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A note on quasi-versions of selection principles

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Abstract. In this paper, we introduce notions of hereditarily weakly selection principles and strongly quasi-selection principles and show that they are different from quasi-selection principles and weakly selection principles which are studied in [1]. By introducing the strongly quasi-separability, the quasi-separability and the weakly separability, we provide relations among these separable properties and weak versions of selection principles. These extend some results of G. Di Maio and Lj.D.R. Kočinac [1].

1. Introduction

Throughout the paper all spaces are assumed to be topological spaces. By $\mathbb{N}$ and $\mathbb{R}$ we denote the sets of natural numbers and real numbers. ω denotes the first infinite cardinal. The continuum is denoted by $\mathfrak{c}$. Most of undefined notion and terminology are as in [3].

Recall two very known selection principles defined in 1996 by M. Scheepers [5]. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of sets of an infinite set X.

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that $b_n \in A_n$ for each $n \in \mathbb{N}$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{\mathsf{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n : n \in \mathbb{N}\}$ such that B_n is a finite subset of A_n for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

G. Di Maio and Lj.D.R. Kočinac [1] defined the following quasi-versions of selection principles:

1. A space X is said to be *quasi-Rothberger* if for each closed set $F \subset X$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X there is a sequence $\{U_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $F \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}$.

2. A space X is said to be *quasi-Menger* if for each closed set $F \subset X$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subset \mathcal{U}_n$ and $F \subset \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n}$.

For a space X and a subset F of X, we denote:

- $\mathcal{O}_F = \{\mathcal{U} : \mathcal{U}$ is a cover of F by sets open in $X\}$;
- $\mathcal{O}_F^D = \{\mathcal{U} : \mathcal{U}$ is a family of open subsets of X such that $F \subset \overline{\bigcup \mathcal{U}}\}$.

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So, a space X is quasi-Rothberger (resp., quasi-Menger) if and only if each closed subset F of X satisfies $\mathsf{S}_1(\mathcal{O}_F, \mathcal{O}_F^D)$ (resp., $\mathsf{S}_{\mathsf{fin}}(\mathcal{O}_F, \mathcal{O}_F^D)$).

We denote

- L : Lindelöf;
- qL : quasi-Lindelöf;
- wL : weakly Lindelöf;
- H : Hurewicz;
- qH : quasi-Hurewicz;
- wH : weakly Hurewicz;
- M : Menger;
- qM : quasi-Menger;
- wM : weakly Menger;
- R : Rothberger;
- qR : quasi-Rothberger;
- wR : weakly Rothberger;
- GN : Gerlits-Nagy;
- qGN : quasi-Gerlits-Nagy;
- wGN : weakly Gerlits-Nagy.

In [1] the authors established the following implications.

$$
\begin{array}{ccccc}
\mathsf{L} & \longrightarrow & \mathsf{qL} & \longrightarrow & \mathsf{wL} \\
\uparrow & & \uparrow & & \uparrow \\
\mathsf{H} & \longrightarrow & \mathsf{qH} & \longrightarrow & \mathsf{wH} \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{M} & \longrightarrow & \mathsf{qM} & \longrightarrow & \mathsf{wM} \\
\uparrow & & \uparrow & & \uparrow \\
\mathsf{R} & \longrightarrow & \mathsf{qR} & \longrightarrow & \mathsf{wR} \\
\uparrow & & \uparrow & & \uparrow \\
\mathsf{GN} & \longrightarrow & \mathsf{qGN} & \longrightarrow & \mathsf{wGN} \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{L} & \longrightarrow & \mathsf{qL} & \longrightarrow & \mathsf{wL}
\end{array}
$$

Diagram 1

In this paper, we introduce hereditarily weakly selection principles and strongly quasi-selection principles (Rows 1-2 in **Diagram** 2) stronger than quasi-selection principles and weakly selection principles (Rows 3-4 in **Diagram** 2 or Columns 2-3 in **Diagram** 1) and investigate the relationships among these selection principles. We also introduce the strongly quasi-separability, the quasi-separability and the weakly separability (Column 1 in **Diagram** 2) and the weak π-base in order to obtain characterizations of these weak selection principles. We give the following implications.

$$\begin{array}{ccccccc}
\mathsf{hs} & \longrightarrow & \mathsf{hwR} & \longrightarrow & \mathsf{hwM} & \longrightarrow & \mathsf{cs} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{sqs} & \longrightarrow & \mathsf{sqR} & \longrightarrow & \mathsf{sqM} & \longrightarrow & \mathsf{CCC} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{qs} & \longrightarrow & \mathsf{qR} & \longrightarrow & \mathsf{qM} & \longrightarrow & \mathsf{q\text{-}1\text{-}s\text{-}L} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{ws} & \longrightarrow & \mathsf{wR} & \longrightarrow & \mathsf{wM} & \longrightarrow & \mathsf{1\text{-}s\text{-}L}
\end{array}$$

Diagram 2 : All cases

This paper is organized as follows. In Section 2, we introduce weakly dense sets and the quasi-separability (qs) to characterize quasi-selection principles (Row 3 in **Diagram** 2). In Section 3, we introduce the strongly quasi-separability (sqs) and study strongly quasi-selection principles (Row 2 in **Diagram** 2). In Section 4, we introduce the weakly separability (ws) to study weakly selection principles (Row 4 in **Diagram** 2). In Section 5, by the hereditarily separability (hs), we study hereditarily weakly selection principles (Row 1 in **Diagram** 2). In Section 6, in order to complete the **Diagram** 2, we compare the hereditarily separability, the strongly quasi-separability, the quasi-separability and the weakly separability (Column 1 in **Diagram** 2) and point that these separable properties are different.

2. Quasi-selection principles

Definition 2.1. A subset D of X is said to be *weakly dense* in X, if for every open neighborhood assignment $\{U_x : x \in D\}$, then $\bigcup_{x\in D} U_x$ is dense in X.

Definition 2.2. A space X is said to be *quasi-separable* if each closed subspace of X has a countable weakly dense subset.

Obviously, each dense subset of X is weakly dense in X.

Theorem 2.3. *If X is a quasi-separable space, then X is quasi-Rothberger.*

Proof. Let F be a closed subset of X and $\{x_n : n \in \mathbb{N}\}$ be a countable weakly dense subset of F. If $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open covers of F by sets open in X, then $\mathcal{V}_n = \{U \cap F : U \in \mathcal{U}_n\}$ is an open cover of F. Take $U_n \cap F \in \mathcal{V}_n$ for each $n \in \mathbb{N}$ such that $x_n \in U_n \cap F$. Let τ_F be the subspace topology of F. Since X is quasi-separable, then $F = \mathrm{Cl}_{\tau_F}(\bigcup_{n\in\mathbb{N}} U_n \cap F) \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}$. So X is quasi-Rothberger. □

The converse of Theorem 2.3 is not true.

Example 2.4. ([6]) There is a quasi-Rothberger space which is not quasi-separable.

Proof. Let X be an uncountable set and $X^* = X \bigcup\{\infty\}$, where $\infty \notin X$. Endow X^* with the following topology τ^*:

$$\tau^* = \{V \cup \{\infty\} : V \subset X\} \bigcup \{\emptyset\}.$$

Then (X^*, τ^*) is quasi-Rothberger but it is not quasi-separable. Indeed, let F be a closed subspace of X^*, $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of covers of F by sets open in X^*. Pick any $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$, then $\infty \in U_n$. For each $x \in F$ and any neighborhood U_x of x, then $\infty \in U_x$. Hence $U_x \cap (\bigcup_{n\in\mathbb{N}} U_n) \neq \emptyset$. Thus $F \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}$. So X^* is quasi-Rothberger. X is a closed uncountable discrete subspace of X^*. So X^* is not quasi-separable. □

Recall that a space X is said to be *hereditarily separable* if each subspace of X is separable. A hereditarily separable space is quasi-separable. The converse is not true.

Example 2.5. ([6]) There is a quasi-separable space which is not hereditarily separable.

Proof. Let $\mathbb{R}$ be real line with usual topology τ, we denote

$$\mathcal{B} = \{V - A : V \in \tau, A \subset \mathbb{R}, |A| \leq \omega\}.$$

The collection $\mathcal{B}$ is a base for a new topology τ' on $\mathbb{R}$.

1. $(\mathbb{R}, \tau')$ is quasi-separable. Let F be a τ'-closed subset of $\mathbb{R}$ and take a countable τ-dense subset $D_F = \{x_n : n \in \mathbb{N}\}$ of F since $(\mathbb{R}, \tau)$ is hereditarily separable. If U_n is a τ'-open neighborhood of x_n for each $n \in \mathbb{N}$, then $F \subset \mathrm{Cl}_{\tau'}(\bigcup_{n\in\mathbb{N}} U_n)$. In fact, let $x \in F$, $U_x = V_x - A_x$ be a τ'-open neighborhood of x, where V_x is a τ-open subset of $\mathbb{R}$ and $|A_x| \leq \omega$. Take $x_{n_0} \in D_F \cap V_x$ and $U_{n_0} = V_{n_0} - A_{n_0} \in \tau'$, then $x_{n_0} \in V_{n_0} \cap V_x \neq \emptyset$, where V_{n_0} is a τ-open subset of $\mathbb{R}$, $|A_{n_0}| \leq \omega$. So $U_x \cap U_{n_0} \neq \emptyset$. Otherwise, $(V_x - A_x) \cap (V_{n_0} - A_{n_0}) = \emptyset$, then $(V_x \cap V_{n_0}) - (A_x \cup A_{n_0}) = \emptyset$. But $|V_{n_0} \cap V_x| > \omega$, and $|A_{n_0} \cup A_x| \leq \omega$. This is a contradiction.

2. For any countable set D, since $\mathbb{R} \backslash D$ is non-empty open and disjoint from D, then D is not dense in $(\mathbb{R}, \tau')$. So $(\mathbb{R}, \tau')$ is not hereditarily separable. □

Remark 2.6. Example 2.5 show that the quasi-separability is weaker than the hereditarily separability. Thus Theorem 2.3 improves Proposition 2.2 of [1].

Recall that a space X is said to be *1-star-Lindelöf* [2] if for every open cover $\mathcal{U}$ of X, there exists a countable subset $\mathcal{V} \subset \mathcal{U}$ such that $X = \mathrm{st}(\bigcup \mathcal{V}, \mathcal{U})$, where $\mathrm{st}(\bigcup \mathcal{V}, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap (\bigcup \mathcal{V}) \neq \emptyset\}$. So we give the following definition.

Definition 2.7. A space X is said to be *quasi-1-star-Lindelöf* if for each clopen subset F of X and every cover $\mathcal{U}$ of F by sets open in X, there exists a countable subset $\mathcal{V} \subset \mathcal{U}$ such that $F \subset \mathrm{st}(\bigcup \mathcal{V}, \mathcal{U})$.

Obviously, each quasi-1-star-Lindelöf space is 1-star-Lindelöf.

Theorem 2.8. *If X is a quasi-Menger space, then X is quasi-1-star-Lindelöf.*

Proof. Suppose that $\mathcal{U}$ is a cover of clopen subset $F \subset X$ by sets open in X. Let $\mathcal{U}_n = \mathcal{U}$, then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open covers of F. There exists a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ such that $F \subset \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n}$. Let $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$, then $\mathcal{V}$ is countable. Thus $F \subset \mathrm{st}(\bigcup \mathcal{V}, \mathcal{U})$. In fact, let $x \in F$, there exists $U \in \mathcal{U}$ such that $x \in U$. There exist $n_0 \in \mathbb{N}$ and $V \in \mathcal{V}_{n_0}$ such that $U \cap V \neq \emptyset$. Thus $x \in U \subset \mathrm{st}(\bigcup \mathcal{V}_{n_0}, \mathcal{U}) \subset \mathrm{st}(\bigcup \mathcal{V}, \mathcal{U})$. So X is quasi-1-star-Lindelöf. □

We denote

- qs : quasi-separable;
- qR : quasi-Rothberger;
- qM : quasi-Menger;
- q-1-s-L : quasi-1-star-Lindelöf.

So we have Row 3 of **Diagram** 2.

$$\mathsf{qs} \overset{\not\leftarrow}{\longrightarrow} \mathsf{qR} \longrightarrow \mathsf{qM} \longrightarrow \mathsf{q\text{-}1\text{-}s\text{-}L}$$

Diagram 3 : Quasi-selection principle case.

Recall that a *π-base* [4] of X is a family $\mathcal{V}$ of non-empty open subsets in X such that for each non-empty open subset U of X, there exists $V \in \mathcal{V}$ such that $V \subset U$. The π-weight of X, denoted $\pi w(X)$, is defined as follows:

$$\pi w(X) = \omega + \min\{|\mathcal{V}| : \mathcal{V} \text{ is a } \pi\text{-base of } X\}.$$

In order to give a new characterization of the quasi-selection principles, we define weak π-bases weaker than π-bases.

Definition 2.9. A family $\mathcal{V}$ ($X \notin \mathcal{V}$) of non-empty open subsets of X is said to be a *weak π-base* of X, if for each non-empty open subset U of X, there exists $V \in \mathcal{V}$ such that $V \cap U \neq \emptyset$.

The *weak π-weight of X*, denoted $w\pi w(X)$, is defined as follows:

$$w\pi w(X) = \omega + \min\{|\mathcal{V}| : \mathcal{V} \text{ is a weak } \pi\text{-base of } X\}.$$

Note that $w\pi w(X) \leq d(X) \leq \pi w(X)$, where $d(X)$ denotes the density of X.

Example 2.10. There is a space X such that $w\pi w(X) < d(X)$.

Proof. Let $\mathbb{R}$ be endowed with discrete topology, we denote

$$\mathcal{V} = \{(x, y) : x < y, x, y \in \mathbb{Q}\}, \text{where } \mathbb{Q} \text{ is the set of rational numbers.}$$

Then $\mathcal{V}$ is a countable weak π-base for discrete topology on $\mathbb{R}$. So $w\pi w(\mathbb{R}) = \omega$. But $d(\mathbb{R}) = \mathfrak{c} > w\pi w(\mathbb{R})$. □

Definition 2.11. Let $F \subseteq X$. A family $\mathcal{V}$ of open subsets of X is said to be a weak π-base on F, if for each open subset U of X with $U \cap F \neq \emptyset$, there exists $V \in \mathcal{V}$ such that $V \cap U \neq \emptyset$.

Note that for a subset $F \subseteq X$, if $F = X$, then a weak π-base of F and a weak π-base on F are the same; if $F \subsetneq X$, then a weak π-base of F and a weak π-base on F are different.

Lemma 2.12. *A family $\mathcal{V}$ of open subsets of X is a weak π-base on $F \subseteq X$ if and only if $F \subseteq \overline{\bigcup \mathcal{V}}$.*

Proof. Suppose $\mathcal{V}$ is a weak π-base on F, let $x \in F$, and suppose U is an open subset of X with $x \in U$. Since $U \cap F \neq \emptyset$, there is some $V \in \mathcal{V}$ so that $V \cap U \neq \emptyset$. That is, $U \cap \bigcup \mathcal{V} \neq \emptyset$. Since U was arbitrary, $x \in \overline{\bigcup \mathcal{V}}$. So $F \subseteq \overline{\bigcup \mathcal{V}}$.

Suppose $F \subseteq \overline{\bigcup \mathcal{V}}$ and let U be an open subset of X with $U \cap F \neq \emptyset$. Now, observe that it must be the case that $U \cap \bigcup \mathcal{V} \neq \emptyset$, so there is some $V \in \mathcal{V}$ with $V \cap U \neq \emptyset$. □

By Lemma 2.12, we can obtain the following Theorems 2.13-2.14.

Theorem 2.13. *For a space X, the following are equivalent:*

(1) X is quasi-Rothberger;

(2) For each closed subset F of X and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X, there exists $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \mathbb{N}\}$ is a weak π-base on F.

Theorem 2.14. *For a space X, the following are equivalent:*

(1) X is quasi-Menger;

(2) For each closed subset F of X and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X, there exists a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\{\bigcup \mathcal{V}_n : n \in \mathbb{N}\}$ is a weak π-base on F.

3. Strongly quasi-selection principles

Definition 3.1. A space X is said to be *strongly quasi-separable* if each subspace of X has a countable weakly dense subset.

Definition 3.2. A space X is said to be

1. *strongly quasi-Rothberger* if for each subset $F \subset X$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X, there exists $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $F \subset \overline{\bigcup_{n \in \mathbb{N}} U_n}$.

2. *strongly quasi-Menger* if for each subset $F \subset X$ and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X there exists a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $F \subset \overline{\bigcup_{n \in \mathbb{N}} \bigcup \mathcal{V}_n}$.

Theorem 3.3. *If X is a strongly quasi-separable space, then X is strongly quasi-Rothberger.*

Theorem 3.4. *If X is a hereditarily weakly Rothberger (resp., hereditarily weakly Menger) space, then X is strongly quasi-Rothberger (resp., strongly quasi-Menger).*

Proof. We only show the case of the hereditarily weakly Rothberger. Suppose that $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of covers of a subset F by sets open in X, then $\mathcal{V}_n = \{U \cap F : U \in \mathcal{U}_n\}$ is an open cover of subspace F of X. There exists a $U_n \cap F \in \mathcal{V}_n$ such that $F = \mathrm{Cl}_{\tau_F}(\bigcup_{n\in\mathbb{N}}(U_n \cap F))$. So $F \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}$. Indeed, let $x \in F$ and U_x be an open neighborhood of x in X. There exists $n_0 \in \mathbb{N}$ such that $(U_x \cap F) \cap (U_{n_0} \cap F) \neq \emptyset$. Thus $U_x \cap U_{n_0} \neq \emptyset$. So X is strongly quasi-Rothberger. □

Obviously, every strongly quasi-Rothberger (resp., strongly quasi-Menger) space is quasi-Rothberger (resp., quasi-Menger). Hence we obtain the following implications (Column 2 of **Diagram** 2). Note that Example 3.5-3.6 and Example 4.5 show that each converse of the implications is not true.

$$\mathsf{hwR} \xrightarrow{\not\leftarrow} \mathsf{sqR} \xrightarrow{\not\leftarrow} \mathsf{qR} \xrightarrow{\not\leftarrow} \mathsf{wR}$$

Diagram 4 : Rothberger case.

Example 3.5. ([6]) There is a quasi-Rothberger space which is not strongly quasi-Rothberger.

Proof. Let $X = [0, 1]$, we denote

$$\tau = \{[0, 1]\} \bigcup \{V : V \subset (0, 1]\}.$$

Then τ is a topology on X. (X, τ) is quasi-Rothberger but is not strongly quasi-Rothberger. In fact, for each closed subspace F of X, we have $0 \in F$. Since $\{0\}$ is a weak dense subset of F, then X is quasi-separable. So X is quasi-Rothberger. But $(0, 1]$ is an uncountable discrete open subset of X. So X is not strongly quasi-Rothberger. □

Example 3.6. There is a strongly quasi-Rothberger space which is not hereditarily weakly Rothberger.

Proof. Let X be the subset of the plane $\mathbb{R}^2$ defined by $y \geq 0$, i.e., $X = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. Endow X with the following topology τ: For every point $P(x, y) \in X$, let

$$U_P = \{(x', y') \in X : |x' - x| \leq y' - y\}.$$

The family $\{U_P : P \in X\}$ is a base of X for the topology τ.

(X, τ) is strongly quasi-Rothberger. In fact, suppose that F is a subset of X and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a family of open covers of F by sets open in X. Pick $D_F = \{x_n : n \in \mathbb{N}\}$ being a countable subset of F. Take $U_n \in \mathcal{U}_n$ such that $x_n \in U_n$ for each $n \in \mathbb{N}$, then $F \subset \overline{\bigcup_{n\in\mathbb{N}} U_n}$. Thus X is a strongly quasi-Rothberger space.

(X, τ) is not hereditarily weakly Rothberger. In fact, pick subset $X_1 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ of X, then X_1 is uncountable and its subspace topology τ_{X_1} is discrete. So (X_1, τ_{X_1}) is not weakly Rothberger. Thus (X, τ) is not hereditarily weakly Rothberger. □

Recall that a space X satisfies *CCC* (*countable chain condition*) [2,4] if each pairwise disjoint collection of non-empty open subsets of X is countable.

From Fig. 1 of [4], we have that if a space X has countable spread, then X satisfies *CCC*.

Theorem 3.7. *If X satisfies CCC, then X is quasi-1-star-Lindelöf.*

Proof. Let $\mathcal{U}$ be a cover of a clopen subset F of X by sets open in X, then $\mathcal{U} \bigcup \{X - F\}$ is an open cover of X. There exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup(\mathcal{U} \bigcup \{X - F\}) \subseteq \overline{\bigcup \mathcal{V} \bigcup \{X - F\}} = \overline{\bigcup \mathcal{V}} \bigcup \{X - F\}$ [4, Proposition 3.4]. Then $F \subset \overline{\bigcup \mathcal{V}}$. For each $x \in F$, take $U \in \mathcal{U}$ such that $x \in U$. Then $U \cap (\bigcup \mathcal{V}) \neq \emptyset$. Thus $F \subseteq \mathrm{st}(\bigcup \mathcal{V}, \mathcal{U})$. So X is quasi-1-star-Lindelöf. □

Question 3.8. *Whether Theorem 3.7 would still hold if the clopenness in the definition of quasi-1-star-Lindelöf can be weakened to just closed subspaces?*

Thus we obtain Column 4 of **Diagram** 2.

Theorem 3.9. *If X is a strongly quasi-Menger space, then X satisfies CCC.*

Proof. Suppose that $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ is a pairwise disjoint family of non-empty open subsets of X. Pick $x_\alpha \in U_\alpha$ for each $\alpha \in \Lambda$ and let $F = \{x_\alpha : \alpha \in \Lambda\}$. Let $\mathcal{U}_n = \mathcal{U}$ for each $n \in \mathbb{N}$, then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open covers of subset F of X. Thus there exists a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$, where $\mathcal{V}_n = \{U_\alpha : \alpha \in \Lambda_n\}$ with $|\Lambda_n| < \omega$ such that $F \subset \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n}$. If $|\Lambda| > \omega$, then there exists $\alpha_0 \in \Lambda - \bigcup_{n\in\mathbb{N}} \Lambda_n$. Since U_{α_0} is a neighborhood of x_{α_0}, then $U_{\alpha_0} \cap (\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n) \neq \emptyset$. There exists $U_\alpha \in \mathcal{V}_n$ for some $n \in \mathbb{N}$ such that $U_{\alpha_0} \cap U_\alpha \neq \emptyset$, a contradiction. Hence $|\Lambda| \leq \omega$. So X is CCC. □

We denote

- sqs : strongly quasi-separable;
- sqR : strongly quasi-Rothberger;
- sqM : strongly quasi-Menger;
- CCC : countable chain condition.

Hence we have Row 2 of **Diagram** 2.

$$\text{sqs} \longrightarrow \text{sqR} \longrightarrow \text{sqM} \longrightarrow \text{CCC}$$

Diagram 5 : Strongly quasi-selection principle case.

Similarly, we can prove:

Theorem 3.10. *For a space X, the following are equivalent:*
(1) X is strongly quasi-Rothberger;
(2) For each subset F of X and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X, there exists $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \mathbb{N}\}$ is a weak π-base on F.

Theorem 3.11. *For a space X, the following are equivalent:*
(1) X is strongly quasi-Menger;
(2) For each subset F of X and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of F by sets open in X, there exists a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\{\bigcup \mathcal{V}_n : n \in \mathbb{N}\}$ is a weak π-base on F.

4. Weakly selection principles

Definition 4.1. A space X is said to be *weakly separable* if X has a countable weakly dense subset.

Definition 4.2. ([1]) A space X is said to be
1. *weakly Rothberger* if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists $U_n \in \mathcal{U}_n$ such that $X = \overline{\bigcup_{n\in\mathbb{N}} U_n}$.
2. *weakly Menger* if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ such that $X = \overline{\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n}$.

Theorem 4.3. *If X is a weakly separable space, then X is weakly Rothberger.*

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be a countable weakly dense subset of X and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. Take $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $x_n \in U_n$. Then $X = \overline{\bigcup_{n\in\mathbb{N}} U_n}$ since X is a weakly separable space. So X is weakly Rothberger. □

Example 4.4. ([6]) There is a weakly Rothberger space which is not weakly separable.

Proof. Let X be an uncountable set and $X^* = X \bigcup \{\infty\}$, where $\infty \notin X$. Endow X^* with the following topology τ^*:

$$\tau^* = \{X^* - A : A \in [X]^{\leq\omega}\} \bigcup \{U : U \subseteq X\}.$$

Then

1. (X^*, τ^*) is weakly Rothberger. In fact, let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X^*. Pick $U_1 = X^* - A \in \mathcal{U}_1$ such that $\infty \in U_1$, where $A = \{x_n : n \in \mathbb{N}\}$. Choose $U_{n+1} \in \mathcal{U}_{n+1}$ for each $n \in \mathbb{N}$ such that $x_n \in U_{n+1}$. Then $X^* = \bigcup_{n\in\mathbb{N}} U_n$. Thus (X^*, τ^*) is Rothberger. So (X^*, τ^*) is weakly Rothberger.

2. (X^*, τ^*) is not weakly separable. Let $C = \{c_n : n \in \mathbb{N}\}$ be any countable subset of X^*. (i) If $\infty \in C$, we can put $c_1 = \infty$. Let $B = \{b_n : n \in \mathbb{N}\} \subset X$ with $B \cap C = \emptyset$. Take $U_1 = X^* - B$ and $U_{n+1} = \{c_{n+1}\}$ for $n \in \mathbb{N}$. Then U_n is an open neighborhood of c_n for each $n \in \mathbb{N}$. Thus $X^* \neq \overline{\bigcup_{n\in\mathbb{N}} U_n}$ since each $b_n \notin \overline{\bigcup_{n\in\mathbb{N}} U_n}$. (ii) If $\infty \notin C$, take $U_n = \{c_n\}$ for each $n \in \mathbb{N}$, then U_n is an open neighborhood of c_n. Thus $X^* \neq \overline{\bigcup_{n\in\mathbb{N}} U_n}$ since $\infty \notin \overline{\bigcup_{n\in\mathbb{N}} U_n}$. So (X^*, τ^*) is not weakly separable. □

Example 4.5. There is a weakly Rothberger space which is not quasi-Rothberger.

Proof. Such a space is described in [1, Example 2.10], where it is shown that X is almost Rothberger but it is not quasi-Rothberger. Thus X is weakly Rothberger since the weakly Rothberger is weaker than the almost Rothberger. □

Similar to Theorem 2.8, one can prove:

Theorem 4.6. *If X is a weakly-Menger space, then X is 1-star-Lindelöf.*

We denote

- ws : weakly separable;
- wR : weakly Rothberger;
- wM : weakly Menger;
- 1-s-L : 1-star-Lindelöf.

Hence we have Row 4 of **Diagram** 2.

$$\mathsf{ws} \xrightarrow{\not\leftarrow} \mathsf{wR} \longrightarrow \mathsf{wM} \longrightarrow \mathsf{1\text{-}s\text{-}L}$$

Diagram 6 : Weakly selection principle case.

Similarly, one proves:

Theorem 4.7. *For a space X, the following are equivalent:*

(1) X is weakly-Rothberger;

(2) For each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \mathbb{N}\}$ is a weak π-base of X.

Theorem 4.8. *For a space X, the following are equivalent:*

(1) X is weakly-Menger;

(2) For each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists a finite subset $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\{\bigcup \mathcal{V}_n : n \in \mathbb{N}\}$ is a weak π-base of X.

5. Hereditarily weak selection principles

Definition 5.1. *A space X is said to be*

1. hereditarily weakly Rothberger *if each subspace of X is weakly Rothberger;*

2. hereditarily weakly Menger *if each subspace of X is weakly Menger.*

Theorem 5.2. *If X is a hereditarily separable space, then X is hereditarily weakly Rothberger.*

Proof. Assume that F is a subspace of X and $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of F. If $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open subset covers of F, take $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $x_n \in U_n$. Then $F = \mathrm{Cl}_{\tau_F}(\bigcup_{n\in\mathbb{N}} U_n)$, where τ_F is the topology of F. So X is hereditarily weakly Rothberger. □

The *spread of X*, denoted $s(X)$, is defined as follows [4]:

$$s(X) = \omega + \sup\{|D| : D \subset X, D \text{ is discrete}\}.$$

Theorem 5.3. *If X is a hereditarily weakly Menger space, then X has countable spread, i.e.,* $s(X) = \omega$.

Proof. Let $F = \{x_\alpha : \alpha \in \Lambda\}$ be a discrete subset of X. Take $\mathcal{U} = \{\{x_\alpha\} : \alpha \in \Lambda\}$ and $\mathcal{U}_n = \mathcal{U}$, then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open subset covers of discrete subspace F. There exists a finite $\Lambda_n \subset \Lambda$ for each $n \in \mathbb{N}$ such that $F = \mathrm{Cl}_{\tau_F}(\bigcup_{n\in\mathbb{N}} \bigcup_{\alpha\in\Lambda_n}\{x_\alpha\}) = \bigcup_{n\in\mathbb{N}} \bigcup_{\alpha\in\Lambda_n}\{x_\alpha\}$. Then $|F| = |\bigcup_{n\in\mathbb{N}} \bigcup_{\alpha\in\Lambda_n}\{x_\alpha\}| = \omega$. So $s(X) = \omega$. □

We denote

- hs : hereditarily separable;
- hwR : hereditarily weakly Rothberger;
- hwM : hereditarily weakly Menger;
- cs : countable spread.

Thus we have Row 1 of **Diagram** 2.

$$\mathsf{hs} \longrightarrow \mathsf{hwR} \longrightarrow \mathsf{hwM} \longrightarrow \mathsf{cs}$$

Diagram 7 : Hereditarily weak selection principle case.

Similar to Theorems 2.13-2.14, one can prove:

Theorem 5.4. *For a space X, the following are equivalent:*

(1) X is hereditarily weakly Rothberger;

(2) For each subspace F of X and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ *of open covers of F, there exists* $U_n \in \mathcal{U}_n$ *such that* $\{U_n : n \in \mathbb{N}\}$ *is a* π*-base of F.*

Theorem 5.5. *For a space X, the following are equivalent:*

(1) X is hereditarily weakly Menger;

(2) For each subspace F of X and each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ *open covers of F, there exists a finite subset* $\mathcal{V}_n \subset \mathcal{U}_n$ *such that* $\{\bigcup \mathcal{V}_n : n \in \mathbb{N}\}$ *is a weak* π*-base of F.*

6. Remarks on separable properties

It is easy to show following implications about the separable property. We point the converse of each implication is not true.

$$\mathsf{hs} \xrightarrow{\not\leftarrow} \mathsf{sqs} \xrightarrow{\not\leftarrow} \mathsf{qs} \xrightarrow{\not\leftarrow} \mathsf{ws} \xleftarrow{\not\rightarrow} \mathsf{s}$$

Diagram 8 : Separability case.

Example 6.1. There is a strongly quasi-separable space which is not hereditarily separable.

Proof. In fact, in Example 2.5, the space $(\mathbb{R}, \tau')$ is strongly quasi-separable, but it is not hereditarily separable. Example 2.5 also means that a strongly quasi-separable space need not be separable. □

Example 6.2. *There is a quasi-separable space which is not strongly quasi-separable.*

Proof. Example 3.5 is a quasi-separable space and not strongly quasi-separable. □

Example 6.3. There is a weakly separable space which is not quasi-separable.

Proof. In Example 2.4, it is shown that (X^*, τ^*) is not quasi-separable. Since $\{\infty\}$ is a weakly dense subset of (X^*, τ^*), then (X^*, τ^*) is weakly-separable. □

Example 6.4. *There is a weakly separable space which is not separable.*

Proof. Let X be an uncountable set endowed with countable complement topology, then X is weakly separable and not separable. □

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References

[1] G. Di Maio, Lj. D. R. Kočinac, *A note on quasi–Menger and similar spaces* Topol. Appl. **179** (2015), 148–155.
[2] E. K. van Douwen, G. M. Reed, A. W. Roscoe, I. J. Tree, *Star covering properties*, Topol. Appl. **39** (1991), 71–103.
[3] R. Engelking, *General Topology* (revised and completed edition), Heldermann, 1989.
[4] R. Hodel, *Cardinal function* I, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., 1984, 3–61.
[5] M. Scheepers, *Combinatorics of open covers* I: *Ramsey theory*, Topol. Appl. **69** (1996), 31–62.
[6] L. A. Steen, J. A. Seebach, *Counterexamples in Topology*, Dover Publications, 1995.