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Further refinement of Young's type inequalities and its reversed using the Kantorovich constants

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Abstract. In this paper, we show a multiple-term refinement of Young's type inequality and its reverse via the Kantorovich constants, which extends and unifies two recent and important results due to L. Nasiri et al. (Result. Math (74), 2019), and C. Yang et al. (Journal. Math. Inequalities, (14), 2020). An application of these scalars results we give a multiple-term refinement of Young's type inequalities for operators, Hilbert-Schmidt norms, traces and the unitarily invariant norms.

1. Introduction and preliminaries

1.1. On (the scalar) Young's type inequality and its refinements

We start by reviewing some important facts concerning some Young's type inequality.

The famous Young inequality (for scalars) says that if $a, b > 0$ and $0 \leq \alpha \leq 1$, then

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b. \tag{1}$$

Even though this inequality looks very simple, it is of great interest in operator theory. Refining this inequality has taken the attention of many researchers in the field, where adding a positive term to the left side is possible.

The first refinements of Young's inequality is the squared version proved in [3] as follows

$$(a^{\alpha}b^{1-\alpha})^2 + r_0^2(a-b)^2 \leq (\alpha a + (1-\alpha)b)^2, \tag{2}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Later, Kittaneh and Manasrah [6], obtained the other interesting refinement of Young's inequality

$$a^{\alpha}b^{1-\alpha} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \alpha a + (1-\alpha)b, \tag{3}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

J. Zhao and J. Wu [12], obtained the following refinement of inequality (1) as follows:

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1. If $0 < \alpha \leq \frac{1}{2}$, then

$$a^{\alpha}b^{1-\alpha} + \alpha(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{b})^2 \leq \alpha a + (1-\alpha)b. \tag{4}$$

2. If $\frac{1}{2} < \alpha \leq 1$, then

$$a^{\alpha}b^{1-\alpha} + (1-\alpha)(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{a})^2 \leq \alpha a + (1-\alpha)b, \tag{5}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$ and $r_1 = \min\{2r_0, 1-2r_0\}$.

Throughout this paper, we will us the notation $r_0 = \min\{\alpha, 1-\alpha\}$, $r_n = \min\{2r_{n-1}, 1-2r_{n-1}\}$, $R_0 = \max\{\alpha, 1-\alpha\}$, and $K\big(t, 2\big) := \frac{(1+t)^2}{4t}$ the Kantorovich constant and for $l, k \in \mathbb{N}$, we define the functions $f_{l,k}(x, y)$ by

$$f_{l,k}(x, y) = \Big(\sqrt{x^{\frac{k-1}{2^l}}y^{1-\frac{k-1}{2^l}}} - \sqrt{x^{\frac{k}{2^l}}y^{1-\frac{k}{2^l}}}\Big)^2.$$

In [2], D. Choi showed a multiple-terms refinement of Young's inequality as follows:

Theorem 1.1. *Let a and b be two positive numbers and* $0 < \alpha < 1$. *Then we have*

$$\begin{aligned} K\Big(\sqrt[2^{N-1}]{h}, 2\Big)^{r_N(\alpha)} a^{\alpha}b^{1-\alpha} \quad &+ \quad \sum_{l=0}^{N-1} r_l(\alpha) \sum_{k=1}^{2^l} f_{l,k}(a,b)\chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha) \\ &\leq \quad \alpha a + (1-\alpha)b. \end{aligned} \tag{6}$$

where $h = \frac{a}{b}$.

In a recent work, Kai [5] gave the following Young type inequality:

$$\Big[(\alpha a)^{2\alpha}b^{2-2\alpha}\chi_{(0,\frac{1}{2}]}(\alpha) + a^{2\alpha}[(1-\alpha)b]^{2-2\alpha}\chi_{(\frac{1}{2},1)}(\alpha)\Big] + r_0^2(a-b)^2 \leq \alpha^2a^2 + (1-\alpha)^2b^2, \tag{7}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$ and $\chi_I(\alpha)$ the charcteristic function.

Recently, Nasiri et al. [10] obtained the following reverses refinement of inequality (7) using the Kantorovich constants as follows

Theorem 1.2. *Let a and b be two positive numbers and* $0 < \alpha < 1$. *Then we have*

(1) If $0 < \alpha < \frac{1}{2}$, *then*

$$\alpha^2a^2 + (1-\alpha)^2b^2 \leq K(h, 2)\Big)^{-r} a^{2\alpha}[(1-\alpha)b]^{2-2\alpha} + (1-\alpha)^2(a-b)^2, \tag{8}$$

where $r = \min\{2\alpha, 1-2\alpha\}$ *and* $h = \frac{(1-\alpha)b}{a}$.

(2) If $\frac{1}{2} \leq \alpha < 1$, *then*

$$\alpha^2a^2 + (1-\alpha)^2b^2 \leq K(h, 2)\Big)^{-r} (\alpha a)^{2\alpha}b^{2-2\alpha} + \alpha^2(a-b)^2, \tag{9}$$

where $r = \min\{2\alpha - 1, 2-2\alpha\}$ *and* $h = \frac{\alpha a}{b}$.

The same authors in [8] showed the following refinement of inequality (7).

Theorem 1.3. *Let a and b be two positive numbers and* $0 < \alpha < 1$. *We have*

(1) If $0 < \alpha < \frac{1}{2}$, *then*

$$(\alpha a)^{2\alpha}b^{2-2\alpha} + \alpha^2(a-b)^2 + rb(\sqrt{\alpha a} - \sqrt{b})^2 \leq \alpha^2a^2 + (1-\alpha)^2b^2, \tag{10}$$

where $r = \min\{2\alpha, 1 - 2\alpha\}$.

(2) *If* $\frac{1}{2} \leq \alpha < 1$, *then*

$$a^{2\alpha}[(1-\alpha)b]^{2-2\alpha} + (1-\alpha)^2(a-b)^2 + ra(\sqrt{a} - \sqrt{(1-\alpha)b})^2 \leq \alpha^2 a^2 + (1-\alpha)^2 b^2, \tag{11}$$

where $r = \min\{2\alpha - 1, 2 - 2\alpha\}$.

A reverses inequalities using the Kantorovich constants of the above theorem is given in [9].

In 2020, C. Yang and Y. Li [11] showed the following refinement of inequalities (8) and (9) and its reverses as follows:

Theorem 1.4. *Let a and b be two positive numbers and* $0 < \alpha < 1$. *We have*

(1) *If* $0 < \alpha < \frac{1}{4}$, *then*

$$\begin{aligned}(\alpha a)^{2\alpha} b^{2-2\alpha} &+ \alpha^2(a-b)^2 + 2\alpha b(\sqrt{\alpha a} - \sqrt{b})^2 + rb(\sqrt[4]{\alpha ab} - \sqrt{b})^2 \\ &\leq \alpha^2 a^2 + (1-\alpha)^2 b^2 \\ &\leq a^{2\alpha}[(1-\alpha)b]^{2-2\alpha} + (1-\alpha)^2(a-b)^2 - 2\alpha a(\sqrt{a} - \sqrt{(1-\alpha)b})^2 \\ &- ra(\sqrt[4]{(1-\alpha)ab} - \sqrt{a})^2,\end{aligned} \tag{12}$$

where $r = \min\{4\alpha, 1 - 4\alpha\}$.

(2) *If* $\frac{1}{4} \leq \alpha < \frac{1}{2}$, *then*

$$\begin{aligned}(\alpha a)^{2\alpha} b^{2-2\alpha} &+ \alpha^2(a-b)^2 + (1-2\alpha)b(\sqrt{\alpha a} - \sqrt{b})^2 + rb(\sqrt[4]{\alpha ab} - \sqrt{\alpha a})^2 \\ &\leq \alpha^2 a^2 + (1-\alpha)^2 b^2 \\ &\leq a^{2\alpha}[(1-\alpha)b]^{2-2\alpha} + (1-\alpha)^2(a-b)^2 - (1-2\alpha)a(\sqrt{a} - \sqrt{(1-\alpha)b})^2 \\ &- ra(\sqrt[4]{(1-\alpha)ab} - \sqrt{(1-\alpha)b})^2,\end{aligned} \tag{13}$$

where $r = \min\{2 - 4\alpha, 4\alpha - 1\}$.

(3) *If* $\frac{1}{2} \leq \alpha < \frac{3}{4}$, *then*

$$\begin{aligned}a^{2\alpha}[(1-\alpha)b]^{2-2\alpha} &+ (1-\alpha)^2(a-b)^2 + (2\alpha-1)a(\sqrt{a} - \sqrt{(1-\alpha)b})^2 \\ &+ ra(\sqrt[4]{(1-\alpha)ab} - \sqrt{(1-\alpha)b})^2 \\ &\leq \alpha^2 a^2 + (1-\alpha)^2 b^2 \\ &\leq (\alpha a)^{2\alpha} b^{2-2\alpha} + \alpha^2(a-b)^2 - (2\alpha-1)b(\sqrt{\alpha a} - \sqrt{b})^2 \\ &- rb(\sqrt[4]{\alpha ab} - \sqrt{\alpha a})^2,\end{aligned} \tag{14}$$

where $r = \min\{4\alpha - 2, 3 - 4\alpha\}$.

(4) *If* $\frac{3}{4} \leq \alpha < 1$, *then*

$$\begin{aligned}a^{2\alpha}[(1-\alpha)b]^{2-2\alpha} &+ (1-\alpha)^2(a-b)^2 + (2-2\alpha)a(\sqrt{a} - \sqrt{(1-\alpha)b})^2 \\ &+ ra(\sqrt[4]{(1-\alpha)ab} - \sqrt{a})^2 \\ &\leq \alpha^2 a^2 + (1-\alpha)^2 b^2 \\ &\leq (\alpha a)^{2\alpha} b^{2-2\alpha} + \alpha^2(a-b)^2 - (2-2\alpha)b(\sqrt{\alpha a} - \sqrt{b})^2 \\ &- rb(\sqrt[4]{\alpha ab} - \sqrt{b})^2,\end{aligned} \tag{15}$$

where $r = \min\{4 - 4\alpha, 4\alpha - 3\}$.

1.2. *Inequalities for operators and Hilbert-Schmidt norms*

We list in this subsection some recent operator inequalities which concern our work.

Let $B(\mathcal{H})$ denote the $\mathbb{C}^*$-Algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $A \in B(\mathcal{H})$ is called positive, denoted as $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. The set of all positive operators is denoted by $B(\mathcal{H})^+$. The set of all invertible operators in $B(\mathcal{H})^+$, is denoted by $B(\mathcal{H})^{++}$.

Let $\mathbf{M}_n(\mathbb{C})$ be the space of $n \times n$ complex matrices. A norm $|||.|||$ on $\mathbf{M}_n(\mathbb{C})$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbf{M}_n(\mathbb{C})$ and all unitary matrices $U, V \in \mathbf{M}_n(\mathbb{C})$.

For $A = (a_{i,j}) \in \mathbf{M}_n(\mathbb{C})$ the Hilbert-Schmidt norm is defined by $\|A\|_2 = \sqrt{\sum_{i,j=1}^{n} a_{i,j}^2}$.

It is well known that the norm $\|.\|_2$ is unitarily invariant. Let $A, B \in B(\mathcal{H})^{++}$ and $\alpha \in [0, 1]$. The $\alpha-$weighted operators geometric mean of A and B, denoted by $A\sharp_\alpha B$, is defined as

$$A\sharp_\alpha B = A^{1/2}\Big(A^{-1/2}BA^{-1/2}\Big)^{\alpha} A^{1/2},$$

and the $\alpha-$weighted operators arithmetic mean of A and B is defined as

$$A\nabla_\alpha B = (1-\alpha)A + \alpha B.$$

If $\alpha = \frac{1}{2}$, these operators can be rewritten by simplification as $A\nabla B$ and $A\sharp B$.

The operators version of Young's inequality states as follows:

$$A\sharp_\alpha B \leq A\nabla_\alpha B.$$

Recently, several important operators inequalities were established. Next we recall some of them.

The following theorem is proved by C. Yang and Y. Li in [11].

Theorem 1.5. *Let $A, B \in B(\mathcal{H})^{++}$, and $\alpha \in [0, 1]$. Then we have*

1. *If $\alpha \in [0, \frac{1}{4}]$, then*
$$\begin{aligned}\alpha^{2\alpha}(A\sharp_{1-\alpha}B) \quad &+ \quad \alpha^2\Big(A + B - 2A\sharp B\Big) + 2\alpha\Big(\alpha(A\sharp B) + B - 2\sqrt{\alpha}(A\sharp_{\frac{3}{4}}B)\Big)\\ &+ \quad r\Big(\sqrt{\alpha}A\sharp_{\frac{3}{4}}B + B - 2\sqrt[4]{\alpha}A\sharp_{\frac{7}{8}}B\Big) \leq \alpha^2 A + (1-\alpha)^2 B,\end{aligned}$$
where $r = \min\{4\alpha, 1 - 4\alpha\}$.
2. *If $\alpha \in [\frac{1}{4}, \frac{1}{2}]$, then*
$$\begin{aligned}\alpha^{2\alpha}(A\sharp_{1-\alpha}B) \quad &+ \quad \alpha^2\Big(A + B - 2A\sharp B\Big) + (1-2\alpha)\Big(\alpha(A\sharp B) + B - 2\sqrt{\alpha}(A\sharp_{\frac{3}{4}}B)\Big)\\ &+ \quad r\Big(\sqrt{\alpha}A\sharp_{\frac{3}{4}}B + \alpha A\sharp B - 2\sqrt[4]{\alpha^3}A\sharp_{\frac{5}{8}}B\Big) \leq \alpha^2 A + (1-\alpha)^2 B,\end{aligned}$$
where $r = \min\{2 - 4\alpha, 4\alpha - 1\}$.
3. *If $\alpha \in [\frac{1}{2}, \frac{3}{4}]$, then*
$$\begin{aligned}&(1-\alpha)^{2-2\alpha}(A\sharp_\alpha B) \quad + \quad (1-\alpha)^2\Big(A + B - 2A\sharp B\Big)\\ &\quad +(2\alpha-1)\Big((1-\alpha)(A\sharp B) + B - 2\sqrt{1-\alpha}(A\sharp_{\frac{3}{4}}B)\Big)\\ &\quad +r\Big(\sqrt{1-\alpha}A\sharp_{\frac{3}{4}}B + (1-\alpha)A\sharp B - 2\sqrt[4]{(1-\alpha)^3}A\sharp_{\frac{5}{8}}B\Big) \leq \alpha^2 B + (1-\alpha)^2 A,\end{aligned}$$
where $r = \min\{4\alpha - 2, 3 - 4\alpha\}$.
4. *If $\alpha \in [\frac{3}{4}, 1]$, then*
$$\begin{aligned}&(1-\alpha)^{2-2\alpha}(A\sharp_\alpha B) \quad + \quad (1-\alpha)^2\Big(A + B - 2A\sharp B\Big)\\ &\quad +(2-2\alpha)\Big((1-\alpha)(A\sharp B) + B - 2\sqrt{1-\alpha}(A\sharp_{\frac{3}{4}}B)\Big)\\ &\quad +r\Big(\sqrt{1-\alpha}A\sharp_{\frac{3}{4}}B + B - 2\sqrt[4]{1-\alpha}A\sharp_{\frac{7}{8}}B\Big) \leq \alpha^2 B + (1-\alpha)^2 A,\end{aligned}$$
where $r = \min\{4 - 4\alpha, 4\alpha - 3\}$.

The following theorem proved by L. Nasiri et al. [10].

Theorem 1.6. *Let* $A, B \in \mathbf{M}^{+}{}_{n}(\mathbb{C})$ *and* $X \in \mathbf{M}_n(\mathbb{C})$ *and* $0 \leq \alpha \leq 1$.

1. *If* $0 \leq \alpha \leq \frac{1}{2}$, *then*

$$\begin{aligned}\|\alpha AX + (1-\alpha)XB\|_2^2 &\geq K^{-r}(1-\alpha)^{2(1-\alpha)}\|A^{\alpha}XB^{1-\alpha}\|_2^2 + (1-\alpha)^2\|AX - XB\|_2^2 \\ &+ 2\alpha(1-\alpha)\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2,\end{aligned}$$

where $K = \min\{K\Big(\frac{(1-\alpha)\mu_j}{\lambda_i}, 2\Big), 1 \leq i, j \leq n\}$ *and* $r = \min\{2\alpha, 1-2\alpha\}$.

2. *If* $\frac{1}{2} < \alpha \leq 1$, *then*

$$\begin{aligned}\|\alpha AX + (1-\alpha)XB\|_2^2 &\geq K^{-r}\alpha^{2\alpha}\|A^{\alpha}XB^{1-\alpha}\|_2^2 + \alpha^2\|AX - XB\|_2^2 \\ &+ 2\alpha(1-\alpha)\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2,\end{aligned}$$

where $K = \min\{K\Big(\frac{\alpha\lambda_i}{\mu_j}, 2\Big), 1 \leq i, j \leq n\}$ *and* $r = \min\{2\alpha - 1, 2 - 2\alpha\}$.

The purpose of this work is devoted to generalize and unify some new important results (and other previous results) concerning both scalar and operator versions of Young's inequality and provide several applications.

In section 2, we establish in Theorem 2.1 a new multiple-term refinement of Young's type inequality involving the Kantorovich constants. This theorem will generalize and unify the result (see Theorem 1.2) obtained by L. Nasiri et al. [10] and the result (see Theorem 1.4) obtained by C. Yang [11].

Section three is devoted to certain applications of the main results of the second section to obtain new and general operator inequalities. This section contain one of the main results of this work (see Theorem 3.2) and (see Theorem 3.3). These theorems is considered as two of the main results of this paper.

In the fourth (and last) section, we give a multiple-term refinement of Young's type inequalities and its reverses for the Hilbert-Schmidt norms (see Theorems 4.1, and 4.2), we end this section by give a multiple-term refinement of Young's type inequalities for unitarily invariant norms and traces (see Theorems 4.4 and 4.5).

2. Refinement of Young's type inequalities via the Kantorovich constants

In following theorem we present a multiple-term refinement of Young's type inequalities due to Kai, Nasiri, and Yang ([5], [8], and [11]) via the Kantorovich constants.

The first result to be proved in this section is the following theorem.

Theorem 2.1. *Let a and b be two positive numbers and* $0 \leq \alpha \leq 1$. *Then for all a positive integer* $N \geq 2$, *we have*

$$\begin{aligned}&\Big[K\Big(\sqrt[2^{N-2}]{\alpha\frac{a}{b}}, 2\Big)^{r_N(\alpha)}(\alpha a)^{2\alpha}b^{2-2\alpha}\chi_{(0,\frac{1}{2}]} \quad + \quad K\Big(\sqrt[2^{N-2}]{\alpha\frac{b}{a}}, 2\Big)^{r_N(\alpha)}a^{2\alpha}[(1-\alpha)b]^{2-2\alpha}\chi_{(\frac{1}{2},1)}\Big] \\ &+r_0^2(a-b)^2 \\ &+\sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}}\Big(bf_{l-1,k}(\alpha a, b)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) + af_{l-1,k}((1-\alpha)b, a)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha)\Big) \\ &\leq \alpha^2a^2 + (1-\alpha)^2b^2. \end{aligned} \tag{16}$$

Proof. Suppose that $0 < \alpha \leq \frac{1}{2}$. We claim that

$$\begin{aligned}K\Big(\sqrt[2^{N-2}]{\alpha\frac{a}{b}}, 2\Big)^{r_N(\alpha)}(\alpha a)^{2\alpha}b^{2-2\alpha} \quad &+ \quad \alpha^2(a-b)^2 + \sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}} bf_{l-1,k}(\alpha a, b)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \\ &\leq \quad \alpha^2a^2 + (1-\alpha)^2b^2. \end{aligned} \tag{17}$$

By using Theorem 1.1, we have

$$\begin{aligned}\alpha^2a^2+(1-\alpha)^2b^2 &- \alpha^2(a-b)^2 = 2\alpha(\alpha ab)+(1-2\alpha)b^2\\ &\geq K\Big(\sqrt[2^{N-2}]{\alpha\frac{a}{b}},2\Big)^{r_{N-1}(2\alpha)}(\alpha ab)^{2\alpha}b^{2(1-2\alpha)}\\ &+ \sum_{l=0}^{N-2} r_l(2\alpha)\sum_{k=1}^{2^l} f_{l,k}(\alpha ab,b^2)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(2\alpha)\\ &= K\Big(\sqrt[2^{N-2}]{\alpha\frac{a}{b}},2\Big)^{r_N(\alpha)}(\alpha a)^{2\alpha}b^{2-2\alpha}\\ &+ \sum_{l=0}^{N-2} r_{l+1}(\alpha)\sum_{k=1}^{2^l} bf_{l,k}(\alpha a,b)\chi_{(\frac{k-1}{2^{l+1}},\frac{k}{2^{l+1}})}(\alpha).\end{aligned}$$

So,

$$\begin{aligned}\alpha^2a^2+(1-\alpha)^2b^2 &\geq K\Big(\sqrt[2^{N-2}]{\alpha\frac{a}{b}},2\Big)^{r_N(\alpha)}(\alpha a)^{2\alpha}b^{2-2\alpha}+\alpha^2(a-b)^2\\ &+ \sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}} bf_{l-1,k}(\alpha a,b)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha).\end{aligned}$$

If $\alpha\in[\frac{1}{2},1]$, then $1-\alpha\in[0,\frac{1}{2}]$. So by changing two elements a,b and two weights $\alpha, 1-\alpha$ in inequality (17), and not that $r_l(1-\alpha)=r_l(\alpha)$ the desired inequality is obtained. □

The next theorem shows a reverse multiple-term refinement of Young's type inequalities due to Nasiri et al. and Yang et al. ([9], [10] and [11]) involving the Kantorovich constants.

The second result to be proved in this section is the following theorem.

Theorem 2.2. *Let a and b be two positive numbers and $0\leq\alpha\leq 1$. Then for all a positive integer $N\geq 2$, we have*

$$\begin{aligned}\alpha^2a^2+(1-\alpha)^2b^2 &\leq R_0^2(a-b)^2+\\ &\Big[K\Big(\sqrt[2^{N-2}]{(1-\alpha)\frac{b}{a}},2\Big)^{-r_N(\alpha)}a^{2\alpha}[(1-\alpha)b]^{2-2\alpha}\chi_{(0,\frac{1}{2})}(\alpha)\\ &+K\Big(\sqrt[2^{N-2}]{\alpha\frac{a}{b}},2\Big)^{-r_N(\alpha)}(\alpha a)^{2\alpha}b^{2-2\alpha}\chi_{(\frac{1}{2},1]}(\alpha)\Big]\\ &-\sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}}\Big(bf_{l-1,k}(\alpha a,b)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha)+af_{l-1,k}((1-\alpha)b,a)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha)\Big)\end{aligned}$$

Proof. Suppose that $0<\alpha\leq\frac{1}{2}$. We claim that

$$\begin{aligned}\alpha^2a^2+(1-\alpha)^2b^2 &\leq K\Big(\sqrt[2^{N-2}]{(1-\alpha)\frac{b}{a}},2\Big)^{-r_N(\alpha)}a^{2\alpha}[(1-\alpha)b]^{2-2\alpha}+(1-\alpha)^2(a-b)^2\\ &- \sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}} af_{l-1,k}((1-\alpha)b,a)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha).\end{aligned} \tag{18}$$

By using Theorem 1.1, we have

$$\begin{aligned}
(1-\alpha)^2(a-b)^2 &- \alpha^2a^2-(1-\alpha)^2b^2\\
&= 2\alpha[(1-\alpha)ab]+(1-2\alpha)a^2-2(1-\alpha)ab\\
&\geq K\Big(\sqrt[2^{N-2}]{(1-\alpha)\frac{b}{a}},2\Big)^{r_{N-1}(2\alpha)}a^{2-2\alpha}[(1-\alpha)b]^{2\alpha}-2(1-\alpha)ab\\
&+ \sum_{l=0}^{N} r_l(2\alpha)\sum_{k=1}^{2^l} f_{l,k}((1-\alpha)ab,a^2)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(2\alpha)\\
&= K\Big(\sqrt[2^{N-2}]{(1-\alpha)\frac{b}{a}},2\Big)^{r_N(\alpha)}a^{2-2\alpha}[(1-\alpha)b]^{2\alpha}-2(1-\alpha)ab\\
&+ \sum_{l=0}^{N} r_{l+1}(\alpha)\sum_{k=1}^{2^l} af_{l,k}((1-\alpha)b,a)\chi_{(\frac{k-1}{2^{l+1}},\frac{k}{2^{l+1}})}(\alpha)\\
&= K\Big(\sqrt[2^{N-2}]{(1-\alpha)\frac{b}{a}},2\Big)^{r_N(\alpha)}a^{2-2\alpha}[(1-\alpha)b]^{2\alpha}-2(1-\alpha)ab\\
&+ \sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}} af_{l-1,k}((1-\alpha)b,a)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha).
\end{aligned}$$

By using the AM-GM inequality we have

$$\begin{aligned}
&K\Big(\sqrt[2^{N-2}]{(1-\alpha)\frac{b}{a}},2\Big)^{r_N(\alpha)}a^{2-2\alpha}[(1-\alpha)b]^{2\alpha}\\
+&K\Big(\sqrt[2^{N-2}]{(1-\alpha)\frac{b}{a}},2\Big)^{-r_N(\alpha)}a^{2\alpha}[(1-\alpha)b]^{2-2\alpha}\\
&\qquad\geq 2(1-\alpha)ab.
\end{aligned}$$

So,

$$\begin{aligned}
\alpha^2a^2+(1-\alpha)^2b^2 &\leq K\Big(\sqrt[2^{N-2}]{(1-\alpha)\frac{b}{a}},2\Big)^{-r_N(\alpha)}a^{2\alpha}[(1-\alpha)b]^{2-2\alpha}+(1-\alpha)^2(a-b)^2\\
&- \sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}} af_{l-1,k}((1-\alpha)b,a)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha).
\end{aligned}$$

If $\alpha\in[\frac{1}{2},1]$, then $1-\alpha\in[0,\frac{1}{2}]$. So by changing two elements a,b and two weights $\alpha, 1-\alpha$ in inequality (18), and not that $r_l(1-\alpha)=r_l(\alpha)$ the desired inequality is obtained. □

3. Inequalities for operators

In this section, we are concerned by the investigation of further refinements of some operator inequalities. Our results will use the results of the first section.

Before stating and proving our results, we need to recall the following basic lemma.

Lemma 3.1. *Let $T\in B(\mathcal{H})$ be self-adjoint. If f and g are both continuous functions with $f(t)\geq g(t)$ for $t\in Sp(T)$ (where the sign $Sp(T)$ denotes the spectrum of operator T,) then $f(T)\geq g(T)$.*

The first main result to be proved in this section is the following theorem.

Theorem 3.2. *Let $A,B\in B(\mathcal{H})^{++}$, and $0\leq\alpha\leq 1$. If all positive numbers m,m' and M,M' satisfy either of the following conditions $0<mI\leq A\leq m'I<M'I\leq B\leq MI$, $0<mI\leq B\leq m'I<M'I\leq A\leq MI$. Then for all positive integer N, we have*

1. *If* $0 < \alpha \le \frac{1}{2}$, *then*

$$
\begin{aligned}
& K\Big(\sqrt[2^{N-1}]{\alpha^2 h}, 2\Big)^{r_N(\alpha)} \alpha^{2\alpha} A\sharp_\alpha B \quad + \quad \alpha^2(A+B-2A\sharp B) \\
& + \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} A\sharp_{\frac{k-1}{2^l}} B + \alpha^{\frac{k}{2^{l-1}}} A\sharp_{\frac{k}{2^l}} B - 2\alpha^{\frac{2k-1}{2^l}} A\sharp_{\frac{2k-1}{2^{l+1}}} B\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \\
& \le (1-\alpha)^2 A + \alpha^2 B,
\end{aligned} \tag{19}
$$

where $h = \frac{M}{m}$.

2. *If* $\frac{1}{2} < \alpha \le 1$, *then*

$$
\begin{aligned}
& K\Big(\sqrt[2^{N-1}]{(1-\alpha)^2 h}, 2\Big)^{r_N(\alpha)} (1-\alpha)^{2-2\alpha} A\sharp_\alpha B \quad + \quad (1-\alpha)^2(A+B-2A\sharp B) \\
& + \sum_{l=1}^{N-1} r_l(\alpha) \Big[\sum_{k=1}^{2^{l-1}} \Big((1-\alpha)^{\frac{k-1}{2^{l-1}}} A\sharp_{\frac{2^l-k+1}{2^l}} B + (1-\alpha)^{\frac{k}{2^{l-1}}} A\sharp_{\frac{2^l-k}{2^l}} B \\
& -2(1-\alpha)^{\frac{2k-1}{2^l}} A\sharp_{\frac{2^{l+1}-2k+1}{2^{l+1}}} B\Big)\Big]\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha) \\
& \le (1-\alpha)^2 A + \alpha^2 B,
\end{aligned} \tag{20}
$$

where $h = \frac{M}{m}$.

Proof. Suppose that $0 < \alpha \le \frac{1}{2}$, we claim that

$$
\begin{aligned}
& K\Big(\sqrt[2^{N-1}]{\alpha^2 h}, 2\Big)^{r_N(\alpha)} \alpha^{2\alpha} A\sharp_\alpha B \quad + \quad \alpha^2(A+B-2A\sharp B) \\
& + \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} A\sharp_{\frac{k-1}{2^l}} B + \alpha^{\frac{k}{2^{l-1}}} A\sharp_{\frac{k}{2^l}} B - 2\alpha^{\frac{2k-1}{2^l}} A\sharp_{\frac{2k-1}{2^{l+1}}} B\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \\
& \le (1-\alpha)^2 A + \alpha^2 B,
\end{aligned} \tag{21}
$$

Taking $b = 1$, $a^2 = t > 0$ in Theorem 2.1, then we have

$$
\begin{aligned}
\alpha^2 t + (1-\alpha)^2 \quad \ge \quad & K\Big(\sqrt[2^{N-1}]{\alpha^2 t}, 2\Big)^{r_N(\alpha)} \alpha^{2\alpha} t^\alpha + \alpha^2(\sqrt{t}-1)^2 \\
& + \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\sqrt{\alpha^{\frac{k-1}{2^{l-1}}}\sqrt{t}^{\frac{k-1}{2^{l-1}}}} - \sqrt{\alpha^{\frac{k}{2^{l-1}}}\sqrt{t}^{\frac{k}{2^{l-1}}}}\Big)^2 \chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \\
= \; & K\Big(\sqrt[2^{N-1}]{\alpha^2 t}, 2\Big)^{r_N(\alpha)} \alpha^{2\alpha} t^\alpha + \alpha^2(t - 2\sqrt{t} + 1) \\
& + \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} t^{\frac{k-1}{2^l}} + \alpha^{\frac{k}{2^{l-1}}} t^{\frac{k}{2^l}} - 2\alpha^{\frac{2k-1}{2^l}} t^{\frac{2k-1}{2^{l+1}}}\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha).
\end{aligned}
$$

So,

$$
\begin{aligned}
\alpha^2 t + (1-\alpha)^2 \quad \ge \quad & K\Big(\sqrt[2^{N-1}]{\alpha^2 t}, 2\Big)^{r_N(\alpha)} \alpha^{2\alpha} t^\alpha \\
& + \alpha^2(t - 2\sqrt{t} + 1) \\
& + \sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} t^{\frac{k-1}{2^l}} + \alpha^{\frac{k}{2^{l-1}}} t^{\frac{k}{2^l}} - 2\alpha^{\frac{2k-1}{2^l}} t^{\frac{2k-1}{2^{l+1}}}\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha).
\end{aligned}
$$

For $X := (A^{-1/2}BA^{-1/2})$, under the first condition, we get $I \leq hI = \frac{M}{m}I \leq X \leq h^{'}I = \frac{M^{'}}{m^{'}}$, and then $Sp(X) \subseteq [h, h^{'}] \subseteq [1, +\infty)$.
Then by Lemma 3.1, we have

$$\begin{aligned}&(1-\alpha)^2 I + \alpha^2 A^{-1/2}BA^{-1/2} \geq \\ &\min_{h \leq t \leq h'} K\Big(\sqrt[2^{N-1}]{\alpha^2 t}, 2\Big)^{r_N(\alpha)} \alpha^{2\alpha} (A^{-1/2}BA^{-1/2})^{\alpha} \\ &+\alpha^2 (A^{-1/2}BA^{-1/2} - 2\sqrt{A^{-1/2}BA^{-1/2}} + I) \\ &+\sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} (A^{-1/2}BA^{-1/2})^{\frac{k-1}{2^l}} + \alpha^{\frac{k}{2^{l-1}}} (A^{-1/2}BA^{-1/2})^{\frac{k}{2^l}} \\ &-2\alpha^{\frac{2k-1}{2^l}} (A^{-1/2}BA^{-1/2})^{\frac{2k-1}{2^{l+1}}}\Big) \chi_{\left(\frac{k-1}{2^l}, \frac{k}{2^l}\right)}(\alpha).\end{aligned}$$

Since the Kantorovich constant $K\big(t, 2\big) = \frac{(1+t)^2}{4t}$ is an increasing function on $(0, +\infty)$, then

$$\begin{aligned}&(1-\alpha)^2 I + \alpha^2 A^{-1/2}BA^{-1/2} \geq \\ &K\Big(\sqrt[2^{N-1}]{\alpha^2 h}, 2\Big)^{r_N(\alpha)} \alpha^{2\alpha} (A^{-1/2}BA^{-1/2})^{\alpha} \\ &+\alpha^2 (A^{-1/2}BA^{-1/2} - 2\sqrt{A^{-1/2}BA^{-1/2}} + I) \\ &+\sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} (A^{-1/2}BA^{-1/2})^{\frac{k-1}{2^l}} + \alpha^{\frac{k}{2^{l-1}}} (A^{-1/2}BA^{-1/2})^{\frac{k}{2^l}} \\ &-2\alpha^{\frac{2k-1}{2^l}} (A^{-1/2}BA^{-1/2})^{\frac{2k-1}{2^{l+1}}}\Big) \chi_{\left(\frac{k-1}{2^l}, \frac{k}{2^l}\right)}(\alpha).\end{aligned}$$

In a similar way, under the second condition, we have $I \leq \frac{1}{h}I = \frac{m}{M}I \leq X \leq \frac{1}{h'}I = \frac{m^{'}}{M^{'}}$, and then $Sp(X) \subseteq [\frac{1}{h}, \frac{1}{h'}] \subseteq (0, 1)$.
By Lemma 3.1, we have

$$\begin{aligned}&(1-\alpha)^2 I + \alpha^2 A^{-1/2}BA^{-1/2} \geq \\ &K\Big(\sqrt[2^{N-1}]{\frac{\alpha^2}{h}}, 2\Big)^{r_N(\alpha)} \alpha^{2\alpha} (A^{-1/2}BA^{-1/2})^{\alpha} \\ &+\alpha^2 (A^{-1/2}BA^{-1/2} - 2\sqrt{A^{-1/2}BA^{-1/2}} + I) \\ &+\sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} (A^{-1/2}BA^{-1/2})^{\frac{k-1}{2^l}} + \alpha^{\frac{k}{2^{l-1}}} (A^{-1/2}BA^{-1/2})^{\frac{k}{2^l}} \\ &-2\alpha^{\frac{2k-1}{2^l}} (A^{-1/2}BA^{-1/2})^{\frac{2k-1}{2^{l+1}}}\Big) \chi_{\left(\frac{k-1}{2^l}, \frac{k}{2^l}\right)}(\alpha).\end{aligned}$$

So, multiplying the above inequality by $A^{\frac{1}{2}}$ on the left-hand side and on the right hand side, we can deduce the result.

If $\alpha \in [\frac{1}{2}, 1]$, then $1 - \alpha \in [0, \frac{1}{2}]$. So by changing two operators A, B and two weights $\alpha, 1 - \alpha$ in inequality (21) and not that $A\sharp_\alpha B = B\sharp_{1-\alpha}A$, $r_l(1 - \alpha) = r_l(\alpha)$ the desired inequality is obtained. □

The next theorem show a reversed multiple-term refinements of Young's type inequalities for operators.

Theorem 3.3. *Let $A, B \in B(\mathcal{H})^{++}$, and $0 \leq \alpha \leq 1$. If all positive numbers m, m' and M, M' satisfy either of the following conditions $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$. Then for all positive integer N, we have*

1. *If* $0 < \alpha \leq \frac{1}{2}$, *then*

$$\alpha^2 A + (1-\alpha)^2 B \leq K\Big(\sqrt[2^{N-1}]{(1-\alpha)^2 h}, 2\Big)^{-r_N(\alpha)} \alpha^{2\alpha} A\sharp_{1-\alpha} B + (1-\alpha)^2 (A + B - 2A\sharp B)$$
$$-\sum_{l=1}^{N-1} r_l(\alpha)\Big[\sum_{k=1}^{2^{l-1}} \Big((1-\alpha)^{\frac{k-1}{2^{l-1}}} A\sharp_{\frac{2^l-k+1}{2^l}} B + (1-\alpha)^{\frac{k}{2^{l-1}}} A\sharp_{\frac{2^l-k}{2^l}} B$$
$$-2(1-\alpha)^{\frac{2k-1}{2^l}} A\sharp_{\frac{2^{l+1}-2k+1}{2^{l+1}}} B\Big)\Big] \times \chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha), \tag{22}$$

where $h = \frac{M}{m}$.

2. *If* $\frac{1}{2} < \alpha \leq 1$, *then*

$$\alpha^2 A + (1-\alpha)^2 B \leq K\Big(\sqrt[2^{N-1}]{\alpha^2 h}, 2\Big)^{-r_N(\alpha)} (1-\alpha)^{2-2\alpha} A\sharp_{1-\alpha} B + \alpha^2 (A + B - 2A\sharp B)$$
$$-\sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} A\sharp_{\frac{k-1}{2^l}} B + \alpha^{\frac{k}{2^{l-1}}} A\sharp_{\frac{k}{2^l}} B - 2\alpha^{\frac{2k-1}{2^l}} A\sharp_{\frac{2k-1}{2^{l+1}}} B\Big)\chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha)$$
$$\times \chi_{(\frac{2^l-k}{2^l}, \frac{2^l-k+1}{2^l})}(\alpha), \tag{23}$$

where $h = \frac{M}{m}$.

Proof. Using the same method of Theorem 3.2, we can get Inequalities (22) and (23) by Theorem 2.2. □

4. Inequalities for matrix

In this section, we focus on the matrix version of Young's type inequality for the Hilbert-Schmidt norms, traces and the unitarily invariant norms.

4.1. Refinement of Young's type inequality for Hilbert-Schmidt norms

In this subsection, we are concerned by establishing a new refinement of Young's type inequality for Hilbert-Schmidt norms.
The following shows matrix inequalities corresponding to Theorem 2.1.

Theorem 4.1. *Let* $0 \leq \alpha \leq 1$ *and let* $A, X, B \in \mathbf{M}_n(\mathbb{C})$ *be such that* $0 < mI \leq A, B < MI$. *We set* $h = \frac{M}{m}$. *Then for all a positive integer* $N \geq 2$, *we have*

1. *If* $0 \leq \alpha \leq \frac{1}{2}$, *then*

$$\|\alpha AX + (1-\alpha)XB\|_2^2 \geq K\Big(\sqrt[2^{N-2}]{\alpha h}, 2\Big)^{r_N(\alpha)} \alpha^{2\alpha} \|A^\alpha X B^{1-\alpha}\|_2^2 + \alpha^2 \|AX - XB\|_2^2$$
$$+\sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big(\alpha^{\frac{k-1}{2^{l-1}}} \|A^{\frac{k-1}{2^l}} X B^{1-\frac{k-1}{2^l}}\|_2^2 + \alpha^{\frac{k}{2^{l-1}}} \|A^{\frac{k}{2^l}} X B^{1-\frac{k}{2^l}}\|_2^2$$
$$-2\alpha^{\frac{2k-1}{2^l}} \|A^{\frac{2k-1}{2^{l+1}}} X B^{1-\frac{2k-1}{2^{l+1}}}\|_2^2\Big)\chi_{(\frac{k-1}{2^l}, \frac{k}{2^l})}(\alpha) \tag{24}$$

2. *If* $\frac{1}{2} < \alpha \leq 1$, *then*

$$\|\alpha AX + (1-\alpha)XB\|_2^2 \geq K\Big(\sqrt[2^{N-2}]{(1-\alpha)h}, 2\Big)^{r_N(\alpha)} (1-\alpha)^{2-2\alpha} \|A^\alpha X B^{1-\alpha}\|_2^2$$
$$+(1-\alpha)^2 \|AX - XB\|_2^2$$
$$+\sum_{l=1}^{N-1} r_l(\alpha) \sum_{k=1}^{2^{l-1}} \Big((1-\alpha)^{\frac{k-1}{2^{l-1}}} \|A^{1-\frac{k-1}{2^l}} X B^{\frac{k-1}{2^l}}\|_2^2 + (1-\alpha)^{\frac{k}{2^{l-1}}} \|A^{1-\frac{k}{2^l}} X B^{\frac{k}{2^l}}\|_2^2$$
$$-2(1-\alpha)^{\frac{2k-1}{2^l}} \|A^{1-\frac{2k-1}{2^{l+1}}} X B^{\frac{2k-1}{2^{l+1}}}\|_2^2\Big)\chi_{(\frac{2^l-k}{2^l}, \frac{2^l-k+1}{2^l})}(\alpha) \tag{25}$$

Proof. Since A and B are positive matrices, then by the spectral decomposition theorem, there exist unitary matrices $U, V \in \mathbf{M}_n(\mathbb{C})$ satisfying $A = UD_1U^*$, $B = VD_2V^*$, where $D_1 = diag(a_1, a_2, \dots, a_n)$, $D_2 = diag(b_1, b_2, \dots, b_n)$, $(a_i \geq 0, b_i \geq 0,\ i = 1, 2, \dots, n)$. Suppose that $Y = U^*XV = [y_{i,j}]$, we have

$$\alpha AX + (1-\alpha)XB = U(\alpha D_1 Y + (1-\alpha)YD_2)V^* = U((\alpha a_i + (1-\alpha)b_j))y_{i,j})V^*,$$

$$A^{\alpha}XB^{1-\alpha} = U(a_i^{\alpha}b_j^{1-\alpha}y_{i,j})V^*,\ AX - XB = U[(a_i - b_j)y_{i,j}]V^*,\ A^{\frac{1}{2}}XB^{\frac{1}{2}} = U(a_i^{\frac{1}{2}}b_j^{\frac{1}{2}}y_{i,j})V^*$$

$$A^{\frac{k-1}{2^l}}XB^{1-\frac{k-1}{2^l}} = U(a_i^{\frac{k-1}{2^l}}b_j^{1-\frac{k-1}{2^l}}y_{i,j})V^* \text{ and } A^{1-\frac{k-1}{2^l}}XB^{\frac{k-1}{2^l}} = U(a_i^{1-\frac{k-1}{2^l}}b_j^{\frac{k-1}{2^l}}y_{i,j})V^*$$

$$A^{\frac{2k-1}{2^{l+1}}}XB^{1-\frac{2k-1}{2^{l+1}}} = U(a_i^{\frac{2k-1}{2^{l+1}}}b_j^{1-\frac{2k-1}{2^{l+1}}}y_{i,j})V^*$$

Now by the Theorem 2.1 and the unitarily invariant of the Hilbert-Schmidt norm, one obtains that

$$\begin{aligned}
\|\alpha AX + (1-\alpha)XB\|_2^2 &= \sum_{i,j=1}^{n}(\alpha a_i + (1-\alpha)b_j))^2|y_{i,j}|^2 \\
&\geq K\Big(\sqrt[2^{N-2}]{\alpha t_{i,j}}, 2\Big)^{r_N(\alpha)}\alpha^{2\alpha}\sum_{i,j=1}^{n}(a_i^{\alpha}b_j^{1-\alpha})^2|y_{i,j}|^2 \\
&+2\alpha(1-\alpha)\sum_{i,j=1}^{n}(a_i^{\frac{1}{2}}b_j^{\frac{1}{2}})^2|y_{i,j}|^2 + \alpha^2\sum_{i,j=1}^{n}(a_i - b_j)^2|y_{i,j}|^2 \\
&+\sum_{l=1}^{N-1}r_l(\alpha)\sum_{k=1}^{2^{l-1}}\Big(\alpha^{\frac{k-1}{2^{l-1}}}a_i^{\frac{k-1}{2^l}}b_i^{1-\frac{k-1}{2^l}} + \alpha^{\frac{k}{2^{l-1}}}a_i^{\frac{k}{2^l}}b_i^{1-\frac{k}{2^l}} \\
&-2\alpha^{\frac{2k-1}{2^l}}a_i^{\frac{2k-1}{2^{l+1}}}b_i^{1-\frac{2k-1}{2^{l+1}}}\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha)
\end{aligned}$$

where $t_{i,j} = \frac{a_i}{b_j}$.

According to the conditions $0 < mI \leq A, B < MI$. and $0 \leq \alpha \leq 1$, $\frac{m}{M} = \frac{1}{h} \leq t_{i,j} = \frac{a_i}{b_j} \leq h = \frac{M}{m}$ and the property of the Kantorovich constant, we have

$$\begin{aligned}
\|\alpha AX + (1-\alpha)XB\|_2^2 &= \sum_{i,j=1}^{n}(\alpha a_i + (1-\alpha)b_j))^2|y_{i,j}|^2 \\
&\geq K\Big(\sqrt[2^{N-2}]{\alpha h}, 2\Big)^{r_N(\alpha)}\alpha^{2\alpha}\sum_{i,j=1}^{n}(a_i^{\alpha}b_j^{1-\alpha})^2|y_{i,j}|^2 \\
&+2\alpha(1-\alpha)\sum_{i,j=1}^{n}(a_i^{\frac{1}{2}}b_j^{\frac{1}{2}})^2|y_{i,j}|^2 + \alpha^2\sum_{i,j=1}^{n}(a_i - b_j)^2|y_{i,j}|^2 \\
&+\sum_{l=1}^{N-1}r_l(\alpha)\sum_{k=1}^{2^{l-1}}\Big(\alpha^{\frac{k-1}{2^{l-1}}}a_i^{\frac{k-1}{2^l}}b_i^{1-\frac{k-1}{2^l}} + \alpha^{\frac{k}{2^{l-1}}}a_i^{\frac{k}{2^l}}b_i^{1-\frac{k}{2^l}} \\
&-2\alpha^{\frac{2k-1}{2^l}}a_i^{\frac{2k-1}{2^{l+1}}}b_i^{1-\frac{2k-1}{2^{l+1}}}\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha).
\end{aligned}$$

So,

$$\begin{aligned} \|\alpha AX+(1-\alpha)XB\|_2^2 \quad &\geq \quad K\Big(\sqrt[2^{N-2}]{\alpha h},2\Big)^{r_N(\alpha)}\alpha^{2\alpha}\|A^{\alpha}XB^{1-\alpha}\|_2^2+\alpha^2\|AX-XB\|_2^2 \\ +\sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}}&\Big(\alpha^{\frac{k-1}{2^{l-1}}}\|A^{\frac{k-1}{2^l}}XB^{1-\frac{k-1}{2^l}}\|_2^2+\alpha^{\frac{k}{2^{l-1}}}\|A^{\frac{k}{2^l}}XB^{1-\frac{k}{2^l}}\|_2^2 \\ &-2\alpha^{\frac{2k-1}{2^l}}\|A^{\frac{2k-1}{2^{l+1}}}XB^{1-\frac{2k-1}{2^{l+1}}}\|_2^2\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha). \end{aligned}$$

By the same process we can show the inequality (25). This completes the proof of our result. □

A direct consequence of Theorem 2.2, we get the following reverses refinement of Young's type inequality for the Hilbert-Schmidt norms.

Theorem 4.2. *Let $0\leq\alpha\leq1$ and let $A,X,B\in\mathbf{M}_n(\mathbb{C})$ be such that $0<mI\leq A,B<MI$. We set $h=\frac{M}{m}$. Then for all a positive integer $N\geq2$, we have*

1. *If $0\leq\alpha\leq\frac{1}{2}$, then*

$$\begin{aligned} \|\alpha AX+(1-\alpha)XB\|_2^2 \quad &\leq \quad K\Big(\sqrt[2^{N-2}]{\alpha h},2\Big)^{-r_N(\alpha)}(1-\alpha)^{2-2\alpha}\|A^{\alpha}XB^{1-\alpha}\|_2^2 \\ +(1-\alpha)^2\|AX-XB\|_2^2& \\ +\sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}}&\Big((1-\alpha)^{\frac{k-1}{2^{l-1}}}\|A^{1-\frac{k-1}{2^l}}XB^{\frac{k-1}{2^l}}\|_2^2+(1-\alpha)^{\frac{k}{2^{l-1}}}\|A^{1-\frac{k}{2^l}}XB^{\frac{k}{2^l}}\|_2^2 \\ &-2(1-\alpha)^{\frac{2k-1}{2^l}}\|A^{1-\frac{2k-1}{2^{l+1}}}XB^{\frac{2k-1}{2^{l+1}}}\|_2^2\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha) \end{aligned}$$

2. *If $\frac{1}{2}<\alpha\leq1$, then*

$$\begin{aligned} \|\alpha AX+(1-\alpha)XB\|_2^2 \quad &\leq \quad K\Big(\sqrt[2^{N-2}]{(1-\alpha)h},2\Big)^{-r_N(\alpha)}\alpha^{2\alpha}\|A^{\alpha}XB^{1-\alpha}\|_2^2 \\ +(1-\alpha)^2\|AX-XB\|_2^2& \\ +\sum_{l=1}^{N-1} r_l(\alpha)\sum_{k=1}^{2^{l-1}}&\Big(\alpha^{\frac{k-1}{2^{l-1}}}\|A^{\frac{k-1}{2^l}}XB^{1-\frac{k-1}{2^l}}\|_2^2+\alpha^{\frac{k}{2^{l-1}}}\|A^{\frac{k}{2^l}}XB^{1-\frac{k}{2^l}}\|_2^2 \\ &-2\alpha^{\frac{2k-1}{2^l}}\|A^{\frac{2k-1}{2^{l+1}}}XB^{1-\frac{2k-1}{2^{l+1}}}\|_2^2\Big)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha) \end{aligned}$$

Proof. Utilising Theorem 2.2, and employing the same ideas as used in the proof of Theorem 4.1, we can get the desired result. □

4.2. Refinement of Young's type inequality for unitarily invariant norms.

In this subsection, we are concerned by establishing a new refinement of Young's type inequality for traces.

To prove our result we need the following lemma.

Lemma 4.3. *[7] Let $A,B\in\mathbf{M}_n(\mathbb{C})$ be positive semidefinite matrices and $X\in\mathbf{M}_n(\mathbb{C})$. Then we have*

$$|||A^{\alpha}XB^{1-\alpha}|||\leq|||AX|||^{\alpha}|||XB|||^{1-\alpha}. \tag{26}$$

In particular

$$tr|A^{\alpha}B^{1-\alpha}|\leq(trA)^{\alpha}(trB)^{1-\alpha}. \tag{27}$$

Theorem 4.4. *Let $0 \le \alpha \le 1$ and Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive semidefinite matrices and $X \in \mathbf{M}_n(\mathbb{C})$. Then for all a positive integer $N \ge 2$, we have*

$$\begin{aligned}
&\Big[K\Big(\sqrt[2^{N-2}]{\alpha\frac{|||AX|||}{|||XB|||}},2\Big)^{r_N(\alpha)}\alpha^{2\alpha}|||A^\alpha XB^{1-\alpha}|||^2\chi_{(0,\frac{1}{2}]}(\alpha)\\
&+K\Big(\sqrt[2^{N-2}]{(1-\alpha)\frac{|||XB|||}{|||AX|||}},2\Big)^{r_N(\alpha)}(1-\alpha)^{2-2\alpha}|||A^\alpha XB^{1-\alpha}|||^2\chi_{(\frac{1}{2},1)}(\alpha)\Big]\\
&+r_0^2(|||AX|||-|||XB|||)^2+2\alpha(1-\alpha)|||AX||||||XB|||\\
&+\sum_{l=1}^{N-1}r_l(\alpha)\sum_{k=1}^{2^{l-1}}\Big[|||XB|||f_{l-1,k}\Big(\alpha|||AX|||,|||XB|||\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha)\\
&+|||AX|||f_{l-1,k}\Big((1-\alpha)|||XB|||,|||AX|||\Big)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha)\Big]\\
&\le(\alpha|||AX|||+(1-\alpha)|||XB|||)^2.
\end{aligned}$$

Proof. By using Theorem 2.1 and Lemma 4.3, we have

$$\begin{aligned}
&K\Big(\sqrt[2^{N-2}]{\alpha\frac{|||AX|||}{|||XB|||}},2\Big)^{r_N(\alpha)}\alpha^{2\alpha}|||A^\alpha XB^{1-\alpha}|||^2\\
&+r_0^2(|||AX|||-|||XB|||)^2+2\alpha(1-\alpha)|||AX||||||XB|||\\
&+\sum_{l=1}^{N-1}r_l(\alpha)\sum_{k=1}^{2^{l-1}}|||XB|||f_{l-1,k}\Big(\alpha|||AX|||,|||XB|||\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha\\
&\le K\Big(\sqrt[2^{N-2}]{\alpha\frac{|||AX|||}{|||XB|||}},2\Big)^{r_N(\alpha)}\alpha^{2\alpha}|||AX|||^{2\alpha}|||XB|||^{2-2\alpha}\\
&+r_0^2(|||AX|||-|||XB|||)^2+2\alpha(1-\alpha)|||AX||||||XB|||\\
&+\sum_{l=1}^{N-1}r_l(\alpha)\sum_{k=1}^{2^{l-1}}\Big[|||XB|||f_{l-1,k}\Big(\alpha|||AX|||,|||XB|||\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha)\\
&\text{(by Lemma 4.3)}\\
&\le(\alpha|||AX|||+(1-\alpha)|||XB|||)^2\ \text{(by Theorem 2.1)}.
\end{aligned}$$

Employing the same ideas as used in the proof in the case $0 \le \alpha \le \frac{1}{2}$, we can get the desired result for the case $\frac{1}{2} \le \alpha \le 1$. This completes the proof of our result. □

4.3. *Refinement of Young's type inequality for traces*

We end this paper by giving an inequality for traces by using Theorem 2.1. Precisely, we show the following result.

Theorem 4.5. *Let $0 \le \alpha \le 1$ and Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive semidefinite matrices. Then for all a positive integer*

$N \geq 2$, *we have*

$$\Big[K\Big(\sqrt[2^{N-2}]{\alpha\frac{tr(A)}{tr(B)}},2\Big)^{r_N(\alpha)}\alpha^{2\alpha}tr(|A^{\alpha}B^{1-\alpha}|)^2\chi_{(0,\frac{1}{2}]}(\alpha)$$
$$K\Big(\sqrt[2^{N-2}]{\alpha\frac{tr(B)}{tr(A)}},2\Big)^{r_N(\alpha)}+(1-\alpha)^{2-2\alpha}tr(|A^{\alpha}B^{1-\alpha}|)^2\chi_{(\frac{1}{2},1)}(\alpha)\Big]$$
$$+r_0^2(tr(A)-tr(B))^2+2\alpha(1-\alpha)tr(A)tr(B)$$
$$+\sum_{l=1}^{N-1}r_l(\alpha)\sum_{k=1}^{2^{l-1}}\Big[tr(A)f_{l-1,k}\Big(\alpha tr(A),tr(B)\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha)$$
$$+tr(A)f_{l-1,k}\Big((1-\alpha)tr(B),tr(A)\Big)\chi_{(\frac{2^l-k}{2^l},\frac{2^l-k+1}{2^l})}(\alpha)\Big]$$
$$\leq\Big[tr(\alpha A+(1-\alpha)B)\Big]^2.$$

Proof. Suppose that $0 \leq \alpha \leq \frac{1}{2}$, by using Theorem 2.1 and Lemma 4.3, we have

$$K\Big(\sqrt[2^{N-2}]{\alpha\frac{tr(A)}{tr(B)}},2\Big)^{r_N(\alpha)}\alpha^{2\alpha}tr(|A^{\alpha}B^{1-\alpha}|)^2$$
$$+r_0^2(tr(A)-tr(B))^2+2\alpha(1-\alpha)tr(A)tr(B)$$
$$+\sum_{l=1}^{N-1}r_l(\alpha)\sum_{k=1}^{2^{l-1}}\Big[tr(A)f_{l-1,k}\Big(\alpha tr(A),tr(B)\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha)$$
$$\leq K\Big(\sqrt[2^{N-2}]{\alpha\frac{tr(A)}{tr(B)}},2\Big)^{r_N(\alpha)}\alpha^{2\alpha}tr(A)^{2\alpha}tr(B)^{2-2\alpha}$$
$$+r_0^2(tr(A)-tr(B))^2+2\alpha(1-\alpha)tr(A)tr(B)$$
$$+\sum_{l=1}^{N-1}r_l(\alpha)\sum_{k=1}^{2^{l-1}}tr(A)f_{l-1,k}\Big(\alpha tr(A),tr(B)\Big)\chi_{(\frac{k-1}{2^l},\frac{k}{2^l})}(\alpha)$$
$$\text{(by Lemma 4.3)}$$
$$\leq\Big[tr(\alpha A+(1-\alpha)B)\Big]^2\ \text{(by Theorem 2.1)}.$$

Employing the same ideas as used in the proof in the case $0 \leq \alpha \leq \frac{1}{2}$, we can get the desired result for the case $\frac{1}{2} \leq \alpha \leq 1$. This completes the proof of our result. □

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