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The nonlinear mixed bi-skew Lie triple derivations on $*$-algebras

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Abstract. Let $\mathcal{A}$ be a unital $*$-algebra. In this paper, under some mild conditions on $\mathcal{A}$, it is shown that a map $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear mixed bi-skew Lie triple derivation if and only if Φ is an additive $*$-derivation. As applications, nonlinear mixed bi-skew Lie triple derivations on prime $*$-algebras, von Neumann algebras with no central summands of type I_1, factor von Neumann algebras and standard operator algebras are characterized.

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. For $A, B \in \mathcal{A}$, define the bi-skew Jordan product of A and B by $A \circ B = A^*B + B^*A$ and the bi-skew Lie product of A and B by $[A, B]_\diamond = A^*B - B^*A$. The bi-skew Jordan product and bi-skew Lie product have attracted many scholars to study (see for example [2–6, 10, 14–17]). Particular attention has been paid to understand maps which preserve the bi-skew Jordan product and the bi-skew Lie product on C*-algebras. M. Wang and G. Ji [15] proved that every bijective map preserving bi-skew Lie product between factor von Neumann algebras is a linear $*$-isomorphism or a conjugate linear $*$-isomorphism. C. Li et al. [10] proved that every bijective map preserving bi-skew Jordan product between von Neumann algebras with no central abelian projections is just the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism. A. Taghavi and S. Gholampoor [14] studied surjective maps preserving bi-skew Jordan product between C^*-algebras.

Recall that an additive map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be an additive derivation if $\Phi(AB) = \Phi(A)B + A\Phi(B)$ holds for all $A, B \in \mathcal{A}$. Furthermore, Φ is said to be an additive $*$-derivation if it is an additive derivation and satisfies $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{A}$. We say that $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear bi-skew Lie derivation or bi-skew Jordan derivation if

$$\Phi([A, B]_\diamond) = [\Phi(A), B]_\diamond + [A, \Phi(B)]_\diamond$$

or

$$\Phi(A \circ B) = \Phi(A) \circ B + A \circ \Phi(B)$$

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hold for all $A, B \in \mathcal{A}$. Recently, many authors have studied nonlinear bi-skew Lie derivations and bi-skew Jordan derivations. For example, L. Kong and J. Zhang [6] proved that any nonlinear bi-skew Lie derivation on factor von Neumann algebra $\mathcal{A}$ with $\dim\mathcal{A} \geq 2$ is an additive $*$-derivation. A. Taghavi and M. Razeghi [15] investigated nonlinear bi-skew Lie derivations on prime $*$-algebras. Let Φ be a nonlinear bi-skew Lie derivation on a unital prime $*$-algebra with a nontrivial projection. They proved that if $\Phi(I)$ and $\Phi(iI)$ are self-adjoint, then Φ is an additive $*$-derivation. V. Darvish et al. [2] proved any nonlinear bi-skew Jordan derivation on prime $*$-algebra is an additive $*$-derivation. A. Khan [5] proved that any nonlinear bi-skew Lie triple derivation on factor von Neumann algebra $\mathcal{A}$ with $\dim\mathcal{A} \geq 2$ is an additive $*$-derivation. V. Darvish et al. [3] proved any nonlinear bi-skew Jordan triple derivation on prime $*$-algebra is an additive $*$-derivation.

Recently, many authors have studied derivations corresponding to some mixed products (see for example [8, 9, 11, 12, 18, 19]). Y. Zhou, Z. Yang and J. Zhang [18] proved any map Φ from a unital $*$-algebra $\mathcal{A}$ containing a non-trivial projection to itself satisfying

$$\Phi([[A, B]_*, C]) = [[\Phi(A), B]_*, C] + [[A, \Phi(B)]_*, C] + [[A, B]_*, \Phi(C)]$$

for all $A, B, C \in \mathcal{A}$, is an additive $*$-derivation, where $[A, B] = AB - BA$ is the usual Lie product of A and B and $[A, B]_* = AB - BA^*$ is the skew Lie product of A and B. Y. Zhou and J. Zhang [19] proved that any map Φ on factor von Neumann algebra $\mathcal{A}$ satisfying

$$\Phi([[A, B], C]_*) = [[\Phi(A), B], C]_* + [[A, \Phi(B)], C]_* + [[A, B], \Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$, is also an additive $*$-derivation. X. Zhao and X. Fang [17] gave similar result on finite von Neumann algebra with no central summands of type I_1. Y. Pang, D. Zhang and D. Ma [11] proved that if Φ is a second nonlinear mixed Jordan triple derivable mapping on a factor von Neumann algebra $\mathcal{A}$, that is, if Φ satisfies

$$\Phi(A \circ B \bullet C) = \Phi(A) \circ B \bullet C + A \circ \Phi(B) \bullet C + A \circ B \bullet \Phi(C)$$

for all $A, B, C \in \mathcal{A}$, then Φ is an additive $*$-derivation, where $A \circ B = AB + BA$ is the usual Jordan product of A and B and $A \bullet B = AB + BA^*$ is the Jordan $*$-product of A and B. Lately, N. Rehman, J. Nisar and M. Nazim [12] generalized the above result to general $*$-algebras. C. Li and D. Zhang [8, 9] studied the derivations corresponding to the mixed products $[A, B]_* \bullet C$ and $[A \bullet B, C]_*$.

Motivated by the above mentioned works, in this paper, we will consider the derivations corresponding to the new product of the mixture of the bi-skew Lie product and the bi-skew Jordan product. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be a nonlinear mixed bi-skew Lie triple derivation if

$$\Phi([A \circ B, C]_\diamond) = [\Phi(A) \circ B, C]_\diamond + [A \circ \Phi(B), C]_\diamond + [A \circ B, \Phi(C)]_\diamond$$

holds for all $A, B, C \in \mathcal{A}$. In this paper, we will give the structure of the nonlinear mixed bi-skew Lie triple derivations on $*$-algebra. Under some mild conditions on a $*$-algebra $\mathcal{A}$, we prove that Φ is a nonlinear mixed bi-skew Lie triple derivation on $\mathcal{A}$ if and only if Φ is an additive $*$-derivation.

2. Main result and corollaries

The following is our main result in this paper.

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with the unit I. Assume that $\mathcal{A}$ contains a nontrivial projection P which satisfies*

$$(\spadesuit) \quad X\mathcal{A}P = 0 \quad \textit{implies} \quad X = 0$$

and

$$(\clubsuit) \quad X\mathcal{A}(I - P) = 0 \quad \textit{implies} \quad X = 0.$$

Then a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies

$$\Phi([A \circ B, C]_\diamond) = [\Phi(A) \circ B, C]_\diamond + [A \circ \Phi(B), C]_\diamond + [A \circ B, \Phi(C)]_\diamond$$

for all $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$-derivation.

Recall that an algebra $\mathcal{A}$ is prime if $A\mathcal{A}B = \{0\}$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$. It is easy to see that prime $*$-algebras satisfy (♠) and (♣). Applying Theorem 2.1 to prime $*$-algebras, we have the following corollary.

Corollary 2.2. *Let $\mathcal{A}$ be a prime $*$-algebra with unit I and P be a nontrivial projection in $\mathcal{A}$. Then a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Phi([A \circ B, C]_\diamond) = [\Phi(A) \circ B, C]_\diamond + [A \circ \Phi(B), C]_\diamond + [A \circ B, \Phi(C)]_\diamond$$

for all $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$-derivation.

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and $\mathcal{F}(\mathcal{H}) \subseteq B(\mathcal{H})$ be the subalgebra of all bounded finite rank operators. A subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$ is called a standard operator algebra if it contains $\mathcal{F}(\mathcal{H})$. Now we have the following corollary.

Corollary 2.3. *Let $\mathcal{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathcal{H}$ containing the identity operator I. Suppose that $\mathcal{A}$ is closed under the adjoint operation. Then $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Phi([A \circ B, C]_\diamond) = [\Phi(A) \circ B, C]_\diamond + [A \circ \Phi(B), C]_\diamond + [A \circ B, \Phi(C)]_\diamond$$

for all $A, B, C \in \mathcal{A}$ if and only if Φ is a linear $*$-derivation. Moreover, there exists an operator $T \in B(\mathcal{H})$ satisfying $T + T^* = 0$ such that $\Phi(A) = AT - TA$ for all $A \in \mathcal{A}$, i.e., Φ is inner.

Proof. Since $\mathcal{A}$ is prime, we have that Φ is an additive $*$-derivation. It follows from [13] that Φ is a linear inner derivation, i.e., there exists an operator $S \in B(\mathcal{H})$ such that $\Phi(A) = AS - SA$. Since $\Phi(A^*) = \Phi(A)^*$, we have

$$A^*S - SA^* = \Phi(A^*) = \Phi(A)^* = -A^*S^* + S^*A^*$$

for all $A \in \mathcal{A}$. Hence $A^*(S + S^*) = (S + S^*)A^*$, and then $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Let $T = S - \frac{1}{2}\lambda I$. It is easy to see that $T + T^* = 0$ such that $\Phi(A) = AT - TA$. □

A von Neumann algebra $\mathcal{M}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. $\mathcal{M}$ is a factor von Neumann algebra if its center only contains the scalar operators. It is well known that a factor von Neumann algebra is prime. Now we have the following corollary.

Corollary 2.4. *Let $\mathcal{M}$ be a factor von Neumann algebra with $dim(\mathcal{M}) \geq 2$. Then a map $\Phi : \mathcal{M} \to \mathcal{M}$ satisfies*

$$\Phi([A \circ B, C]_\diamond) = [\Phi(A) \circ B, C]_\diamond + [A \circ \Phi(B), C]_\diamond + [A \circ B, \Phi(C)]_\diamond$$

if and only if Φ is an additive $*$-derivation.

It is shown in [1] and [7] that if a von Neumann algebra $\mathcal{M}$ has no central summands of type I_1, then $\mathcal{M}$ satifies (♠) and (♣). Now we have the following corollary.

Corollary 2.5. *Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1. Then a map $\Phi : \mathcal{M} \to \mathcal{M}$ satisfies*

$$\Phi([A \circ B, C]_\diamond) = [\Phi(A) \circ B, C]_\diamond + [A \circ \Phi(B), C]_\diamond + [A \circ B, \Phi(C)]_\diamond$$

if and only if Φ is an additive $*$-derivation.

3. The proof of main result

The proof of Theorem 2.1. In the following, let $P_1 = P$ and $P_2 = I - P$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j (i, j = 1, 2)$. Then $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$. Let $\mathcal{N} = \{A \in \mathcal{A} : A^* = -A\}, \mathcal{N}_{12} = \{P_1NP_2 + P_2NP_1 : N \in \mathcal{N}\}, \mathcal{N}_{ii} = P_i\mathcal{N}P_i (i = 1, 2)$. Thus, for every $N \in \mathcal{N}$, $N = N_{11} + N_{12} + N_{22}$, where $N_{11} \in \mathcal{N}_{11}, N_{12} \in \mathcal{N}_{12}, N_{22} \in \mathcal{N}_{22}$.

Proof. Clearly, we only need to prove the necessity. We will complete the proof by several claims.

Claim 1.. $\Phi(0) = 0$.

Indeed, we have

$$\Phi(0) = \Phi([0 \circ 0, 0]_\diamond) = [\Phi(0) \circ 0, 0]_\diamond + [0 \circ \Phi(0), 0]_\diamond + [0 \circ 0, \Phi(0)]_\diamond = 0.$$

Claim 2.. For every $N \in \mathcal{N}$, we have $\Phi(N) \in \mathcal{N}$.

For any $N \in \mathcal{N}$, $N = [N \circ \frac{i}{2}I, \frac{i}{2}I]_\diamond$. Since $[A \circ B, C]_\diamond \in \mathcal{N}$ for all $A, B, C \in \mathcal{A}$, we get

$$\begin{aligned}\Phi(N) &= \Phi([N \circ \frac{i}{2}I, \frac{i}{2}I]_\diamond)\\ &= [\Phi(N) \circ \frac{i}{2}I, \frac{i}{2}I]_\diamond + [N \circ \Phi(\frac{i}{2}I), \frac{i}{2}I]_\diamond + [N \circ \frac{i}{2}I, \Phi(\frac{i}{2}I)]_\diamond \in \mathcal{N}.\end{aligned}$$

Claim 3.. For every $C_{11} \in \mathcal{N}_{11}, N_{12} \in \mathcal{N}_{12}$ and $D_{22} \in \mathcal{N}_{22}$, we have

$$\Phi(C_{11} + N_{12}) = \Phi(C_{11}) + \Phi(N_{12})$$

and

$$\Phi(N_{12} + D_{22}) = \Phi(N_{12}) + \Phi(D_{22}).$$

Let $T = \Phi(C_{11} + N_{12}) - \Phi(C_{11}) - \Phi(N_{12})$. By Claim 2, we have $T^* = -T$. Since $[I \circ P_2, C_{11}]_\diamond = 0$, we obtain

$$\begin{aligned}&[\Phi(I) \circ P_2, C_{11} + N_{12}]_\diamond + [I \circ \Phi(P_2), C_{11} + N_{12}]_\diamond + [I \circ P_2, \Phi(C_{11} + N_{12})]_\diamond\\ &= \Phi([I \circ P_2, C_{11} + N_{12}]_\diamond)\\ &= \Phi([I \circ P_2, C_{11}]_\diamond) + \Phi([I \circ P_2, N_{12}]_\diamond)\\ &= [\Phi(I) \circ P_2, C_{11} + N_{12}]_\diamond + [I \circ \Phi(P_2), C_{11} + N_{12}]_\diamond + [I \circ P_2, \Phi(C_{11}) + \Phi(N_{12})]_\diamond.\end{aligned}$$

This implies that $[I \circ P_2, T]_\diamond = 0$, and hence $P_1TP_2 = P_2TP_1 = P_2TP_2 = 0$.

Next, it follows from $[I \circ (P_2 - P_1), N_{12}]_\diamond = 0$ that

$$\begin{aligned}&[\Phi(I) \circ (P_2 - P_1), C_{11} + N_{12}]_\diamond + [I \circ \Phi(P_2 - P_1), C_{11} + N_{12}]_\diamond\\ &+ [I \circ (P_2 - P_1), C_{11} + N_{12}]_\diamond\\ &= \Phi([I \circ (P_2 - P_1), C_{11} + N_{12}]_\diamond)\\ &= \Phi([I \circ (P_2 - P_1), C_{11}]_\diamond) + \Phi([I \circ (P_2 - P_1), N_{12}]_\diamond)\\ &= [\Phi(I) \circ (P_2 - P_1), C_{11} + N_{12}]_\diamond + [I \circ \Phi(P_2 - P_1), C_{11} + N_{12}]_\diamond\\ &+ [I \circ (P_2 - P_1), \Phi(C_{11}) + \Phi(N_{12})]_\diamond.\end{aligned}$$

So $[I \circ (P_2 - P_1), T]_\diamond = 0$, and it yields that $P_1TP_1 = 0$. Hence $T = 0$.

Similarly, we can get that $\Phi(N_{12} + D_{22}) = \Phi(N_{12}) + \Phi(D_{22})$.

Claim 4.. For every $C_{11} \in \mathcal{N}_{11}, N_{12} \in \mathcal{N}_{12}$ and $D_{22} \in \mathcal{N}_{22}$, we have

$$\Phi(C_{11} + N_{12} + D_{22}) = \Phi(C_{11}) + \Phi(N_{12}) + \Phi(D_{22}).$$

Let $T = \Phi(C_{11} + N_{12} + D_{22}) - \Phi(C_{11}) - \Phi(N_{12}) - \Phi(D_{22})$. By Claim 2, we have $T^* = -T$. Since $[P_1 \circ I, D_{22}]_\diamond = 0$, it follows from Claim 3 that

$$\begin{aligned}&[\Phi(P_1) \circ I, C_{11} + N_{12} + D_{22}]_\diamond + [P_1 \circ \Phi(I), C_{11} + N_{12} + D_{22}]_\diamond\\ &+ [P_1 \circ I, \Phi(C_{11} + N_{12} + D_{22})]_\diamond\\ &= \Phi([P_1 \circ I, C_{11} + N_{12} + D_{22}]_\diamond)\\ &= \Phi([P_1 \circ I, C_{11} + N_{12}]_\diamond) + \Phi([P_1 \circ I, D_{22}]_\diamond)\\ &= [\Phi(P_1) \circ I, C_{11} + N_{12} + D_{22}]_\diamond + [P_1 \circ \Phi(I), C_{11} + N_{12} + D_{22}]_\diamond\\ &+ [P_1 \circ I, \Phi(C_{11}) + \Phi(N_{12}) + \Phi(D_{22})]_\diamond.\end{aligned}$$

This yields that $[P_1 \circ I, T]_\diamond = 0$, and then $P_1TP_1 = P_1TP_2 = 0$. In the similar manner, we can show that $P_2TP_1 = P_2TP_2 = 0$. Hence $T = 0$.

Claim 5.. For every $N_{12}, B_{12} \in \mathcal{N}_{12}$, we have

$$\Phi(N_{12} + B_{12}) = \Phi(N_{12}) + \Phi(B_{12}).$$

Let $N_{12}, B_{12} \in \mathcal{N}_{12}$. Then $N_{12} = P_1NP_2 + P_2NP_1$, $B_{12} = P_1BP_2 + P_2BP_1$, where $N, B \in \mathcal{N}$. Since

$$[(iP_1 + N_{12}) \circ (iP_2 + B_{12}), \frac{i}{2}I]_\diamond = N_{12} + B_{12} - iN_{12}B_{12} - iB_{12}N_{12},$$

where

$$N_{12} + B_{12} \in \mathcal{N}_{12}$$

and

$$-iN_{12}B_{12} - iB_{12}N_{12} = P_1(-i(NP_2B + BP_2N))P_1 + P_2(-i(NP_1B + BP_1N))P_2 \in \mathcal{N}_{11} + \mathcal{N}_{22},$$

we can get from Claim 4 that

$$\begin{aligned}
&\Phi(N_{12} + B_{12}) + \Phi(-iN_{12}B_{12} - iB_{12}N_{12}) \\
&= \Phi(N_{12} + B_{12} - iN_{12}B_{12} - iB_{12}N_{12}) \\
&= \Phi([(iP_1 + N_{12}) \circ (iP_2 + B_{12}), \frac{i}{2}I]_\diamond) \\
&= [(\Phi(iP_1) + \Phi(N_{12})) \circ (iP_2 + B_{12}), \frac{i}{2}I]_\diamond + [(iP_1 + N_{12}) \circ (\Phi(iP_2) + \Phi(B_{12})), \frac{i}{2}I]_\diamond \\
&+ [(iP_1 + N_{12}) \circ (iP_2 + B_{12}), \Phi(\frac{i}{2}I)]_\diamond \\
&= \Phi([(iP_1) \circ (iP_2), \frac{i}{2}I]_\diamond) + \Phi([(iP_1) \circ B_{12}, \frac{i}{2}I]_\diamond) + \Phi([N_{12} \circ (iP_2), \frac{i}{2}I]_\diamond) \\
&+ \Phi([N_{12} \circ B_{12}, \frac{i}{2}I]_\diamond) \\
&= \Phi(B_{12}) + \Phi(N_{12}) + \Phi(-iN_{12}B_{12} - iB_{12}N_{12}).
\end{aligned}$$

This implies that

$$\Phi(N_{12} + B_{12}) = \Phi(N_{12}) + \Phi(B_{12}).$$

Claim 6.. For every $C_{ii}, D_{ii} \in \mathcal{N}_{ii}$ $(i = 1, 2)$, we have

$$\Phi(C_{ii} + D_{ii}) = \Phi(C_{ii}) + \Phi(D_{ii}).$$

Let $T = \Phi(C_{11} + D_{11}) - \Phi(C_{11}) - \Phi(D_{11})$. By Claim 2, we have $T^* = -T$. Since $[P_2 \circ I, C_{11}]_\diamond = [P_2 \circ I, D_{11}]_\diamond = 0$, we obtain

$$\begin{aligned}
&[\Phi(P_2) \circ I, C_{11} + D_{11}]_\diamond + [P_2 \circ \Phi(I), C_{11} + D_{11}]_\diamond + [P_2 \circ I, \Phi(C_{11} + D_{11})]_\diamond \\
&= \Phi([P_2 \circ I, C_{11} + D_{11}]_\diamond) \\
&= \Phi([P_2 \circ I, C_{11}]_\diamond) + \Phi([P_2 \circ I, D_{11}]_\diamond) \\
&= [\Phi(P_2) \circ I, C_{11} + D_{11}]_\diamond + [P_2 \circ \Phi(I), C_{11} + D_{11}]_\diamond + [P_2 \circ I, \Phi(C_{11}) + \Phi(D_{11})]_\diamond.
\end{aligned}$$

Hence $[P_2 \circ I, T]_\diamond = 0$, and then $P_1TP_2 = P_2TP_1 = P_2TP_2 = 0$. Now we have $T = P_1TP_1$.

For every $A_{12} \in \mathcal{A}_{12}$, let $N = A_{12} - A_{12}^*$. Then

$$[C_{11} \circ N, \frac{i}{2}I]_\diamond,\ [D_{11} \circ N, \frac{i}{2}I]_\diamond \in \mathcal{N}_{12}.$$

In view of Claim 5, we find that

$$
\begin{aligned}
&[\Phi(C_{11}+D_{11})\circ N,\frac{i}{2}I]_\diamond+[(C_{11}+D_{11})\circ\Phi(N),\frac{i}{2}I]_\diamond\\
&+[(C_{11}+D_{11})\circ N,\Phi(\frac{i}{2}I)]_\diamond\\
&=\Phi([(C_{11}+D_{11})\circ N,\frac{i}{2}I]_\diamond)\\
&=\Phi([C_{11}\circ N,\frac{i}{2}I]_\diamond)+\Phi([D_{11}\circ N,\frac{i}{2}I]_\diamond)\\
&=[(\Phi(C_{11})+\Phi(D_{11}))\circ N,\frac{i}{2}I]_\diamond+[(C_{11}+D_{11})\circ\Phi(N),\frac{i}{2}I]_\diamond\\
&+[(C_{11}+D_{11})\circ N,\Phi(\frac{i}{2}I)]_\diamond.
\end{aligned}
$$

This yields that $[T\circ N,\frac{i}{2}I]_\diamond=0$, that is, $A_{12}^*T-TA_{12}=0$. Multiplying the above equation by P_1 from the left, we have $P_1TA_{12}=0$ for all $A_{12}\in\mathcal{A}_{12}$. It follows from (♣) that $P_1TP_1=0$, and hence $T=0$.

Similarly, we can show that $\Phi(C_{22}+D_{22})=\Phi(C_{22})+\Phi(D_{22})$.

By using Claims 4-6, one can obtain the following claim easily.

Claim 7.. Φ is additive on $\mathcal{N}$.

Claim 8..

1. $\Phi(I)=\Phi(iI)=0$;
2. For any $M\in\mathcal{A}$ such that $M^*=M$, we have $\Phi(M)^*=\Phi(M)$ and $\Phi(iM)=i\Phi(M)$.

For any $M\in\mathcal{M}$, it follows from Claims 2 and 7 that

$$
\begin{aligned}
4\Phi(iI)=\Phi(4iI)&=\Phi([(iI)\circ(iI),iI]_\diamond)\\
&=[\Phi(iI)\circ(iI),iI]_\diamond+[(iI)\circ\Phi(iI),iI]_\diamond+[(iI)\circ(iI),\Phi(iI)]_\diamond\\
&=2[-2i\Phi(iI),iI]_\diamond+[2I,\Phi(iI)]_\diamond\\
&=12\Phi(iI).
\end{aligned}
$$

This implies that $\Phi(iI)=0$.

For any $M\in\mathcal{A}$ such that $M^*=M$,

$$0=\Phi([M\circ(iI),iI]_\diamond)=[\Phi(M)\circ(iI),iI]_\diamond=2(\Phi(M)-\Phi(M)^*).$$

Hence $\Phi(M)^*=\Phi(M)$ for all $M^*=M$.

Now, we can get that

$$
\begin{aligned}
0=4\Phi(iI)&=\Phi(4iI)=\Phi([I\circ I,iI]_\diamond)\\
&=[\Phi(I)\circ I,iI]_\diamond+[I\circ\Phi(I),iI]_\diamond\\
&=8i\Phi(I).
\end{aligned}
$$

This yields that $\Phi(I)=0$.

For any $M\in\mathcal{A}$ such that $M^*=M$, we have

$$
\begin{aligned}
4\Phi(iM)=\Phi(4iM)&=\Phi([I\circ M,iI]_\diamond)\\
&=[I\circ\Phi(M),iI]_\diamond\\
&=4i\Phi(M).
\end{aligned}
$$

Thus $\Phi(iM)=i\Phi(M)$ for all $M^*=M$.

Claim 9.. For any $A_1, A_2 \in \mathcal{A}$ such that $A_1^* = A_1, A_2^* = A_2$, we have

$$\Phi(A_1 + A_2) = \Phi(A_1) + \Phi(A_2)$$

and

$$\Phi(A_1 + iA_2) = \Phi(A_1) + i\Phi(A_2).$$

Let $A_1^* = A_1, A_2^* = A_2$. It follows from Claims 7 and 8 that

$$i\Phi(A_1 + A_2) = \Phi(i(A_1 + A_2)) = \Phi(iA_1) + \Phi(iA_2) = i(\Phi(A_1) + \Phi(A_2)).$$

That is, $\Phi(A_1 + A_2) = \Phi(A_1) + \Phi(A_2)$.

Now, on the one hand, we have

$$\begin{aligned}4i\Phi(A_1) = \Phi(4iA_1) &= \Phi([(A_1 + iA_2) \circ I, iI]_\diamond) \\ &= [\Phi(A_1 + iA_2) \circ I, iI]_\diamond \\ &= 2i(\Phi(A_1 + iA_2) + \Phi(A_1 + iA_2)^*).\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}4i\Phi(A_2) = \Phi(4iA_2) &= \Phi([(A_1 + iA_2) \circ (iI), iI]_\diamond) \\ &= [\Phi(A_1 + iA_2) \circ (iI), iI]_\diamond \\ &= 2(\Phi(A_1 + iA_2) - \Phi(A_1 + iA_2)^*).\end{aligned}$$

Comparing the above two equations, we obtain $\Phi(A_1 + iA_2) = \Phi(A_1) + i\Phi(A_2)$.

Claim 10..

1. For every $A \in \mathcal{A}$, we have $\Phi(iA) = i\Phi(A)$ and $\Phi(A^*) = \Phi(A)^*$;
2. Φ is additive on $\mathcal{A}$.

For any $A \in \mathcal{A}$, we have $A = A_1 + iA_2$, where $A_1^* = A_1, A_2^* = A_2$. It follows from Claim 9 that

$$\begin{aligned}\Phi(iA) &= \Phi(iA_1 - A_2) = i\Phi(A_1) - \Phi(A_2) \\ &= i(\Phi(A_1) + i\Phi(A_2)) = i\Phi(A_1 + iA_2) \\ &= i\Phi(A).\end{aligned}$$

Next, from Claims 8 and 9, we find that

$$\begin{aligned}\Phi(A^*) &= \Phi(A_1 - iA_2) = \Phi(A_1) - i\Phi(A_2) \\ &= (\Phi(A_1) + i\Phi(A_2))^* = (\Phi(A_1 + iA_2))^* \\ &= \Phi(A)^*.\end{aligned}$$

For any $A, B \in \mathcal{A}$, we have $A = A_1 + iA_2$ and $B = B_1 + iB_2$, where $A_1^* = A_1, A_2^* = A_2, B_1^* = B_1, B_2^* = B_2$. Then we can obtain from Claim 9 that

$$\begin{aligned}\Phi(A + B) &= \Phi((A_1 + B_1) + i(A_2 + B_2)) \\ &= \Phi(A_1 + B_1) + i\Phi(A_2 + B_2) \\ &= (\Phi(A_1) + i\Phi(A_2)) + (\Phi(B_1) + i\Phi(B_2)) \\ &= \Phi(A) + \Phi(B).\end{aligned}$$

Claim 11.. Φ is an additive $*$-derivation on $\mathcal{A}$.

For every $A, B \in \mathcal{A}$, on the one hand, by Claims 8 (1) and 10, we have

$$\begin{aligned} 2i\Phi(A^*B + B^*A) &= \Phi(2i(A^*B + B^*A)) \\ &= \Phi([A \circ B, iI]_\diamond) \\ &= [\Phi(A) \circ B, iI]_\diamond + [A \circ \Phi(B), iI]_\diamond \\ &= 2i(\Phi(A)^*B + B^*\Phi(A) + A^*\Phi(B) + \Phi(B)^*A). \end{aligned}$$

This yields that

$$\Phi(A^*B + B^*A) = \Phi(A)^*B + B^*\Phi(A) + A^*\Phi(B) + \Phi(B)^*A.$$

On the other hand, we also have

$$\begin{aligned} -2(\Phi(A^*B - B^*A)) &= \Phi(-2(A^*B - B^*A)) \\ &= \Phi([A \circ iB, iI]_\diamond) \\ &= [\Phi(A) \circ iB, iI]_\diamond + [A \circ \Phi(iB), iI]_\diamond \\ &= -2(\Phi(A)^*B - B^*\Phi(A) + A^*\Phi(B) - \Phi(B)^*A). \end{aligned}$$

This yields that

$$\Phi(A^*B - B^*A) = \Phi(A)^*B - B^*\Phi(A) + A^*\Phi(B) - \Phi(B)^*A.$$

By summing the above two equations, we have

$$\Phi(A^*B) = \Phi(A)^*B + A^*\Phi(B).$$

Replacing A by A^* in the above equation and using Claim 10 (1), we obtain

$$\Phi(AB) = \Phi(A)B + A\Phi(B).$$

Hence Φ is an additive $*$-derivation on $\mathcal{A}$ by Claim 10. This completes the proof of Theorem 2.1. $\square$

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