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Comparison of proper shape and proper shape over finite coverings

Nikita Shekutkovski[a], Abdulla Buklla[a]

[a]*Institute of Mathematics, Faculty of Natural Sciences and Mathematics, Sts. Cyril and Methodius University, 1000 Skopje, N. Macedonia*

Abstract. The theory of proper shape over finite coverings, defined in [8], uses only finite coverings to compare noncompact spaces. In this paper we investigate the relations between this theory and the proper shape defined by Ball and Sherr in [3]. We show that if two spaces have same proper shape they belong to the same class in theory of proper shape over finite coverings, but the opposite doesn't hold in general.

1. Introduction

Homotopy theory is a strong tool for classifying compact locally nice spaces, but it is not very accurate when dealing with noncompact spaces. For example, the line and point are too distant geometric spaces to be in a same class. This problem was solved by defining proper homotopy.

For comparing spaces with local problems, shape theory by Borsuk in [4] is shown to be more appropriate than homotopy.

The proper shape was defined by Ball and Sherr in [3] as shape analogue of proper homotopy for spaces with local difficulties. There exist different approaches to proper shape, and it is shown that they are equivalent theories in general [2].

Instead of using the original approach from [3], we will work with the intrinsic definition of the proper shape from [1], defined by Akaike and Sakai. Actually, we use the modified equivalent definition for the proper shape from [7]. Some equivalences between these different approaches were established in papers [6, 7].

On the other hand, in [3] a new theory of shape is defined using intrinsic techniques and only finite coverings of spaces. This theory of proper shape is called proper shape over finite coverings and it is shown to be more suitable for locally-compact, separable metric spaces with compact spaces of quasicomponents.

2. Proper homotopy over a covering

Along this paper by a covering we mean an open covering of the space.

For arbitrary space W we denote by $Cov_F(W)$ the set of all finite open coverings of W consisting of sets with compact boundary and by $Cov(W)$ we denote the set of all open coverings of W.

In this setion we will introduce the intrinsic definitions for proper shape and proper shape over finite coverings.

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Email addresses: nikita@pmf.ukim.com (Nikita Shekutkovski), abdullabuklla@hotmail.com (Abdulla Buklla)

If $\mathcal{U}, \mathcal{V}$ are two coverings of the space X, then $\mathcal{V}$ is refinement of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. We write $\mathcal{V} \prec \mathcal{U}$.

If $U \in \mathcal{U}$, then the star of U is the set $st(U, \mathcal{U}) = \cup\{W \in \mathcal{U} | W \cap U \neq \emptyset\}$ and by $st\mathcal{U}$ will be denoted the collection of all $st(U, \mathcal{U}), U \in \mathcal{U}$.

Let $f : X \to Y$ be a function and let $\mathcal{V}$ be a covering of Y. We say that $g : X \to Y$ is $\mathcal{V}$−near to f if for every $x \in X$, $f(x)$ and $g(x)$ lie in the same member of $\mathcal{V}$. It is denoted by $f =_{\mathcal{V}} g$.

Definition 2.1. Let $\mathcal{V}$ be a covering of Y. A function $f : X \to Y$ is $\mathcal{V}$ - continuous at the point $x \in X$ if there exists a neighborhood U_x of x and $V \in \mathcal{V}$ such that $f(U_x) \subseteq V$. A function $f : X \to Y$ is $\mathcal{V}$-continuous on X if it is $\mathcal{V}$-continuous at every point $x \in X$.

(The family of all such U_x form a covering $\mathcal{U}$ of X. Shortly, we say that $f : X \to Y$ is $\mathcal{V}$−continuous, if there exists $\mathcal{U}$ such that $f(\mathcal{U}) \prec \mathcal{V}$.)

Observation: When X and Y are paracompact spaces, it is enough to take $\mathcal{U}$ and $\mathcal{V}$ to be locally finite coverings, since locally finite coverings are cofinal in the set of all coverings.

Now, we define a proper (noncontinuous) function.

Definition 2.2. A function $f : X \to Y$ is proper if for every compact D in Y there exists compact set C in X such that $f(X \backslash C) \subseteq Y \backslash D$.

Definition 2.3. For arbitrary covering $\mathcal{V}$ of the space Y, we say that two proper functions $f, g : X \to Y$ are $\mathcal{V}$−properly homotopic, if there exists a proper function $F : X \times I \to Y$ such that:

1) $F : X \times I \to Y$ is $st\mathcal{V}$−continuous.

2) $F : X \times I \to Y$ is $\mathcal{V}$−continuous at all points of $X \times \partial I$.

3) $F(x, 0) = f(x), F(x, 1) = g(x)$.

We denote this by $f \overset{\mathcal{V}}{\sim}_p g$.

Proposition 2.4. *The relation "* $\overset{\mathcal{V}}{\sim}_p$ *" is an equivalence relation.*

Proof. See [9]. □

Proposition 2.5. *Let X, Y, Z be topological spaces, $g : Y \to Z$ be $\mathcal{W}$- continuous function and $\mathcal{V}$ be a covering of Y, such that $g(\mathcal{V}) \prec \mathcal{W}$. If two $\mathcal{V}$- continuous functions $f_1, f_2 : X \to Y$ are properly $\mathcal{V}$- homotopic functions, i.e. $f_1 \overset{\mathcal{V}}{\sim}_p f_2$, then $g \circ f_1 \overset{\mathcal{W}}{\sim}_p g \circ f_2$.*

Proof. By definitions it follows that the compositions $g \circ f_1, g \circ f_2$ are proper and $\mathcal{W}$-continuous functions. Since $f_1, f_2 : X \to Y$ are properly $\mathcal{V}$-homotopic, then there exists a function $F : X \times I \to Y$ such that:

(1) F is proper $st(\mathcal{V})$ - continuous function, which is $\mathcal{V}$- continuous on $X \times \partial I$;

(2) $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$ for all points $x \in X$.

Let us consider the function $K : X \times I \to Z$ defined by $K(x, s) = (g \circ F)(x, s)$. Since $g(\mathcal{V}) \prec \mathcal{W}$ it implies $g(st(\mathcal{V})) \prec st(\mathcal{W})$. Also, F is $st(\mathcal{V})$ - continuous so there exists an open covering $\mathcal{U}$, such that $F(\mathcal{U}) \prec st(\mathcal{V})$. We conclude that $(g \circ F)(\mathcal{U}) = g(F(\mathcal{U})) \prec g(st(\mathcal{V})) \prec st(\mathcal{W})$. Therefore, the function K is $st(\mathcal{W})$- continuous. Since F is $\mathcal{V}$- continuous on $X \times \partial I$, $g(\mathcal{V}) \prec \mathcal{W}$ and g is $\mathcal{W}$-continuous function then it follows that $K = g \circ F$ is $\mathcal{W}$-continuous on $X \times \partial I$.

From the fact that composition of proper functions is proper it follows that $K = g \circ F$ is proper.

If $x \in X$ is an arbitrary point, then $K(x, 0) = (g \circ F)(x, 0) = g(F(x, 0)) = g(f_1(x)) = (g \circ f_1)(x)$ and $K(x, 1) = (g \circ F)(x, 1) = g(F(x, 1)) = g(f_2(x)) = (g \circ f_2)(x)$.

Therefore, we showed that the functions $g \circ f_1, g \circ f_2$ are properly -$\mathcal{W}$ properly homotopic, i.e., $g \circ f_1 \overset{\mathcal{W}}{\sim}_p g \circ f_2$ □

Proposition 2.6. *Let $G : Y \times I \to Z$ be a proper $st(\mathcal{W})$- continuous function and $\mathcal{W}$- continuous on $Y \times \partial I$. Then there exists a covering $\mathcal{V}$ of Y, such that for each proper $\mathcal{V}$- continuous function $f : X \to Y$, the function $G(f \times id) : X \times I \to Z$ is proper and $st(\mathcal{W})$- continuous, and $\mathcal{W}$- continuous on $X \times \partial I$.*

Proof. Similar form of the theorem is proven for compact metric spaces in [9], Theorem 3.0.5. In noncompact case the proof actually remains the same. The additional condition of functions being proper doesn't violate the proof since composition and direct product of two proper functions is a proper function. □

3. Proper proximate nets. Intrinsic proper shape

Here we will give a short description of proper shape originally defined by Akaike and Sakai in the modified version given in [10]. That the definitions are equivalent it is shown in [6].

For arbitrary space M we define ordering in the set of all open coverings of M by $(CovM, \geq)$ by $\mathcal{U} \geq \mathcal{V} \Leftrightarrow \mathcal{U} \prec \mathcal{V}$):

Consider two paracompact topological spaces X and Y. Now, we will define proper proximate net from X to Y.

Definition 3.1. A proper proximate net from X to Y is a family $\underline{f} = (f_{\mathcal{V}} | \mathcal{V} \in CovY)$ of $\mathcal{V}$ - continuous proper functions $f_{\mathcal{V}} : X \to Y$, such that $f_{\mathcal{V}} \overset{\mathcal{V}}{\sim}_p f_{\mathcal{W}}$ whenever $\mathcal{W} \prec \mathcal{V}$

Definition 3.2. Two proper proximate nets $\underline{f}$ and $\underline{g}$ from X to Y are properly homotopic if $f_{\mathcal{V}} \overset{\mathcal{V}}{\sim}_p g_{\mathcal{V}}$ for all coverings $\mathcal{V} \in CovY$. We denote $\underline{f} \sim_p \underline{g}$.

Proposition 3.3. *The relation of proper homotopy of proper proximate nets is an equivalence relation. The proper homotopy class of proximate net $\underline{f}$ from X to Y will be denoted by $[\underline{f}]_p$.*

Proof. Let $\underline{f} = (f_{\mathcal{V}} | \mathcal{V} \in CovY)$ and $\underline{g} = (g_{\mathcal{V}} | \mathcal{V} \in CovY)$ be proper homotopic proper proximate nets from X to Y. It follows that, for all coverings $\mathcal{V} \in CovY$ the $\mathcal{V}$- continuous functions $f_{\mathcal{V}}$ and $g_{\mathcal{V}}$ are properly $\mathcal{V}$-homotopic. For every covering $\mathcal{V} \in CovY$ by Proposition 2.4 the relation of proper $\mathcal{V}$- homotopy $f_{\mathcal{V}} \overset{\mathcal{V}}{\sim}_p g_{\mathcal{V}}$ of $\mathcal{V}$- continuous proper functions is an equivalence relation. So, by the definition the relation of proper homotopy of proper proximate nets is an equivalence relation. □

Now, lets introduce the notion of composition of proper proximate nets $\underline{f} : X \to Y$ and $\underline{g} : Y \to Z$. Let $\underline{f} = (f_{\mathcal{V}} | \mathcal{V} \in CovY)$ and $\underline{g} = (g_{\mathcal{W}} | \mathcal{W} \in CovZ)$. From the fact that $g_{\mathcal{W}}$ is $\mathcal{W}$-continuous function it follows that there exists a covering $\mathcal{V} \in Y$ such that $g_{\mathcal{W}}(\mathcal{V}) \prec \mathcal{W}$. For $\mathcal{W}$ we define $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} : X \to Z$. The function h is proper and $\mathcal{W}$-continuous. Although the definition depends on the choice of $\mathcal{V}$, the next Lemma shows that for two coverings $\mathcal{V}, \mathcal{V}' \in CovY$ such that $g_{\mathcal{W}}(\mathcal{V}), g_{\mathcal{W}}(\mathcal{V}') \prec \mathcal{W}$ is true that $g_{\mathcal{W}} \circ f_{\mathcal{V}} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} \circ f_{\mathcal{V}'}$.

Lemma 3.4. *If $\underline{f}$ is proper proximate net and $\mathcal{V}, \mathcal{V}' \in CovY$ such that $g_{\mathcal{W}}(\mathcal{V}), g_{\mathcal{W}}(\mathcal{V}') \prec \mathcal{W}$, $\mathcal{W} \in CovZ$. Then $g_{\mathcal{W}} \circ f_{\mathcal{V}} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} \circ f_{\mathcal{V}'}$.*

Proof. Let $\mathcal{V}'' \in CovY$ be a common refinement of $\mathcal{V}$ and $\mathcal{V}'$, i.e., $\mathcal{V}'' \prec \mathcal{V}, \mathcal{V}'$. Since $\underline{f}$ is proper proximate net, by definition it follows that $f_{\mathcal{V}''} \overset{\mathcal{V}}{\sim}_p f_{\mathcal{V}}$ and $f_{\mathcal{V}''} \overset{\mathcal{V}'}{\sim}_p f_{\mathcal{V}'}$. By Proposition 2.5. it follows that $g_{\mathcal{W}} \circ f_{\mathcal{V}''} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} \circ f_{\mathcal{V}}$ and $g_{\mathcal{W}} \circ f_{\mathcal{V}''} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} \circ f_{\mathcal{V}'}$. From the transitivity of the proper homotopy we conclude that $g_{\mathcal{W}} \circ f_{\mathcal{V}} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} \circ f_{\mathcal{V}'}$. □

Now, we will show that the functions $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}}$ generate a proper proximate net $\underline{h} = (h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} | \mathcal{W} \in CovZ)$, i.e., we will show that for every $\mathcal{W}' \prec \mathcal{W}$ it follows that $h_{\mathcal{W}'} \overset{\mathcal{W}}{\sim}_p h_{\mathcal{W}}$.

Let $\mathcal{W}' \prec \mathcal{W}$. From the fact that $(g_{\mathcal{W}})$ is proper proximate net, there exists proper homotopy G from Y to Z such that G is $st(\mathcal{W})$-continuous on $Y \times I$ and $\mathcal{W}$-continuous on $Y \times \partial I$. From Proposition 2.6. there exists a covering $\mathcal{V}''$ of $CovY$ such that for every proper $\mathcal{V}''$-continuous function $f_{\mathcal{V}''} : X \to Y$, the function $G(f_{\mathcal{V}''} \times id) : X \times I \to Z$ is $st(\mathcal{W})$-continuous on $X \times I$ and $\mathcal{W}$-continuous on $X \times \partial I$. It follows that $g_{\mathcal{W}'} \circ f_{\mathcal{V}''} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} \circ f_{\mathcal{V}''}$.

Now, consider $h_{\mathcal{W}'} = g_{\mathcal{W}'} \circ f_{\mathcal{V}'}$ and $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}}$ for some $\mathcal{V}' \in CovY, g_{\mathcal{W}'}(\mathcal{V}') \prec \mathcal{W}'$ and a covering $\mathcal{V} \in CovY, g_{\mathcal{W}}(\mathcal{V}) \prec \mathcal{W}$. By Lemma 3.4., since $g_{\mathcal{W}}(\mathcal{V}), g_{\mathcal{W}}(\mathcal{V}'') \prec \mathcal{W}$ it follows that:

$$g_{\mathcal{W}} \circ f_{\mathcal{V}} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} \circ f_{\mathcal{V}''} \text{ (1).}$$

Now, consider a covering $\mathcal{V}_1$ of Y, such that $\mathcal{V}_1 \prec \mathcal{V}', \mathcal{V}''$. Since $g_{\mathcal{W}'}(\mathcal{V}_1), g_{\mathcal{W}'}(\mathcal{V}') \prec \mathcal{W}'$, by Lemma 3.4. it follows that $g_{\mathcal{W}'} \circ f_{\mathcal{V}_1} \overset{\mathcal{W}'}{\sim}_p g_{\mathcal{W}'} \circ f_{\mathcal{V}'}$.

Since $\mathcal{W}' \prec \mathcal{W}$ it follows $g_{\mathcal{W}'} \circ f_{\mathcal{V}_1} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}'} \circ f_{\mathcal{V}'}$. By Proposition 2.5. and (1) since $\underline{f}$ is a proper proximate net then from $f_{\mathcal{V}_1} \overset{\mathcal{V}''}{\sim}_p f_{\mathcal{V}''}$ and $g_{\mathcal{W}'}(\mathcal{V}'') \prec \mathcal{W}$ it follows that:

$$g_{\mathcal{W}'} \circ f_{\mathcal{V}'} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}'} \circ f_{\mathcal{V}_1} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}'} \circ f_{\mathcal{V}''} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} \circ f_{\mathcal{V}''} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} \circ f_{\mathcal{V}} ,$$

i.e., we showed that $h_{\mathcal{W}'} \overset{\mathcal{W}}{\sim}_p h_{\mathcal{W}}$ □.

Now we will give the following definition about compositions:

Definition 3.5. Let $[\underline{f}]_p, [\underline{g}]_p$ be two proper homotopy classes of proper proximate nets. We define a composition of proper homotopy classes $[\underline{f}]_p$ and $[\underline{g}]_p$ by $[\underline{g}]_p \circ [\underline{f}]_p = [\underline{g} \circ \underline{f}]_p$

From the discussion above in order to show that this composition is well defined we have only to show that if $\underline{f} \sim_p \underline{f'}$ and $\underline{g} \sim_p \underline{g'}$ then $\underline{h} \sim_p \underline{h'}$, where $\underline{h}$ and $\underline{h'}$ are the compositions of proper proximate nets $\underline{f}$ and $\underline{g}$, $\underline{f'}$ and $\underline{g'}$, respectively.

Since $\underline{g} \sim_p \underline{g'}$ by a proper homotopy then for every covering $\mathcal{W} \in CovZ$ it follows that $g_{\mathcal{W}} \overset{\mathcal{W}}{\sim}_p g'_{\mathcal{W}}$ and by Proposition 2.5. there exists a covering $\mathcal{U} \in CovY$ such that $g_{\mathcal{W}}(\mathcal{U}) \prec \mathcal{W}$, $g'_{\mathcal{W}}(\mathcal{U}) \prec \mathcal{W}$ such that for $\mathcal{U}$-continuous function $f_{\mathcal{U}} : X \to Y$ it is true that $g_{\mathcal{W}} \circ f_{\mathcal{U}} \overset{\mathcal{W}}{\sim}_p g'_{\mathcal{W}} \circ f_{\mathcal{U}}$. From the definition of the composition of two proper proximate nets there exists coverings $\mathcal{V}, \mathcal{V}'$ of Y such that $g_{\mathcal{W}}(\mathcal{V}) \prec \mathcal{W}$ and $g'_{\mathcal{W}}(\mathcal{V}') \prec \mathcal{W}$ such that $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}}, h'_{\mathcal{W}} = g'_{\mathcal{W}} \circ f'_{\mathcal{V}}$. Since $\underline{f} \sim_p \underline{f'}$ then $f_{\mathcal{U}} \overset{\mathcal{U}}{\sim}_p f'_{\mathcal{U}}$, so by this fact, Lemma 3.4 and Proposition 2.5 we can conclude that $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} \circ f_{\mathcal{U}} \overset{\mathcal{W}}{\sim}_p g'_{\mathcal{W}} \circ f_{\mathcal{U}} \overset{\mathcal{W}}{\sim}_p g'_{\mathcal{W}} \circ f'_{\mathcal{V}'} = h'_{\mathcal{W}'}$ for all $\mathcal{V} \in CovZ$. Therefore $\underline{h} \sim_p \underline{h'}$.

By the definition of the composition of proper proximate nets and $\mathcal{U}$- continuous function the following Theorem is valid.

Theorem 3.6. *Let $[\underline{f}]_p, [\underline{g}]_p$ and $[\underline{h}]_p$ be three proper homotopy classes of proper proximate nets. Then $[\underline{h}]_p \circ ([\underline{g}]_p \circ [\underline{f}]_p) = ([\underline{h}]_p \circ [\underline{g}]_p) \circ [\underline{f}]_p$*

In this way we proved that the topological spaces and proper homotopy classes of proper proximate nets form category of proper intrinsic shape. We say that proper topological spaces X and Y have the same proper intrinsic shape if they are isomorphic in this category.

4. Proper shape over finite coverings

Now we will give a short description of the category of proper shape over finite coverings from [8].

About the notion of space of quasicomponents, we refer to [8].

Let X be a topological space. By $Cov_F X$, we denote the set that represents the collection of all finite coverings for a topological space X, where each covering is composed of open sets with compact boundaries.

A cofinal sequence of (finite) coverings $\mathcal{V}_1 \prec \mathcal{V}_2 ... \prec \mathcal{V}_n ...$, for a given topological space, is a sequence of (finite) coverings of the space, such that for any covering $\mathcal{V}$, there exists n, such that $\mathcal{V}_n \prec \mathcal{V}$.

Definition 4.1. A sequence (f_n) of proper functions $f_n : X \to Y$ is proper proximate sequence over finite coverings from X to Y, if there exists cofinal sequence $\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3 > ...$ from $Cov_F(Y)$ such that for every $m \geq n$, f_n and f_m are $\mathcal{V}_n-$ properly homotopic. We say that (f_n) is proper proximate sequence over $(\mathcal{V}_n)$.

Definition 4.2. Two proximate sequences $(f_n), (g_n)$ are homotopic if there exists a cofinal sequence $\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3 > ...$ in $Cov_F(Y)$, such that $(f_n), (g_n)$ are proximate sequences over $(\mathcal{V}_n)$ and for all $n \in \mathbb{N}$, f_n and g_n are properly $\mathcal{V}_n-$homotopic.

We denote this by $(f_n) \sim_{pF} (g_n)$.

In the paper [5], it is proven that locally-compact separable metric spaces with compact spaces of quasicomponents and homotopy classes of proximate sequences over finite coverings form a category of proper shape over finite coverings.

Two spaces X, Y have same proper shape over finite coverings if there exists a proper sequence over finite coverings $(f_n) : X \to Y$ and a proper proximate sequence $(f_n{}^*) : Y \to X$ such that:

$$(f_n)(f_n{}^*) \sim_{pF} 1_Y, \quad (f_n{}^*)(f_n) \sim_{pF} 1_X.$$

We denote this by $Sh_{pF}(X) = Sh_{pF}(Y)$.

5. Comparison

First, we will show that if two spaces X, Y have same proper shape they have same proper shape over finite coverings.

Theorem 5.1. *Let X, Y be locally-compact separable metric spaces with compact spaces of quasicomponents. If $Sh_p(X) = Sh_p(Y)$ then $Sh_{pF}(X) = Sh_{pF}(Y)$.*

Proof. Let $Sh_p(X) = Sh_p(Y)$. It follows there exist proximate nets:

$$(f_{\mathcal{V}} \big| \mathcal{V} \in Y) : X \to Y, (g_{\mathcal{W}} \big| \mathcal{W} \in X) : Y \to X \text{ such that } \underline{f} \circ \underline{g} \sim_p \underline{1_Y} \text{ and } \underline{g} \circ \underline{f} \sim_p \underline{1_X}.$$

We have that for arbitrary coverings $\mathcal{V} \in CovY$, $\mathcal{W} \in CovX$, there exists coverings $\mathcal{V}' \in CovY, \mathcal{W}' \in CovX$ such that $f_{\mathcal{V}}(\mathcal{W}') < \mathcal{V}$, $g_{\mathcal{W}}(\mathcal{V}') < \mathcal{W}$:

$$f_{\mathcal{V}} g_{\mathcal{W}'} \overset{\mathcal{V}}{\sim}_p \underline{1_Y} \text{ and } g_{\mathcal{W}} f_{\mathcal{V}'} \overset{\mathcal{W}}{\sim}_p \underline{1_X} \quad (1)$$

From [8] there exists a cofinal sequence $\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3 > ...$ from $Cov_F(Y)$.

Fix the covering $\mathcal{V}_1$. For every point $y \in Y$ there exists an integer $n_{1,y}$ such that $B_{\frac{1}{n_{1,y}}}(y) \subseteq V_1$ for some $V_1 \in \mathcal{V}_1$. Take $\mathcal{V}'_1 = \left\{ B_{\frac{1}{n_{1,y}}}(y) | y \in Y \right\}$, so from the covering $\mathcal{V}_1$ we obtained a covering $\mathcal{V}'_1$ consisting of sets with compact closure i.e. $\mathcal{V}'_1 \in covY$.

Now, fix $\mathcal{V}_2$ and let $y \in Y$. There exists an integer $n_{2,y}$ such that $n_{2,y} > n_{1,y}$ and $B_{\frac{1}{n_{2,y}}}(y) \subseteq V_2$ for all $V_2 \in \mathcal{V}_2$.

Following the same principle, we obtain a sequence of coverings $(\mathcal{V}'_n)$ such that $\mathcal{V}'_1 > \mathcal{V}'_2 > \mathcal{V}'_3 > ...$, $\mathcal{V}_j > \mathcal{V}'_j$ for all integers j and $\mathcal{V}'_j$ consists of sets with compact closure.

We construct sequence $(f_n) : X \to Y$ from the net $(f_{\mathcal{V}}) : X \to Y$ by defining:

$$f_n = f_{\mathcal{V}'_n} \text{ for all } n \in N.$$

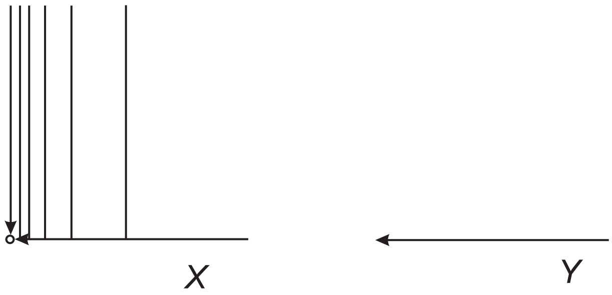
We have that the sequence $(f_n) : X \to Y$ is proximate sequence over finite coverings $(\mathcal{V}'_n)$. In similar way we could construct a proximate sequence $(g_n) : Y \to X$ over a cofinal sequence $\mathcal{U}_1 > \mathcal{U}_2 > \mathcal{U}_3 > ...$ from $Cov_F(X)$. From (1) we have that $(f_n)(g_n) \sim_p \underline{1_Y}$ and $(g_n)(f_n) \sim_p \underline{1_X}$. □

Note. In the previous theorem constructing the coverings with compact closure was not necessary, we could work with the first cofinal sequence of finite coverings from $Cov_F(Y)$ and to consider the functions from the proximate net having those coverings as index.

We need the next theorem from [7] (Theorem 2.1). We assume that spaces are locally compact and paracompact.

Theorem 5.2. *Let* $X = \oplus\{X_\alpha \,|\alpha \in A\}$*. Then* $X{\sim}_p Y$ *if and only if* $Y = \oplus\{Y_\alpha \,|\alpha \in A\}$ *and* $X_\alpha{\sim}_p Y_\alpha$*, for every* $\alpha \in A$*.*

Example 5.3. There are locally-compact separable metric spaces X, Y such that $Sh_{pF}(X) = Sh_{pF}(Y)$, but $Sh_p(X) \neq Sh_p(Y)$.



Proof. Take

$$X = \left\{\left(\tfrac{1}{n+1}, y\right) \middle| y \in [0,1],\, n \in N\right\} \cup [(0,1] \times \{0\}] \cup [\{0\} \times (0,1]]$$

and

$$Y = (0,1] \times \{0\}.$$

First, we will prove that $Sh_{pF}(X) = Sh_{pF}(Y)$. We will define cofinal sequence $(\mathcal{U}_n)$ from $Cov_F(X)$ and a cofinal sequence $(\mathcal{V}_n)$ from $Cov_F(Y)$.

Inductively by n, we define a cofinal sequence $(\mathcal{U}_n)$ from $Cov_F(X)$ as follows:

$$\mathcal{U}_n = \left\{B_{\frac{1}{n}}(0,0)\right\} \cup \left\{B_{\alpha_n}\left(x_1^n\right), B_{\alpha_n}\left(x_2^n\right), ..., B_{\alpha_n}\left(x_{l_n}^n\right)\right\}$$

where $\mathcal{U}_n^0 = \left\{B_{\alpha_n}\left(x_1^n\right), B_{\alpha_n}\left(x_2^n\right), ..., B_{\alpha_n}\left(x_{l_n}^n\right)\right\}$ is the covering of the compact part $X \setminus B_{\frac{1}{n+1}}(0,0)$, $\alpha_1 = \frac{1}{4}$ and for $n > 1$, $\alpha_n \leq min\left\{\frac{1}{2n(n+1)}, \frac{\delta_{n-1}}{2}\right\}$ and δ_{n-1} is the Lebesque number for the covering $\mathcal{U}_{n-1}^0$.

Now, if $B_{\alpha_{n+1}}(x) \in \mathcal{U}_{n+1}$ it follows that $B_{\alpha_{n+1}}(x) \subset U_n$ for some $U_n \in \mathcal{U}_n$. Namely, if an arbitrary ball $B_{\alpha_{n+1}}(x) \in \mathcal{U}_{n+1}$ is not contained in $B_{\frac{1}{n}}(0,0)$ then $B_{\alpha_{n+1}}(x)$ is contained in $X \setminus B_{\frac{1}{n+1}}(0,0)$ in which case we have $B_{\alpha_{n+1}}(x) \subseteq B_{\alpha_n}(x')$, for some $B_{\alpha_n}(x') \in \mathcal{U}_n^0 \subseteq \mathcal{U}_n$. So, $\mathcal{U}_{n+1} \prec \mathcal{U}_n$ i.e. $(\mathcal{U}_n)$ is a cofinal sequence from $Cov_F(X)$.

On the other hand, the coverings of Y are defined by:

$$\mathcal{V}_n = \left\{B_{\frac{1}{n}}(0,0)\right\} \cup \left\{B_{\beta_n}\left(y_1^n\right), B_{\beta_n}\left(y_2^n\right), ..., B_{\beta_n}\left(y_{l_n}^n\right)\right\}$$

where $\left\{B_{\beta_n}\left(y_1^n\right), B_{\beta_n}\left(y_2^n\right), ..., B_{\beta_n}\left(y_{l_n}^n\right)\right\}$ is the covering of the compact part $Y \setminus B_{\frac{1}{n+1}}(0,0)$, where $B_{\beta_n} = B_{\alpha_n} \cap Y$. We will denote by $U_n^\infty = \left\{B_{\frac{1}{n}}(0,0)\right\} \cap X$ and $V_n^\infty = \left\{B_{\frac{1}{n}}(0,0)\right\} \cap Y$.

It is not hard to see that the sequences $(\mathcal{U}_n)$, $(\mathcal{V}_n)$ are cofinal sequences from $Cov_F X$, $Cov_F Y$ respectively.

If we fix $n \in N$, then there exists $a_n \in R$ such that $(0, a_n) \in U_n^\infty$.

Now, for every $n \in N$ we will define functions $f_n : X \to Y$, $g_n : Y \to X$ as follows:

$$g_n(\overline{y}) = \overline{y} \text{ for all } \overline{y} \in Y$$

and

$$f_n(\overline{x}) = \begin{cases} \overline{x}, & if\ \overline{x} \in Y \\ (p_X(\overline{x}), 0), & if\ \overline{x} \notin Y \wedge \overline{x} \notin \{0\} \times (0,1] \\ (a_n, 0), & if\ \overline{x} \in \{0\} \times (0,1] \wedge p_Y(\overline{x}) > a_n \\ (p_Y(\overline{x}), 0), & if\ \overline{x} \in \{0\} \times (0,1] \wedge p_Y(\overline{x}) \leq a_n \end{cases},$$

where p_X, p_Y are projections of plane over vertical, horizontal axes, respectively.
It is not hard to prove that $(f_n)(g_n)\sim_p \underline{1_Y}$ and $(g_n)(f_n)\sim_p \underline{1_X}$ i.e. $Sh_{pF}(X) = Sh_{pF}(Y)$.

(1). f_n is a proper function:

Let $D_n \subset Y$ be compact. Without losing the generality we can assume that $D_n = \left(B_{\frac{1}{n}}(0,0)\right)^C \cap Y$.

i) If $(a_n, 0) \in D_n$, then there exists $b_n < a_n$ such that $(b_n, 0) \notin D_n$ and for the compact set $C_n = X \setminus B_{b_n}(0,0)$ we have:

$$f_n\left(X \setminus (X \setminus B_{b_n}(0,0))\right) = f_n\left(B_{b_n}(0,0)\right) \subset Y \setminus D_n$$

ii) If $(a_n, 0) \notin D_n$, in that case $C' = X \setminus B_{\frac{1}{n}}(0,0)$ is a compact subset of X and $f_n\left(X \setminus C'\right) \subset Y \setminus D_n$

(2). f_n is $\mathcal{V}_n$−continuous function:

It is clear that $f_n(x)$ is $\mathcal{V}_n$−continuous for $x \in X \setminus (\{0\} \times (0,1])$ as a projection. Now, let $\overline{x} \in \{0\} \times (0,1]$. We have the following situations:

i) If $p_Y(\overline{x}) > a_n$ then there exists a neighborhood $U_{\overline{x}} \times V_{\overline{x}}$ of $\overline{x}$ such that if some $y \in V_{\overline{x}}$ it will follow that $y > a_n$ and $f_n\left(U_{\overline{x}} \times V_{\overline{x}}\right) \subseteq B_{\frac{1}{n}}(0,0) \cap Y$.

ii) Let $p_Y(\overline{x}) \le a_n$. There exists a real number b_n, $0 < b_n < a_n$ and a neighborhood $U_{\overline{x}}$ of $p_X(\overline{x})$ such that $(0, b_n) \in B_{\frac{1}{n}}(0,0)$ and $f_n\left(U_{\overline{x}} \times (0, b_n)\right) \subseteq B_{\frac{1}{n}}(0,0) \cap Y$

Thus, f_n is $\mathcal{V}_n$−continuous function for all x of X.

(3). Homotopies:

For fixed natural number $n \in N$ we will construct a homotopy $H_n : X \times I \to X$ with property:

$$H_n(\overline{x}, 0) = 1_X \text{ and } H_n(\overline{x}, 1) = (g_n \circ f_n)(\overline{x}) \text{ for every } \overline{x} \in X$$

Let $\overline{x} \in Y$, in this case we define $H_n(\overline{x}, t) = \overline{x}$ for all $t \in [0,1]$,

If $\overline{x} \in X \setminus [(\{0\} \times (0,1]) \cup Y]$, then we define:

$$H_n(\overline{x}, t) = (1-t)\overline{x} + t \cdot p_X(\overline{x}) \text{ for all } t \in [0,1].$$

If $\overline{x} \in (\{0\} \times (0,1])$ then we have the following two cases:

i) For $p_Y(\overline{x}) > a_n$ we define:

$$H_n(\overline{x}, t) = (1-t)\overline{x} + t \cdot (0,0) \text{ for all } t \in [0,1) \text{ and } H_n(\overline{x}, 1) = (a_n, 0).$$

ii) For $p_Y(\overline{x}) \le a_n$ we define:

$$H_n(\overline{x}, t) = (1-t)\overline{x} + t \cdot (0,0) \text{ for all } t \in [0,1) \text{ and } H_n(\overline{x}, 1) = (p_Y(\overline{x}), 0).$$

It suffices to show that H_n is $\mathcal{V}_n$−continuous proper function in $X \times [0,1]$. Lets firstly show that for fixed n H_n is a proper function.

Let $D_n = Y \setminus B_{\frac{1}{n}}(0,0)$ be a compact subset of Y. The set $B_{a_n}(0,0) \cap X$ is subset of U_n^∞ and $H_n((B_{a_n}(0,0) \cap X) \times I) \subseteq B_{\frac{1}{n}}(0,0) \cap X$. Thus, if we choose the compact set $C_n = X \setminus (B_{a_n}(0,0)) \times I$ we have $H_n(X \setminus C_n) \subseteq Y \setminus D_n$ i.e H_n is proper.

Fix $n \in N$. The function H_n is continuous for all $\overline{x} \in X \setminus (\{0\} \times (0,1])$ as linear combination of continuous functions.

Let now $\overline{x_0} \in \{0\} \times (0,1]$.

i) If $p_Y(\overline{x_0}) > a_n$ then for the points $(\overline{x_0}, t)$ such that $t < 1$ the function H_n will be continuous again as linear combination of continuous functions. For the point $(\overline{x_0}, 1)$ we have $H_n(\overline{x_0}, 1) = f_n(\overline{x_0}) = (a_n, 0)$. From the definition of H_n it is not hard to see that there exists $t_0 \in (0,1)$ and a neighborhood $O_{\overline{x_0}}$ of $\overline{x_0}$ such that $H_n(\overline{x}, t) \in B_{\frac{1}{n}}(0,0)$ for $(\overline{x}, t) \in O_{\overline{x_0}} \times (t_0, 1]$.

We proved that $H_n(O_{\overline{x_0}} \times (t_0, 1]) \subseteq B_{\frac{1}{n}}(0,0) \cap X = U_n^\infty$.

i) If $p_Y(\overline{x_0}) \le a_n$, there exists b_n such that $p_Y(\overline{x_0}) < b_n < a_n$ and for the set $B_{b_n}(0,0) \cap X$ as a neighborhood of $\overline{x_0}$ we have:

$$H_n((B_{b_n}(0,0) \cap X) \times I) \subseteq B_{\frac{1}{n}}(0,0) \cap X = U_n^\infty .$$

For the opposite part, suppose that $Sh_p(X) = Sh_p(Y)$. Take $X = X_0 \cup X_1$ such that $X_0 = Y$ and $X_1 = \left\{\left(\frac{1}{n+1}, y\right) \middle| y \in (0,1], n \in N\right\} \cup [\{0\} \times (0,1]]$. From Theorem 5.2. there exists a disjoint union $Y = Y_0 \cup Y_1$ such that $Sh_p(X_i) = Sh_p(Y_i)$ for $i \in \{0,1\}$. Now, from $Sh_p(X_1) = Sh_p(Y_1)$ and from ([8] Proposition 4.9.) it follows that the spaces X_1, Y_1 have same number of ends.

From $Sh_p(X_0) = Sh_p(Y_0)$ and considering the fact that homotopy preserves connectivity it follows that Y_0 must be an interval with one end. Hence, $Y_1 = Y \setminus Y_0$ could not have infinitely many ends. □

Conclusion

In [5] it is proved that in the category Sh_{pF} the shape of product space is directly determined from the shape of component spaces. Also most of the important properties of the category of Sh_p are also valid for Sh_{pF}.

In this paper we proved that this category provides a coarser classification of spaces than the theory Sh_p.

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