb{K}|$ implies that there exists $c \in \mathbb{K}\backslash\{0\}$ such that $|c| = \|x\|$. Setting $y = c^{-1}x$ we implies that $y \in X$ and $\|y\| = 1$. Hence,*

$$\|T\| = \sup_{\substack{x\in X\\ \|x\|=1}} \|Tx\| = \|T\|_0.$$

Remark 2.8. *Let $\mathbb{K}$ be a quadratically closed field (i.e., every element of $\mathbb{K}$ is a square). Then, we have*

$$\|\mathbb{E}_\omega\| \subseteq |\mathbb{K}|.$$

Indeed, the fact that $\mathbb{K}$ is a quadratically closed and $\omega_i \in \mathbb{K}$ for all $i \in \mathbb{N}$ implies that $\omega_i^{\frac{1}{2}} \in \mathbb{K}$ for all $i \in \mathbb{N}$. Then,

$$\begin{aligned} |\omega_i| &= |(\omega_i^{\frac{1}{2}})^2| \\ &= |\omega_i^{\frac{1}{2}} \times \omega_i^{\frac{1}{2}}| \\ &= |\omega_i^{\frac{1}{2}}|^2. \end{aligned}$$

Taking square root yields $|\omega_i^{\frac{1}{2}}| = |\omega_i|^{\frac{1}{2}}$. It follows from Remark 1.41 (iii) that $\|e_i\| = |\omega_i^{\frac{1}{2}}|$. This enables us to conclude that $\|\mathbb{E}_\omega\| \subseteq |\mathbb{K}|$.

Proposition 2.9. *Consider, for $T \in \mathcal{L}(X, Y)$, the decent norm $\|\cdot\|_0$ defined by*

$$\|T\|_0 = \sup_{\substack{x\in X\\ \|x\|\le 1}} \|Tx\|.$$

Then, $\|\cdot\|_0$ is a non-Archimedean norm equivalent to $\|\cdot\|$, i.e.,

$$\|T\|_0 \le \|T\| \le \frac{1}{\zeta}\|T\|_0, \tag{17}$$

where $\zeta = \sup\{|\lambda| : \lambda \in \mathbb{K}$ and $|\lambda| < 1\}$.

Proof. It is clear that $\|\cdot\|_0$ is a non-Archimedean norm on $\mathcal{L}(X, Y)$. On the one hand, since for all $x \in X$, we have

$$\|Tx\| \le \|T\|\|x\|,$$

then, we obtain

$$\|Tx\| \le \|T\|, \text{ for all } x \in X \text{ such that } \|x\| \le 1.$$

This implies that

$$\|T\|_0 \le \|T\|. \tag{18}$$

On the other hand, if $\lambda \in \mathbb{K}$ such that $0 < |\lambda| < 1$, then the fact that $\lim_{n\to+\infty} |\lambda|^n = 0$ and $\lim_{n\to-\infty} |\lambda|^n = +\infty$ implies that

$$\mathbb{R}_+^* = \bigcup_{n\in\mathbb{Z}} \left]|\lambda|^{n+1}, |\lambda|^n\right].$$

Hence, for $x \neq 0$, there exists $m \in \mathbb{Z}$ such that

$$|\lambda|^{m+1} < \|x\| \leq |\lambda|^m. \tag{19}$$

This is equivalent that $|\lambda| < \|\lambda^{-m}x\| \leq 1$. This implies from definition of the descent norm that

$$\|T(\lambda^{-m}x)\| \leq \|T\|_0. \tag{20}$$

Based on the assumption $\|T(\lambda^{-m}x)\| = \|\lambda^{-m}Tx\| = |\lambda|^{-m}\|Tx\|$, we infer from (20) that

$$\|Tx\| \leq |\lambda|^m \|T\|_0.$$

Thus, the use of (19) makes us conclude that

$$\|Tx\| \leq \frac{\|x\|}{|\lambda|} \|T\|_0, \text{ for all } x \neq 0.$$

This leads to deduce that

$$\|T\| \leq \frac{1}{|\lambda|} \|T\|_0, \text{ for all } \lambda \in \mathbb{K} \text{ and } 0 < |\lambda| < 1. \tag{21}$$

As a result, by using (18), we infer that (17) holds. □

Proposition 2.10. *Let $T \in \mathcal{L}(X, Y)$. Then, the norms $\| \cdot \|_0$ and $\| \cdot \|$ are equal if one of the following holds:*

(i) *The valuation of $\mathbb{K}$ is dense.*

(ii) $\|X\| \subseteq |\mathbb{K}|$

Proof. (i) If the valuation of $\mathbb{K}$ is dense, then there exists a sequence $(\lambda_n)_{n \geq 1}$ such that $|\lambda_n| < 1$ and $\lim_{n \to +\infty} |\lambda_n| = 1$. This implies that $\sup\{|\lambda_n| : (\lambda_n)_{n\geq 1} \subset \mathbb{K} \text{ and } |\lambda_n| < 1\} = 1$. This leads from (17) to

$$\|T\|_0 \leq \|T\| \leq \|T\|_0.$$

(ii) If $\|X\| \subseteq |\mathbb{K}|$, then there exists $c \in \mathbb{K}^*$ such that $\|x\| = |c|$. Then, setting $x_1 = c^{-1}x \in X$ one indeed has $\|x_1\| \leq 1$. This implies that

$$\|T\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\| = \|T\|_0.$$

As a result, $\| \cdot \|_0$ and $\| \cdot \|$ are equal, as desired. □

Definition 2.11. *Let X be a non-Archimedean Banach space. If $T \in \mathcal{L}(X)$, then*
(i) *T is said to be injective, if $N(T) = \{0\}$.*
(ii) *T is said to be surjective, if $R(T) = X$.*
(iii) *T is said to be invertible (or bijective), if it is both injective and surjective.*

Remark 2.12. *Let X be a non-Archimedean Banach space and let $T \in \mathcal{L}(X)$. If T is invertible, then its inverse T^{-1} is a bounded operator. This is the famous Banach inverse bounded operator theorem.*

Lemma 2.13. *Let X be a non-Archimedean Banach space and let $T \in \mathcal{L}(X)$.*
(i) *If $\|T\| < 1$, then $(I_X - T)$ is an isometric operator, $(I_X - T)^{-1} = I_X + \sum_{n=1}^{\infty} T^n$ and*

$$\|(I_X - T)^{-1}\| = 1.$$

(ii) *If $E \subseteq \mathcal{L}(X)$ such that $\{Tx : T \in E\}$ is a bounded set in X, for every $x \in X$. Then, E is a bounded set in $\mathcal{L}(X)$.*

Proof. (*i*) Let $n \in \mathbb{N}$. Since $T \in \mathcal{L}(X)$, then we have

$$\|T^n\| \leq \|T\|^n. \tag{22}$$

Based on the assumption $\|T\| < 1$, we infer that

$$\|T\|^n \longrightarrow 0 \text{ as } n \to +\infty. \tag{23}$$

This implies that the series $\sum_{k=0}^{\infty} T^k$ is absolutely convergent. Putting

$$S_n = \sum_{k=0}^{n} T^k,$$

we have

$$\begin{aligned} S_n(I_X - T) &= \sum_{k=0}^{n} T^k - \sum_{k=0}^{n} T^{k+1} \\ &= I_X - T^{n+1}, \end{aligned} \tag{24}$$

and

$$\begin{aligned} (I_X - T)S_n &= \sum_{k=0}^{n} T^k - \sum_{k=0}^{n} T^{k+1} \\ &= I_X - T^{n+1}. \end{aligned} \tag{25}$$

It follows from (23) and (22) that $T^{n+1} \longrightarrow 0$ as $n \to +\infty$. Letting $n \to +\infty$ in (24) and (25), we obtain

$$(I_X - T)\sum_{k=0}^{\infty} T^k = \sum_{k=0}^{\infty} T^k(I_X - T) = I_X.$$

On the other hand, one sees that

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} T^k \right\| &\leq \sup_{n\geq 1} \|T^n\|. \\ &= \max_{n\geq 1} \|T^n\| \\ &< 1. \end{aligned}$$

Therefore

$$\left\|(I_X - T)^{-1}\right\| = \left\| I_X + \sum_{n=1}^{\infty} T^n \right\|.$$

$$\begin{aligned} \left\|(I_X - T)^{-1}\right\| &= \left\| I_X + \sum_{n=1} T^n \right\| \\ &= 1. \end{aligned}$$

A consequence of these observations is that $I_X - T$ is an isometric operator.

(*ii*) Let $E \subset \mathcal{L}(X)$ be such that for every $x \in X$ the set $\{Tx : T \in E\}$ is bounded. For each positive integer n, let

$$F_n = \{x \in X : \text{ for all } T \in E, \|Tx\| \leq n\}. \tag{26}$$

Clearly, F_n is closed in X. Then, $X = \bigcup_n F_n$, one of F_n has nonempty interior. Hence, there exists $n \in \mathbb{N}$ and $\varepsilon > 0$ such that $B_X(y, \varepsilon) \subset F_n$. Moreover, each F_n is a group. Therefore, we obtain $B_X(0, \varepsilon) \subset F_n$. Let $c \in \mathbb{K}$ such that $0 < |c| < 1$. For every $x \in X$ there is an $m \in \mathbb{Z}$ for which

$$\varepsilon|c| \leq |c|^m\|x\| \leq \varepsilon. \tag{27}$$

This implies that $c^m x \in B_X(0, \varepsilon)$. By virtue of (27) and (26), we have

$$\begin{aligned} \|Tx\| &= \|T(c^{-m}c^m x)\| \\ &= |c|^{-m}\ \|T(c^m x)\| \\ &\leq n|c|^{-m} \\ &\leq \frac{n}{\varepsilon|c|}\ \|x\|. \end{aligned}$$

This implies that $E \subset \left\{T \in \mathcal{L}(X) : \|T\| \leq \frac{n}{\varepsilon|c|}\right\}$. This completes the proof. □

Remark 2.14. *Let X be a non-Archimedean Banach space. Then, $\mathcal{L}(X, \mathbb{K})$ which is called the dual of X and denoted X', is a non-Archimedean Banach space.* ◊

The following result may be found in [16]

Theorem 2.15. *Let $\mathbb{K}$ be spherically complete. Let T be a closed absolutely convex subset of a locally convex space X. For any non zero $x \in X$ and $x \notin T$, there exists $x^* \in X'$ such that $x^*(x) = 1$, $|x'(T)| < 1$.*

2.2. The adjoint operator on $\mathbb{E}_\omega$

Definition 2.16. *Let $T \in \mathcal{L}(\mathbb{E}_\omega)$. The linear operator S is called the adjoint of T if $\langle Tx, y\rangle = \langle x, Sy\rangle$, for all $x, y \in \mathbb{E}_\omega$, where $\langle\cdot,\cdot\rangle$ is the inner product of $\mathbb{E}_\omega$.*

Remark 2.17. *Let $T \in \mathcal{L}(\mathbb{E}_\omega)$. Then,*

(*i*) *In the classical Banach space, any bounded linear operator admit an adjoint, but in the non-Archimedean Banach space, it is not true.*

(*ii*) *The adjoint of an operator T is denoted by T^*. If T^* exists, then it is unique.*

(*iii*) *Let $(e_i)_i \in \mathbb{N}$ the canonical basis of $(\mathbb{E}_\omega)$. Then, T^* is an adjoint for T if, and only if, $\langle Te_i, e_j\rangle = \langle e_i, T^*e_j\rangle$, for all $i, j \in \mathbb{N}$.*

(*iv*) *If M is a subspace of $\mathbb{E}_\omega$, then $M^\perp = \{x \in \mathbb{E}_\omega : \langle x, y\rangle = 0, \text{ for all } y \in \mathbb{E}_\omega\}$.*

The collection of all bounded linear operators on $\mathbb{E}_\omega$ whose adjoint operators do exist is denoted by $\tilde{\mathcal{L}}(\mathbb{E}_\omega)$. Furthermore $\tilde{\mathcal{L}}(\mathbb{E}_\omega)$ is a closed unitary subalgebra of $\mathcal{L}(\mathbb{E}_\omega)$.

Proposition 2.18. [3, proposition 3.20] *If $T \in \tilde{\mathcal{L}}(\mathbb{E}_\omega)$ and for $\lambda \in \mathbb{K}$, then*

(*i*) $(\lambda + T)^* = \lambda + T^*$.

(*ii*) $\|T\| = \|T^*\|$.

Lemma 2.19. *If $T \in \tilde{\mathcal{L}}(\mathbb{E}_\omega)$, then $N(T^*) = R(T)^\perp$.*

Proof. Let us assume that $x \in N(T^*)$. Then, we obtain $T^*x = 0$. This leads to $\langle y, T^*x\rangle = 0$, for all $y \in \mathbb{E}_\omega$. Consequently, $\langle Ty, x\rangle = 0$, for all $y \in \mathbb{E}_\omega$, which yields $x \in R(T)^\perp$. It is show that $N(T^*) \subset R(T)^\perp$. Conversely, let $x \in R(T)^\perp$. Then, we infer that $\langle Ty, x\rangle = 0$ for all $y \in \mathbb{E}_\omega$. This implies that $\langle y, T^*x\rangle = 0$, for all $y \in \mathbb{E}_\omega$. Hence, we deduce that $x \in N(T^*)$. □

Definition 2.20. *Let X be a non-Archimedean Banach space and let $T \in \mathcal{L}(X)$.*
(i) *A sequence $(T_n)_n$ of bounded linear operators mapping on X is said to be norm convergent, denoted by $T_n \to T$, if $\|T_n - T\| \to 0$ as $n \to \infty$.*
(ii) *A sequence $(T_n)_n$ of bounded linear operators mapping on X is said to be pointwise convergent to T, denoted by $T_n \xrightarrow{p} T$, if $\|T_n x - Tx\| \to 0$ for every $x \in X$ as $n \to \infty$.* ◊

3. Non-Archimedean closed and closable linear operators

Since we are going to deal with graphs of linear operators, it will be necessary to consider the cartesian product $X \times Y$ which is a non-Archimedean normed space with the usual definition of addition and multiplication by scalars. The norm is defined by

$$\left\|(x, y)\right\| = \max\left\{\left\|x\right\|, \left\|y\right\|\right\} \text{ for all } x \in X \text{ and } y \in Y.$$

Definition 3.1. *Let X and Y be two non-Archimedean Banach spaces. An unbounded linear operator $T : \mathcal{D}(T) \subset X \longrightarrow Y$ is said to be closed if its graph*

$$\mathcal{G}(T) = \left\{(x, Tx) \in X \times Y : x \in \mathcal{D}(T)\right\},$$

as a subset of $X \times Y$, is closed.

The following proposition gives a characterization for closedness of unbounded linear operator T acting from X into Y.

Proposition 3.2. *Let $T : \mathcal{D}(T) \subset X \longrightarrow Y$ be a closed linear operator. If $(x_n) \subset \mathcal{D}(T)$ such that $\|x_n - x\| \to 0$ and $\|Tx_n - y\| \to 0$ as $n \to \infty$, for some $x \in X$ and $y \in Y$, then $x \in \mathcal{D}(T)$ and $y = Tx$.*

The collection of closed linear operators from X into Y is denoted by $\mathcal{C}(X, Y)$. When $X = Y$, this is simply denoted by $\mathcal{C}(X)$.

Remark 3.3. *Note that if $T \in \mathcal{L}(X, Y)$, then it is closed. Indeed, since T is bounded, then $\mathcal{D}(T) = X$. Moreover, if $(x_n) \subset X$ such that $x_n \to x$ on X as $n \to \infty$, then by the boundedness of T, we infer that $Tx_n \to Tx$ as $n \to \infty$. This implies that $(x_n, Tx_n) \to (x, Tx)$ on $X \times Y$ as $n \to \infty$. Therefore, $G(T)$ is closed.*

Example 3.4. *Let D be a linear operator on $\mathbb{E}_\omega$ defined by $De_j = \lambda_j e_j$ for all $j \in \mathbb{N}$ and whose domain is*

$$\mathcal{D}(D) = \left\{x = (x_j)_{j\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_{j\to+\infty} |\lambda_j||x_j|\|e_j\| = 0\right\}.$$

More precisely, if $x \in \mathcal{D}(D)$, one has $Dx = \sum_{j\in\mathbb{N}} \lambda_j x_j e_j$. Then, D is closed. Indeed, let $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(D)$ such that $x_n \to x$ and $Dx_n \to y$ as $n \to \infty$, for some $x, y \in \mathbb{E}_\omega$.
Write

$$x_n = \sum_{j\in\mathbb{N}} a_j^n e_j, \quad x = \sum_{j\in\mathbb{N}} a_j e_j \text{ and } y = \sum_{j\in\mathbb{N}} b_j e_j,$$

where $a_j^n, a_j, b_j \in \mathbb{K}$, for all $n, j \in \mathbb{N}$ and

$$\lim_{j\to+\infty} |a_j^n|\|e_j\| = \lim_{j\to+\infty} |a_j|\|e_j\| = \lim_{j\to+\infty} |b_j|\|e_j\| = 0. \tag{28}$$

The fact that $x_n \to x$ and $Dx_n \to y$ as $n \to \infty$ implies from [3, proposition 1.40] *that*

$$|a_j^n - a_j| \to 0 \ \text{ and } |\lambda_j a_j^n - b_j| \to 0, \ \text{ as } n \to \infty \ \text{ and for all } j \in \mathbb{N}.$$

This yields that $\lambda_j a_j = b_j$ for all $j \in \mathbb{N}$. It follows from (28) *that*

$$\lim_{j\to+\infty} |\lambda_j||a_j|\|e_j\| = \lim_{j\to+\infty} |\lambda_j a_j|\|e_j\| = \lim_{j\to+\infty} |b_j|\|e_j\| = 0.$$

Consequently, $x \in \mathcal{D}(D)$ and

$$Dx = \sum_{j\in\mathbb{N}} \lambda_j a_j e_j = \sum_{j\in\mathbb{N}} b_j e_j = y.$$

As a result, D is closed linear operator, as desired.

Definition 3.5. *Let T and S be two unbounded linear operators acting from X into Y such that $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $Tx = Sx$ for all $x \in \mathcal{D}(T)$, then S is called an extension of T.*

Definition 3.6. *An unbounded linear operator $T : \mathcal{D}(T) \subset X \longrightarrow Y$ is said to be closable, if it has a closed extension.*

Remark 3.7. *When T is closable, there is a closed operator $\overline{T}$ with $G(\overline{T}) = \overline{G(T)}$. It follows immediately that $\overline{T}$ is the smallest closed extension of T.*

Proposition 3.8. *Let X, Y and Z be non-Archimedean Banach spaces. Assume that $S \in \mathcal{L}(X, Y)$ and T is an unbounded linear operator.*

(i) *For $S + T$ to be closed it is necessary and sufficient that $T : X \longrightarrow Y$ is closed.*

(ii) *For $S + T$ to be closable it is necessary and sufficient that $T : X \longrightarrow Y$ is closable and $\overline{S+T} = S + \overline{T}$.*

(iii) *Let $T : Y \longrightarrow Z$ be a closed linear operator* (*respectively, closable*)*. Then, TS is closed* (*respectively, closable*).

(iv) *Let $T : X \longrightarrow Y$ be a closed linear operator* (*respectively, closable*) *and S be an invertible bounded operator such that $\mathcal{D}(S^{-1}) = Y$. Then, $S^{-1}T$ is closed* (*respectively, closable*).

Proof. (i) Let (x_n) be a sequence of $\mathcal{D}(T + S)$ such that

$$\begin{cases} x_n \to x, \\ (T+S)x_n \to y. \end{cases}$$

Since $\mathcal{D}(S) = X$, then we have $\mathcal{D}(T + S) = \mathcal{D}(T) \cap \mathcal{D}(S) = \mathcal{D}(T)$. Based on the assumptions $x_n \to x$ and S is a bounded linear operator, we get $Sx_n \to Sx$. Hence, by writing, $Tx_n = (T + S)x_n - Sx_n$ and using the fact that T is a closed operator, we deduce that

$$\begin{cases} x \in \mathcal{D}(T), \\ y - Sx = Tx. \end{cases}$$

(ii) If T is a closable operator, then $T + S$ admits the closed extension $S + \overline{T}$. It follows from (i) that $S + T$ is closable and

$$\overline{S+T} \subset \overline{S+\overline{T}} = S + \overline{T}. \tag{29}$$

By replacing in (29) T by $T + S$ and S by $-S$, we infer that $\overline{T} \subset \overline{T+S} - S$. So

$$S + \overline{T} \subset S + \overline{S+T} - S = \overline{S+T}.$$

As a result, $\overline{S+T} = S + \overline{T}$, as desired.

(*iii*) We have $\mathcal{D}(TS) = \{x \in \mathcal{D}(S) : Sx \in \mathcal{D}(T)\}$. Let (x_n) be a sequence of $\mathcal{D}(TS)$ such that

$$\begin{cases} x_n \to x \\ (TS)x_n \to y \end{cases}.$$

Let us show that $x \in \mathcal{D}(TS)$ and $y = (TS)x$. Since $S \in \mathcal{L}(X, Y)$, then we have $Sx_n \to Sx$. Hence, the fact that $T(Sx_n) \to y$ and T is a closed operator implies that

$$\begin{cases} Sx \in \mathcal{D}(T), \\ y = TSx. \end{cases}$$

Therefore, TS is closed. Moreover, if S is closable, then ST admits the closed extension $\overline{S}T$.

(*iv*) Let (x_n) be a sequence of $\mathcal{D}(S^{-1}T)$ such that:

$$\begin{cases} x_n \to x \\ (S^{-1}T)x_n \to y. \end{cases}$$

Our purpose is to show that $x \in \mathcal{D}(S^{-1}T)$ and $y = S^{-1}Tx$. As we have $(S^{-1}T)x_n \to y$ and S is a linear bounded operator, so $Tx_n \to Sy$. Since T is a closed operator, then one obtains

$$\begin{cases} x \in \mathcal{D}(T), \\ Tx = Sy. \end{cases}$$

This implies that

$$\begin{cases} x \in \mathcal{D}(T), \\ Tx \in \mathcal{D}(S^{-1}), \\ y = (S^{-1}T)x. \end{cases}$$

In addition, if T is closable, $S^{-1}T$ admits the closed extension $S^{-1}\overline{T}$. □

For more details related to the results of non-Archimedean linear operators, we may refer to [2, 3].

4. Relative Boundedness in non-Archimedean Banach spaces

Let us give the definition of the gap between two non-Archimedean closed linear operators. We proceed in the same spirit as in Tosio Kato's book [12], but extend the definition to closed linear operators in a non-Archimedean Banach while T. Kato in his monograph [12] limits himself to the case of closed operators in a classical Banach space.

Definition 4.1. *Let X and Y be two non-Archimedean Banach spaces such that $\|X \times Y\| \subseteq |\mathbb{K}|$. Let $T, S \in \mathcal{C}(X, Y)$. Then, we define*

$$\delta(T, S) = \delta\Big(\mathcal{G}(T), \mathcal{G}(S)\Big) \text{ and } \widehat{\delta}(T, S) = \widehat{\delta}\Big(\mathcal{G}(T), \mathcal{G}(S)\Big).$$

More explicitly,

$$\delta(T, S) = \sup_{\substack{x \in \mathcal{D}(T) \\ \|(x, Tx)\| = 1}} \Big[\inf_{y \in D(S)} \Big(\max\Big\{\|x - y\|, \|Tx - Sy\|\Big\}\Big)\Big].$$

At this level, we shall introduce a new concept of $\mathbb{K}$-relatively bounded linear operators on non-Archimedean Banach spaces.

Definition 4.2. *Let X, Y and Z be three non-Archimedean Banach spaces and T, S be linear operators from X to Y and from X to Z, respectively. Then, S is called T-$\mathbb{K}$-bounded (or $\mathbb{K}$-relatively bounded with respect to T) if, $\mathcal{D}(T) \subset \mathcal{D}(S)$ and there exist non-negative constants a_S and b_S, such that*

$$\|Sx\| \leq \max\left\{a_S\|x\|; b_S\|Tx\|\right\}, \text{ for all } x \in \mathcal{D}(T). \tag{30}$$

In that case, the infimum β_S of the constant b_S which satisfies (30) is called the T-$\mathbb{K}$-bound of S (or $\mathbb{K}$-relative bound with respect to T).

Remark 4.3. (*i*) *The concept of relative boundedness over a Banach space can be found in the literature (see [12]).*

(*ii*) *Obviously, if S is T-$\mathbb{K}$-bounded, then S is T-bounded i.e.,*

$$\|Sx\| \leq a_S\|x\| + b_S\|Tx\|, \text{ for all } x \in \mathcal{D}(T). \qquad \diamond$$

Remark 4.4. (*i*) *A bounded operator S is T-$\mathbb{K}$-bounded for any T with $\mathcal{D}(T) \subset \mathcal{D}(S)$ and T-$\mathbb{K}$-bound equal to zero.*

(*ii*) *If $\max\left\{a_S\|x\|; b_S\|Tx\|\right\} = a_S\|x\|$, for all $x \in \mathcal{D}(T)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$, then S is a bounded operator on $\mathcal{D}(T)$.*

Proposition 4.5. *Let X and Y be non-Archimedean Banach spaces and K, S, T, V be linear operators from X to Y.*

(*i*) *If S is T-$\mathbb{K}$-bounded with T-$\mathbb{K}$-bound β_1 and T is V-$\mathbb{K}$-bounded with V-$\mathbb{K}$-bound β_2, then S is V-$\mathbb{K}$-bounded with V-$\mathbb{K}$-bound $\beta_1\beta_2$.*

(*ii*) *If S is T-$\mathbb{K}$-bounded with T-$\mathbb{K}$-bound β_1 and K is T-$\mathbb{K}$-bounded with T-$\mathbb{K}$-bound β_3, then $S \pm K$ is T-$\mathbb{K}$-bounded with T-$\mathbb{K}$-bound $\max\{\beta_1; \beta_3\}$.*

(*iii*) *If S is T-$\mathbb{K}$-bounded with T-$\mathbb{K}$-bound $\beta_1 < 1$, then S is $T + S$-$\mathbb{K}$-bounded with $T + S$-$\mathbb{K}$-bound less than $\dfrac{\beta_1}{1-\beta_1}$.*

Proof. (*i*) Since S is T-$\mathbb{K}$-bounded and T is V-$\mathbb{K}$-bounded, then there exist positive constants a_S, a_T, b_S and b_T such that $\mathcal{D}(V) \subset \mathcal{D}(T) \subset \mathcal{D}(S)$,

$$\|Sx\| \leq \max\left\{a_S\|x\|; b_S\|Tx\|\right\} \text{ for } x \in \mathcal{D}(T), \tag{31}$$

and β_1 the infmum of the constant b_S which satisfies (31).

$$\|Tx\| \leq \max\left\{a_T\|x\|; b_T\|Vx\|\right\} \text{ for } x \in \mathcal{D}(V), \tag{32}$$

and β_2 the infmum of the constant b_t which satisfies (32). Combining (31) and (32), we infer for $x \in \mathcal{D}(V)$ that

$$\begin{aligned} \|Sx\| &\leq \max\left\{a_S\|x\|; b_S \max\left\{a_T\|x\|; b_T\|Vx\|\right\}\right\} \\ &\leq \max\left\{\max\left\{a_S, b_S a_T\right\}\|x\|; b_S b_T\|Vx\|\right\}. \end{aligned}$$

Thus, S is V-$\mathbb{K}$-bounded with V-$\mathbb{K}$-bound $\beta_1\beta_2$.

(*ii*) The fact that S is T-$\mathbb{K}$-bounded and K is T-$\mathbb{K}$-bounded implies that there exist positive constants a_S, a_K, b_S and b_K such that $\mathcal{D}(T) \subset \mathcal{D}(S) \cap \mathcal{D}(K)$

$$\|Sx\| \leq \max\left\{a_S\|x\|; b_S\|Tx\|\right\} \text{ for } x \in \mathcal{D}(T) \text{ and} \tag{33}$$

$$\|Kx\| \leq \max\left\{a_K\|x\|; b_K\|Tx\|\right\} \text{ for } x \in \mathcal{D}(T). \tag{34}$$

and β_3 the infmum of the constant b_t which satisfies (34). On the one hand, we have $\mathcal{D}(T) \subset \mathcal{D}(S) \cap \mathcal{D}(K) = \mathcal{D}(S \pm K)$. On the other hand, by using (33) and (34), we conclude for $x \in \mathcal{D}(T)$ that

$$\begin{aligned} \|(S \pm K)x\| &\leq \max\Big\{\|Sx\|; \|Kx\|\Big\} \\ &\leq \max\Big\{a_S\|x\|; a_K\|x\|; b_S\|Tx\|; b_K\|Tx\|\Big\} \\ &\leq \max\Big\{\max\Big\{a_S; a_K\Big\}\|x\|; \max\Big\{b_S; b_K\Big\}\|Tx\|\Big\}. \end{aligned}$$

Therefore, $S \pm K$ is T-$\mathbb{K}$-bounded with T-$\mathbb{K}$-bound $\max\{\beta_1; \beta_3\}$.

(*iii*) Let us assume that $\mathcal{D}(T) \subset \mathcal{D}(S)$ and there exist non-negative constants a_S and b_S, such that for all $x \in \mathcal{D}(T)$, we have

$$\begin{aligned} \|Sx\| &\leq \max\Big\{a_S\|x\|; b_S\|Tx\|\Big\} \\ &\leq \max\Big\{a_S\|x\|; b_S\|(T+S)x\|; b_S\|Sx\|\Big\} \\ &\leq \max\Big\{\max\{a_S\|x\|; b_S\|(T+S)x\|\}; b_S\|Sx\|\Big\} \\ &\leq \max\Big\{a_S\|x\|; b_S\|(T+S)x\|\Big\} + b_S\|Sx\|. \end{aligned}$$

This implies that

$$\|Sx\| \leq (1 - b_S)^{-1} \max\Big\{a_S\|x\|; b_S\|(T+S)x\|\Big\}.$$

This completes the proof □

Remark 4.6. *Let X, Y and Z be non-Archimedean Banach spaces. Let* $S : \mathcal{D}(S) \subseteq X \longrightarrow Z$ *be a linear operator and* $T \in \mathcal{C}(X, Y)$ *such that* $\mathcal{D}(T) \subset \mathcal{D}(S)$. *By the same reasoning as* [12, Remark 1.4, p. 191], *we can conclude that* $(\mathcal{D}(T), \|\cdot\|_{\mathcal{D}(T)})$, *with the graph norm*

$$\|x\|_{\mathcal{D}(T)} = \max\Big\{\|x\|; \|Tx\|\Big\}, \text{ for all } x \in \mathcal{D}(T),$$

is a non-Archimedean Banach space. Obviously, S is T-$\mathbb{K}$*-bounded if, and only if, S is a bounded operator from* $(\mathcal{D}(T), \|\cdot\|_{\mathcal{D}(T)})$.

Theorem 4.7. *Let X, Y and Z be non-Archimedean Banach spaces. Let* $T : \mathcal{D}(T) \subseteq X \longrightarrow Y$ *and* $S : \mathcal{D}(S) \subseteq X \longrightarrow Z$ *be two linear operators with* $\mathcal{D}(T) \subset \mathcal{D}(S)$. *If T is closed and S is closable, then S is T-*$\mathbb{K}$*-bounded.*

Proof. Suppose that S_0 be a restriction of S to $\mathcal{D}(T)$, that is, $S_0 : (\mathcal{D}(T), \|\cdot\|_{\mathcal{D}(T)}) \longrightarrow Z$ defined by $S_0u = Su$ for $u \in \mathcal{D}(T)$. In order to show that S is T-$\mathbb{K}$-bounded, it suffices to show that S_0 is bounded. By referring to [16, Theorem 3.5], it is enough to show that S_0 is closed. Fix any sequence $(x_n) \subset \mathcal{D}(S_0)$ with $x_n \to x$ in norm $\|\cdot\|_{\mathcal{D}(T)}$ and $S_0x_n \to y$ in the norm of Z as $n \to \infty$. Using the fact T is closed, we infer that $x \in \mathcal{D}(T)$ and $Tx_n \to Tx$.

Noting that

$$\|x_n - x\|_{\mathcal{D}(T)} = \max\Big\{\|x_n - x\|; \|Tx_n - Tx\|\Big\} \text{ for } n \geq 0,$$

we get that $x_n \to x$ in the norm of X. In addition, we have $S_0x_n = Sx_n$ for $(x_n) \subset \mathcal{D}(T)$. By the assumptions that S is closable and $x \in \mathcal{D}(T) \subset \mathcal{D}(S)$, we conclude that

$$y = \overline{S}x = Sx = S_0x.$$

Thus, S_0 is closed. This completes the proof. □

Remark 4.8. *Let X, Y be two non-Archimedean Banach spaces and T, S be two linear operator from X into Y. Assume that T is closable operator. If S is T-$\mathbb{K}$-bounded, then we get $\widetilde{S}$ the extension of the operator S to the domain $\mathcal{D}(\overline{T}) \cup \mathcal{D}(S)$. Indeed, let $(x_n) \subset \mathcal{D}(T)$ such that $x_n \to x \in \mathcal{D}(\overline{T})$ and $Tx_n \to y$. This implies from (30) that (Sx_n) is a Cauchy sequence in the non-Archimedean Banach space Y. Then, there exists $z \in Y$ such that $Sx_n \to z$. This z is uniquely determined, which is a consequence of the closability of the operator T. By setting $Sx = z$, we get an extension of S to the domain $\mathcal{D}(\overline{T}) \cup \mathcal{D}(S)$.*

In the following result, we shall study stability of closedness and closability for linear operators under $\mathbb{K}$-relatively bounded perturbation. This result is useful in the perturbation theory, which was studied by T. Kato in the classical Banach spaces (see [12]).

Definition 4.9. *Let X, Y be two non-Archimedean Banach spaces and let T be a linear operator from X into Y. A sequence $(x_n) \in \mathcal{D}(T)$ is said to be T-convergent to $x \in X$ if both (x_n) and (Tx_n) are Cauchy sequences and $x_n \to x$.*

Theorem 4.10. *Let X and Y be two non-Archimedean Banach spaces. Let T, S be two linear operators from X to Y. If $\mathcal{D}(T) \subset \mathcal{D}(S)$ and there exist positive constants a_S and b_S such that $b_S < 1$ and we have*

$$\|Sx\| \leq \max\left\{a_S\|x\|; b_S\|Tx\|\right\}, \text{ for all } x \in \mathcal{D}(T), \tag{35}$$

then, $S + T$ is closable if, and only if, T is closable. In this case, the closure of T and $T + S$ have the same domain.

Proof. On the one hand, for all $x \in \mathcal{D}(T + S) = \mathcal{D}(S) \cap \mathcal{D}(T) = \mathcal{D}(T)$ we have

$$\begin{aligned} \|(S+T)x\| &\leq \max\left\{\|Sx\|; \|Tx\|\right\} \\ &\leq \max\left\{\max\{a_S\|x\|; b_S\|Tx\|\}; \|Tx\|\right\} \\ &\leq \max\left\{a_S\|x\|; \max\{1; b_S\}\|Tx\|\right\} \\ &\leq \max\left\{a_S\|x\|; \|Tx\|\right\}. \end{aligned} \tag{36}$$

On the other hand, we can write

$$\begin{aligned} \|Tx\| &= \|Tx + Sx - Sx\| \\ &\leq \max\left\{\|(T+S)x\|; \|Sx\|\right\} \\ &\leq \|(T+S)x\| + \|Sx\| \end{aligned}$$

Hence, for all $x \in \mathcal{D}(T)$ we have

$$\|Tx\| - \|Sx\| \leq \|(T+S)x\|.$$

In addition, it follows from (35) that

$$\begin{aligned} \|Tx\| - \|Sx\| &\geq \|Tx\| - \max\left\{a_S\|x\|; b_S\|Tx\|\right\} \\ &\geq \|Tx\| - \left(a_S\|x\| + b_S\|Tx\|\right) \\ &\geq -a_S\|x\| + (1 - b_S)\|Tx\|. \end{aligned} \tag{37}$$

Combining (36) and (37), we deduce that for any $x \in \mathcal{D}(T)$

$$-a_S\|x\| + (1 - b_S)\|Tx\| \leq \|(T+S)x\| \leq \max\left\{a_S\|x\|; \|Tx\|\right\}. \tag{38}$$

Let (x_n) be a sequence in $\mathcal{D}(T+S) = \mathcal{D}(T)$ such that $x_n \to 0$ and $(T+S)x_n \to y$. By applying the first inequality in the left hand side of (38), we infer that (Tx_n) is a Cauchy sequence in the non-Archimedean Banach space Y. Therefore, there exists $y_0 \in Y$ such that $Tx_n \to y_0$. Since T is closable, then $y_0 = 0$. It follows from the second inequality in the right hand side of (38) that $(T+S)x_n \to 0$, which yields that $y = 0$. This implies that $T+S$ is closable. Similarly, T is closable if $T+S$ is also closable.

Let $\overline{T}$ and $\overline{T+S}$ be the closures of T and $T+S$, respectively. Our purpose is to show that $\mathcal{D}(\overline{T}) = \mathcal{D}(\overline{T+S})$. Let $x \in \mathcal{D}(\overline{T+S})$, then there exists a sequence $(x_n) \in \mathcal{D}(T+S)$ such that $x_n \to x$ and $(T+S)x_n \to (\overline{T+S})x$. This implies that (x_n) is $(T+S)$-convergent to x. By using the first inequality of (38), we deduce that (x_n) is T-convergent to x. This implies that $x \in \mathcal{D}(\overline{T})$, so that $\mathcal{D}(\overline{T+S}) \subset \mathcal{D}(\overline{T})$. Similarly, the opposite inclusion is proved. □

The following result is a direct consequence of theorem 4.10.

Corollary 4.11. *Let X and Y be two non-Archimedean Banach spaces. Let T, S be two linear operators from X to Y and S be T-$\mathbb{K}$-bounded with T-$\mathbb{K}$-bounded smaller than* 1. *Then,*

$$S+T \text{ is closed if, and only if, } T \text{ is closed.}$$

Remark 4.12. (*i*) *Let us assume that the hypotheses of theorem 4.10 are satisfied. If S is bounded with $\mathcal{D}(T) \subset \mathcal{D}(S)$, then*

$$\widehat{\delta}(T, T+S) \le \|S\|. \tag{39}$$

Indeed, if S is bounded with $\mathcal{D}(T) \subset \mathcal{D}(S)$, then it follows from Remark 4.4 (*i*) *that S is T-$\mathbb{K}$-bounded with $a_S = \|S\|$ and $b_S = 0$ in* (30). *Hence,* (39) *holds.*

(*ii*) *The results of theorem 4.10 extend those of* [12, theorems IV.1.1 and IV.2.14] *for linear operators on Banach spaces to linear operators on non-Archimedean Banach spaces.*

The following result is obtained from Theorem 4.10.

Theorem 4.13. *Let X and Y be two non-Archimedean Banach spaces such that $\|X \times Y\| \subseteq |\mathbb{K}|$. Let (S_n) be a sequence of linear operators from X into Y and let T be a closable linear operator from X into Y. Let (S_n) be T-$\mathbb{K}$-bounded and satisfy that*

$$\|S_n x\| \le \max\left\{a_n\|x\|; b_n\|Tx\|\right\}, \text{ for every } x \in \mathcal{D}(T) \text{ and for each } n \ge 1.$$

If $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$, then $T_n = T + S_n$ are closable for sufficiently large n and $T_n \xrightarrow{g} T$.

Lemma 4.14. *Let X and Y be two non-Archimedean Banach spaces and let S, T, and K be three operators from X into Y satisfying $\mathcal{D}(T) \subset \mathcal{D}(S) \subset \mathcal{D}(K)$. If*

(*i*) *there exist two constants $a, b > 0$ such that*

$$\|Sx\| \le \max\left(a\|x\|, b\|Tx\|\right) \text{ for all } x \in \mathcal{D}(T),$$

(*ii*) *there exist two constants $e, d > 0$ such that $\mu = \max(b, bd) < 1$ and*

$$\|Kx\| \le \max\left(e\|x\|, d\|Sx\|\right) \text{ for all } x \in \mathcal{D}(S).$$

Then,

$$\|Tx\| \le (1-\mu)^{-1} \max\left(\|(S+T+K)x\|, \nu\|x\|\right),$$

where $\nu = \max(a, ad, e) > 0$.

Proof. Let $x \in \mathcal{D}(S+T+K) = \mathcal{D}(T)$. We have

$$\begin{aligned} \|(S+T+K)x\| &\le \max\Big(\|Sx\|, \|Tx\|, \|Kx\|\Big) \\ &\le \max\Big(\max(a\|x\|, b\|Tx\|), \|Tx\|, \max(e\|x\|, d\|Sx\|)\Big) \\ &\le \max\Big(\max(a, e, da)\|x\|, \max(1, b, bd)\|Tx\|\Big). \end{aligned} \tag{40}$$

Similarly, we have for all $x \in \mathcal{D}(T)$

$$\begin{aligned} \|(S+K)x\| &\le \max\Big(\|Sx\|, \|Kx\|\Big) \\ &\le \max\Big(\max(a\|x\|, b\|Tx\|), \max(e\|x\|, d\|Sx\|)\Big) \\ &\le \max\Big(\max(a, e, da)\|x\|, \max(b, bd)\|Tx\|\Big). \end{aligned} \tag{41}$$

Since,

$$\begin{aligned} \|Tx\| &= \|(S+T+K)x - (S+K)x\| \\ &\le \max\Big(\|(S+T+K)x\|, \|(S+K)x\|\Big) \\ &\le \max\Big(\|(S+T+K)x\|, \max\Big(\max(a, e, da)\|x\|, \max(b, bd)\|Tx\|\Big)\Big) \\ &= \max\Big(\|(S+T+K)x\|, \max(a, e, da)\|x\|, \max(b, bd)\|Tx\|\Big) \\ &\le \max\Big(\|(S+T+K)x\|, \max(a, e, da)\|x\|\Big) + \max(b, bd)\|Tx\|, \end{aligned}$$

which yields $(1 - \max(b, bd))\|Tx\| \le \max\Big(\|(S+T+K)x\|, \max(a, e, da)\|x\|\Big)$. So, we deduce that

$$\|Tx\| \le (1-\mu)^{-1} \max\Big(\|(S+T+K)x\|, \nu\|x\|\Big),$$

where $\mu = \max(b, bd)$ and $\nu = \max(a, ad, e)$. This completes the proof. □

Theorem 4.15. *Let X and Y be two non-Archimedean Banach spaces and let S, T, and K be three operators from X into Y satisfying $\mathcal{D}(T) \subset \mathcal{D}(S) \subset \mathcal{D}(K)$. If*
(i) there exist two constants $a, b > 0$ such that

$$\|Sx\| \le \max\Big(a\|x\|, b\|Tx\|\Big) \text{ for all } x \in \mathcal{D}(T),$$

(ii) there exist two constants $e, d > 0$ such that $\mu = \max(b, bd) < 1$ and

$$\|Kx\| \le \max\Big(e\|x\|, d\|Sx\|\Big) \text{ for all } x \in \mathcal{D}(S).$$

Then,

$$S+T+K \text{ is closable if and only if } T \text{ is closable.}$$

Further, if $\|X \times Y\| \subset |\mathbb{K}|$, then $\mathcal{D}(\overline{S+T+K}) = \mathcal{D}(\overline{T})$ and

$$\hat{\delta}(S+T+K, T) \le (1-\mu)^{-1} \max(\mu, \nu), \tag{42}$$

where $\nu = \max(a, ad, e)$ $(\nu > 0)$ and $\hat{\delta}(.,.)$ is the gap between two operators.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(S+T+K)=\mathcal{D}(T)$ such that $x_n \longrightarrow 0$. and $(S+T+K)x_n \longrightarrow \chi$. By using lemma 4.14, it follows that $(Tx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space Y and therefore, there exists $\Psi_1 \in Y$ such that $Tx_n \longrightarrow \Psi_1$. Since T is closable, then $\Psi_1 = 0$. It follows from (40) that $(S+T+K)x_n \longrightarrow 0$. Then, $\chi = 0$ and $S+T+K$ is closable. Similarly, T is closable if $S+T+K$ is also closable.
In order to prove that

$$\mathcal{D}(\overline{S+T+K}) = \mathcal{D}(\overline{T}).$$

Suppose that $x \in \mathcal{D}(\overline{S+T+K})$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $(x_n)_{n\in\mathbb{N}}$ is $(S+T+K)-$*convergent* to x. From lemma 4.14, we deduce that $(x_n)_{n\in\mathbb{N}}$ is $T-$ *convergent* to x. Hence, $x \in \mathcal{D}(\overline{T})$. As a result, we get $\mathcal{D}(\overline{S+T+K}) \subset \mathcal{D}(\overline{T})$. The opposite inclusion can be checked in the same way.
Now, let $\varphi = (u, (\overline{S+T+K})u) \in G(\overline{S+T+K})$ with $\|\varphi\| = 1$, it follows that

$$\max\Big(u, (\overline{S+T+K})u\Big) = \|\varphi\| = 1. \tag{43}$$

Since, $G(\overline{S+T+K}) = \overline{G(S+T+K)}$, so there exists a sequence $\varphi_n \subset G(S+T+K)$ such that $\varphi_n \longrightarrow \varphi$. Then, $\varphi_n = (u_n, (\overline{S+T+K})u_n)$ with $u_n \in \mathcal{D}(S+T+K)$, $u_n \longrightarrow u$ and $(S+T+K)u_n \longrightarrow (\overline{S+T+K})u$. Next, let $\psi_n = (u_n, Tu_n) \in G(T)$ By using lemma 4.14, we deduce (u_n) is $T-$ *convergent* to u. Then,

$$\psi_n \longrightarrow \psi_0 \in G(\overline{T}).$$

Based on lemma 4.14, we get

$$\begin{aligned}\|\varphi_n - \psi_n\| &= \|(S+K)u_n\| \\ &\le (1-\mu)^{-1}\max\Big(\mu\|(S+T+K)u_n\|, \nu\|u_n\|\Big).\end{aligned}$$

We obtain when $n \longrightarrow \infty$

$$\|\varphi - \psi_0\| \le (1-\mu)^{-1}\max\Big(\mu\|(\overline{S+T+K})u\|, \nu\|u\|\Big).$$

Using Eq. (43), we infer that

$$\|\varphi - \psi_0\| \le (1-\mu)^{-1}\max\Big(\mu, \nu\Big).$$

Hence,

$$\mathrm{dist}(\varphi, G(\overline{T}) \le (1-\mu)^{-1}\max\Big(\mu, \nu\Big).$$

The fact that φ is arbitrary in the unit sphere of $G(\overline{S+T+K})$, then

$$\delta(S+T+K, T) \le (1-\mu)^{-1}\max\Big(\mu, \nu\Big).$$

Similarly, we can estimate $\delta(T, S+T+K)$. As a consequence, we get (42). This completes the proof. □

Remark 4.16. *If the hypotheses of theorem 4.15 are satisfied, then*

$$S+T+K \text{ is closed if, and only if, } T \text{ is closed.}$$

References

[1] T. Diagana, Towards a theory of some unbounded linear operators on p-adic Hilbert spaces and applications, Ann. Math. Blaise Pascal 12, no. 1, 205-222, (2005).
[2] T. Diagana, Non-Archimedean linear operators and applications, Nova Science Publishers, Inc., Huntington, NY, xiv+92 pp. ISBN: 978-1-60021-405-9, 1-60021-405-3, (2007).
[3] T. Diagana and F. Ramaroson, Non-Archimedean operator theory. SpringerBriefs in Mathematics. Springer, Cham, xiii+156 pp. ISBN: 978-3-319-27322-8, 978-3-319-27323-5, (2016).
[4] B. Diarra, An operator on some ultrametric Hilbert spaces. J. Anal. 6, 55-74, (1998).

[5] B. Diarra, Bounded linear operators on ultrametric Hilbert spaces. Afr. Diaspora J. Math. (N.S.) 8, no. 2, 173-181, (2009).
[6] B. Dwork, On the rationality of the zeta function of an algebraic variety. Amer. J. Math. 82, 631-648, (1960).
[7] L. Gruson, Categories d'espaces de Banach ultrametriques. (French) Bull. Soc. Math. France 94, 287-299, (1966).
[8] L. Gruson, Theorie de Fredholm p-adique. (French) Bull. Soc. Math. France 94, 67-95, (1966).
[9] F. Q. Gouvra, p-adic numbers, An introduction. Second edition. Universitext. Springer-Verlag, Berlin, vi+298 pp. ISBN: 3-540-62911-4, (1997).
[10] R. Janz, Holomorphic families of subspaces of a Banach Space. Special classes of linear operators and other topics (Bucharest, 1986), 155-167, Oper. Theory Adv. Appl., 28, Birkhauser, Basel, (1988).
[11] K. Hensel, Eber eine neue Begrundung der Theorie der algebraischen Zahlen. (German) J. Reine Angew. Math. 128 , 1-32, (1905).
[12] T. Kato, Perturbation theory for linear operators, Second ed., in: Grundlehren der Mathematischen Wissenschaften, Band 132, Springer-Verlag, Berlin-Heidelberg-New York, (1976).
[13] S. Ludkovsky and B. Diarra Spectral integration and spectral theory for non-Archimedean Banach spaces. Int. J. Math. Math. Sci. 31 (2002), no. 7, 421-442.
[14] A. F. Monna, On non-Archimedean linear spaces. (Dutch) Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 308-321, (1943).
[15] A. F. Monna, Analyse non-Archimedienne. (French) Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 56. Springer-Verlag, Berlin-New York, vii+119 pp, (1970).
[16] A. C. M. Van Rooij, Non-Archimedean functional analysis. Monographs and Textbooks in Pure and Applied Math., 51. Marcel Dekker, Inc., New York, x+404 pp. ISBN: 0-8247-6556-7, (1978).
[17] W. H. Schikhof, Ultrametric calculus. An introduction to p-adic analysis. Cambridge Studies in Advanced Mathematics, 4. Cambridge University Press, Cambridge viii+306 pp. ISBN: 0-521-24234-7, 1984.
[18] W. H. Schikhof, Banach spaces over nonarchimedean valued fields. Proceedings of the 14th Summer Conference on General Topology and its Applications (Brookville, NY, 1999). Topology Proc. 24 (1999), Summer, 547-581 (2001).
[19] J. P. Serre, Endomorphismes completement continus des espaces de Banach p-adiques. (French) Inst. Hautes Etudes Sci. Publ. Math. No. 12, (1962), 69-85.
[20] A. Weil, Basic number theory. Reprint of the second (1973) edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. xviii+315 pp. ISBN: 3-540-58655-5.