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Operators with complex Gaussian kernels: asymptotic behaviours

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Abstract. In this paper we derive Abelian theorems for the operators with complex Gaussian kernels. Specifically, we establish some results in which known the behaviour of the function and its domain variable approaches to $-\infty$ or $+\infty$ is used to infer the asymptotic behaviour of the transform as its domain variable approaches to $+\infty$ or $-\infty$. For this purpose we use a formula concerning the computation of potential functions by means of these operators with complex Gaussian kernels. This formula allows us to analyse the asymptotic behaviour of these operators in both cases: when the variable approaches to $+\infty$ or $-\infty$. Our results include systematically the noncentered and centered cases of these operators. Here we analyse the Gauss-Weierstrass semigroup on $\mathbb{R}$ as a particular case. We also point out Abelian theorems for other kinds of operators which have been studied in several papers.

1. Introduction

In this paper we consider the following operators with complex Gaussian kernels

$$(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y) = \int_{-\infty}^{+\infty} f(x) \exp\left[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x\right] dx, \tag{1}$$

where $y \in \mathbb{R}$, $\beta, \varepsilon, \delta, \xi, \gamma \in \mathbb{C}$ and f is a suitable complex-valued function defined on $\mathbb{R}$.

These operators has been studied in several papers (see [1], [6], [7], [9], [10], [11], [20], amongst others).

Some interesting contributions in the context of this work are given in [13], [14], [16], [17] and [19].

The subject of this paper was originally of interest in the context of Quantum Field Theory (see [2]). The complex Gaussian operator (1) has an intrinsic interest due to the basic role of the extended oscillator semigroup introduced by Howe [8] (see also Folland [4, Chapter 5]). In his important paper [9], Lieb extends the operator (1) to n dimensions and develops an extensive study of (1) in the context of the spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Here we use formula (1.1) in [11] given by

$$H_{\beta,\varepsilon,\delta,\xi,\gamma,n}(y) = \int_{-\infty}^{+\infty} x^n \exp\left[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x\right] dx$$

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$$= \sqrt{\frac{\pi}{\varepsilon}} \exp\left[(-\beta + \frac{\delta^2}{\varepsilon})y^2 + (\xi + \frac{\delta\gamma}{\varepsilon})y + \frac{\gamma^2}{4\varepsilon}\right] \cdot \sum_{m=0}^{[n/2]} \frac{n!}{(n-2m)!m!2^m}(2\delta y + \gamma)^{n-2m}(2\varepsilon)^{m-n} \tag{2}$$

for $\Re\varepsilon > 0$, $y \in \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$ and where $[n/2]$ denotes the entire part of $n/2$.

In section 2 we obtain Abelian theorems for the operator (1). Specifically, we establish some results in which known the behaviour of the function as its domain variable approaches to $-\infty$ or $+\infty$ is used to infer the asymptotic behaviour of the transform (1) as its domain variable approaches to $+\infty$ or $-\infty$.

In section 3 we analyse the Gauss-Weierstrass semigroup on $\mathbb{R}$ as a particular case (see [1] and [18], amongst others).

Abelian theorems for other integral operators have been also studied in several papers (see [5], [12], and [15], amongst others).

2. Abelian theorems for Operators with Complex Gaussian Kernels

The next result establishes the asymptotic behaviour of the transform (1) as its domain variable approaches to $+\infty$ or $-\infty$ as long as the domain variable approaches to $-\infty$.

Theorem 2.1. *Set $n \in \mathbb{N} \cup \{0\}$ and $\Re\varepsilon > 0$. Let f be a complex-valued measurable function on $\mathbb{R}$ such that $\exp\left(-\Re\varepsilon x^2\right) f(x)$ be Lebesgue integrable on every interval $(T, +\infty)$, for all T. Assume that*

$$\lim_{x\to-\infty} [x^{-n} f(x)] = \lambda, \tag{3}$$

where $\lambda \in \mathbb{C}$. Then

(i) For $(\Re\delta)^2 < \Re\beta\Re\varepsilon$, $\Re\delta \le 0$ and all $\xi, \gamma \in \mathbb{C}$, or alternatively, $(\Re\delta)^2 = \Re\beta\Re\varepsilon$, $\Re\delta \le 0$ and $\Re\xi\Re\varepsilon + \Re\delta\Re\gamma < 0$, one has

$$\lim_{y\to+\infty} \left[F(y) - \lambda H_{\beta,\varepsilon,\delta,\xi,\gamma,n}(y)\right] = 0.$$

(ii) For $(\Re\delta)^2 < \Re\beta\Re\varepsilon$, $\Re\delta \ge 0$ and all $\xi, \gamma \in \mathbb{C}$, or alternatively, $(\Re\delta)^2 = \Re\beta\Re\varepsilon$, $\Re\delta \ge 0$ and $\Re\xi\Re\varepsilon + \Re\delta\Re\gamma > 0$, one has

$$\lim_{y\to-\infty} \left[F(y) - \lambda H_{\beta,\varepsilon,\delta,\xi,\gamma,n}(y)\right] = 0.$$

In both cases $F = \mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f$ is given by (1) and $H_{\beta,\varepsilon,\delta,\xi,\gamma,n}$ is given by (2).

Proof. From (3) one obtains that for a fixed $\varepsilon^* > 0$ there exists a $T(\varepsilon^*)$ depending on ε^* such that

$$\sup_{-\infty<x\le T(\varepsilon^*)} \left|x^{-n} f(x) - \lambda\right| < \varepsilon^*.$$

Observe that

$$\begin{aligned}
&|F(y) - \lambda H_{\beta,\varepsilon,\delta,\xi,\gamma,n}(y)| \\
&= \left|\int_{-\infty}^{+\infty} (x^{-n} f(x) - \lambda) x^n \exp\left[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x\right] dx\right| \\
&\le \int_{-\infty}^{T(\varepsilon^*)} \left|x^{-n} f(x) - \lambda\right| |x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right] dx \\
&+ \int_{T(\varepsilon^*)}^{+\infty} \left|x^{-n} f(x) - \lambda\right| |x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right] dx
\end{aligned}$$

$$\le \sup_{-\infty<x\le T(\varepsilon^*)} \left|x^{-n}f(x)-\lambda\right| \int_{-\infty}^{T(\varepsilon^*)} |x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right] dx$$

$$+ \int_{T(\varepsilon^*)}^{+\infty} \left|x^{-n}f(x)-\lambda\right| |x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right] dx$$

$$\le \varepsilon^* \int_{-\infty}^{+\infty} |x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right] dx$$

$$+ \int_{T(\varepsilon^*)}^{+\infty} \left|x^{-n}f(x)-\lambda\right| |x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right] dx. \tag{4}$$

Since $\Re\delta \le 0$ and taking $y > 0$, expression (4) is less than or equal to

$$\varepsilon^* \exp\left[-\Re\beta y^2 + \Re\xi y\right] \int_{-\infty}^{+\infty} |x|^n \exp\left[-\Re\varepsilon x^2 + 2\Re\delta xy + \Re\gamma x\right] dx$$

$$+\exp\left[-\Re\beta y^2 + \Re\xi y + 2\Re\delta T(\varepsilon^*) y\right] \int_{T(\varepsilon^*)}^{+\infty} \left|x^{-n}f(x)-\lambda\right| |x|^n \exp\left[-\Re\varepsilon x^2 + \Re\gamma x\right] dx. \tag{5}$$

Observe that for $n = 0$ or n even, one has

$$\int_{-\infty}^{+\infty} |x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right] dx = H_{\Re\beta,\Re\varepsilon,\Re\delta,\Re\xi,\Re\gamma,n}(y).$$

Also, for n odd and by using Mathematica Version 9, Wolfram Research, Champaign, IL., one has

$$\int_{-\infty}^{+\infty} |x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right] dx$$

$$= (\Re\varepsilon)^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \exp(-\Re\beta y^2 + \Re\xi y) {}_1F_1\left(\frac{n+1}{2}; \frac{1}{2}; \frac{(2\Re\delta y + \Re\gamma)^2}{4\Re\varepsilon}\right),$$

where ${}_1F_1$ is the confluent hypergeometric function (see [3, Chap. 6]).

Now, using the conditions on $\Re\beta$, $\Re\varepsilon$, $\Re\delta$, $\Re\xi$, $\Re\gamma$, the hypothesis on f and having into account the behaviour of the function ${}_1F_1$ (see [3, formula (3), p. 278]), expression (5) tends to 0 as y tends to $+\infty$.

Therefore

$$\lim_{y\to+\infty} \left[F(y) - \lambda H_{\beta,\varepsilon,\delta,\xi,\gamma,n}(y)\right] = 0.$$

This establishes (i).

Analogously one obtains (ii).

□

The next result establishes the asymptotic behaviour of the transform (1) as its domain variable approaches to $+\infty$ or $-\infty$ as long as the domain variable approaches to $+\infty$.

Theorem 2.2. *Set $n \in \mathbb{N} \cup \{0\}$ and $\Re\varepsilon > 0$. Let f be a complex-valued measurable function on $\mathbb{R}$ such that $\exp\left(-\Re\varepsilon x^2\right) f(x)$ be Lebesgue integrable on every interval $(-\infty, T)$, for all T. Assume that*

$$\lim_{x\to+\infty} \left[x^{-n} f(x)\right] = \lambda, \tag{6}$$

where $\lambda \in \mathbb{C}$. Then

(i) For $(\Re\delta)^2 < \Re\beta\Re\varepsilon$, $\Re\delta \geq 0$ *and all* $\xi, \gamma \in \mathbb{C}$, *or alternatively,* $(\Re\delta)^2 = \Re\beta\Re\varepsilon$, $\Re\delta \geq 0$ *and* $\Re\xi\Re\varepsilon + \Re\delta\Re\gamma < 0$, *one has*

$$\lim_{y\to+\infty}\left[F(y) - \lambda H_{\beta,\varepsilon,\delta,\xi,\gamma,n}(y)\right] = 0.$$

(ii) For $(\Re\delta)^2 < \Re\beta\Re\varepsilon$, $\Re\delta \leq 0$ *and all* $\xi, \gamma \in \mathbb{C}$, *or alternatively,* $(\Re\delta)^2 = \Re\beta\Re\varepsilon$, $\Re\delta \leq 0$ *and* $\Re\xi\Re\varepsilon + \Re\delta\Re\gamma > 0$, *one has*

$$\lim_{y\to-\infty}\left[F(y) - \lambda H_{\beta,\varepsilon,\delta,\xi,\gamma,n}(y)\right] = 0.$$

In both cases $F = \mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f$ *is given by (1) and* $H_{\beta,\varepsilon,\delta,\xi,\gamma,n}$ *is given by (2).*

Proof. From (6) one obtains that for a fixed $\varepsilon^* > 0$ there exists a $T(\varepsilon^*)$ depending on ε^* such that

$$\sup_{T(\varepsilon^*)\leq x<+\infty}\left|x^{-n}f(x) - \lambda\right| < \varepsilon^*.$$

Observe that

$$\begin{aligned}
&|F(y) - \lambda H_{\beta,\varepsilon,\delta,\xi,\gamma,n}(y)|\\
&= \left|\int_{-\infty}^{+\infty}(x^{-n}f(x) - \lambda)x^n \exp\left[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x\right]dx\right|\\
&\leq \int_{-\infty}^{T(\varepsilon^*)}\left|x^{-n}f(x) - \lambda\right||x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right]dx\\
&+ \int_{T(\varepsilon^*)}^{+\infty}\left|x^{-n}f(x) - \lambda\right||x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right]dx\\
&\leq \int_{-\infty}^{T(\varepsilon^*)}\left|x^{-n}f(x) - \lambda\right||x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right]dx\\
&+ \sup_{T(\varepsilon^*)\leq x<+\infty}\left|x^{-n}f(x) - \lambda\right|\int_{T(\varepsilon^*)}^{+\infty}|x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right]dx\\
&\leq \int_{-\infty}^{T(\varepsilon^*)}\left|x^{-n}f(x) - \lambda\right||x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right]dx\\
&+\varepsilon^*\int_{-\infty}^{+\infty}|x|^n \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta xy + \Re\xi y + \Re\gamma x\right]\,dx. \qquad (7)
\end{aligned}$$

Since $\Re\delta \geq 0$ and taking $y > 0$, expression (7) is less than or equal to

$$\begin{aligned}
&\exp\left[-\Re\beta y^2 + \Re\xi y + 2\Re\delta T(\varepsilon^*)y\right]\int_{-\infty}^{T(\varepsilon^*)}\left|x^{-n}f(x) - \lambda\right||x|^n \exp\left[-\Re\varepsilon x^2 + \Re\gamma x\right]dx\\
&+\varepsilon^*\exp\left[-\Re\beta y^2 + \Re\xi y\right]\int_{-\infty}^{+\infty}|x|^n \exp\left[-\Re\varepsilon x^2 + 2\Re\delta xy + \Re\gamma x\right]dx. \qquad (8)
\end{aligned}$$

Now, arguing as in the proof of (i) in Theorem 2.1 above, one obtains that expression (8) tends to 0 as y tends to $+\infty$.

Therefore

$$\lim_{y\to+\infty}\left[F(y) - \lambda H_{\beta,\varepsilon,\delta,\xi,\gamma,n}(y)\right] = 0.$$

This establishes (i).

Analogously one obtains (ii).

□

Remark 1. The functions on $\mathbb{R}$ given by $f(x) = \lambda x^n$, for $\lambda \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, satisfy the conditions of Theorems 2.1 and 2.2.

3. A particular case: the Gauss-Weierstrass semigroup

The Gauss-Weierstrass semigroup on $\mathbb{R}$ (see [20]) is given by

$$\left(e^{z\Delta} f\right)(y) = (4\pi z)^{-1/2} \int_{-\infty}^{+\infty} \exp\left[-\frac{(y-x)^2}{4z}\right] f(x)dx$$

where $\Re z \geq 0$ (and $z \neq 0$).

Excepting for the factor $(4\pi z)^{-1/2}$, this operator corresponds to the case when $\beta = \varepsilon = \delta = \dfrac{1}{4z}$ and $\xi = \gamma = 0$.

In Elliott H. Lieb terminology [9], the kernel of this operator corresponds to a centered Gaussian kernel.

By virtue of Theorem 2.1 (ii) we get

Corollary 3.1. *Set $n \in \mathbb{N} \cup \{0\}$ and $\Re z > 0$. Let f be a complex-valued measurable function on $\mathbb{R}$ such that $\exp\left(-\frac{\Re z}{4|z|^2} x^2\right) f(x)$ be Lebesgue integrable on every interval $(T, +\infty)$, for all T. Assume that*

$$\lim_{x \to -\infty} \left[x^{-n} f(x)\right] = \lambda,$$

where $\lambda \in \mathbb{C}$. Then for $\Re a > 0$ one has

$$\lim_{y \to -\infty} \left[e^{ay} \left(e^{z\Delta} f\right)(y)\right] = 0.$$

Proof. By using Theorem 2.1 (ii) for the operator with complex Gaussian kernel given by $(4\pi z)^{1/2} e^{ay} \left(e^{z\Delta} f\right)(y)$ one obtains that

$$\lim_{y \to -\infty} \left[(4\pi z)^{1/2} e^{ay} \left(e^{z\Delta} f\right)(y) - \lambda H_{\frac{1}{4z}, \frac{1}{4z}, \frac{1}{4z}, a, 0, n}(y)\right] = 0.$$

Now observe that

$$H_{\frac{1}{4z}, \frac{1}{4z}, \frac{1}{4z}, a, 0, n}(y) = (4\pi z)^{1/2} e^{ay} \sum_{m=0}^{[n/2]} \frac{n!}{(n-2m)!m!} z^m y^{n-2m},$$

which tends to zero as $y \to -\infty$.

Thus the result holds. □

Now, by virtue of Theorem 2.2 (i) we get

Corollary 3.2. *Set $n \in \mathbb{N} \cup \{0\}$ and $\Re z > 0$. Let f be a complex-valued measurable function on $\mathbb{R}$ such that $\exp\left(-\frac{\Re z}{4|z|^2} x^2\right) f(x)$ be Lebesgue integrable on every interval $(-\infty, T)$, for all T. Assume that*

$$\lim_{x \to +\infty} \left[x^{-n} f(x)\right] = \lambda,$$

where $\lambda \in \mathbb{C}$. Then for $\Re a < 0$ one has

$$\lim_{y \to +\infty} \left[e^{ay} \left(e^{z\Delta} f\right)(y)\right] = 0.$$

Proof. By using Theorem 2.2 (i) for the operator with complex Gaussian kernel given by $(4\pi z)^{1/2}e^{ay}\left(e^{z\Delta}f\right)(y)$ one obtains that

$$\lim_{y\to+\infty}\left[(4\pi z)^{1/2}e^{ay}\left(e^{z\Delta}f\right)(y)-\lambda H_{\frac{1}{4z},\frac{1}{4z},\frac{1}{4z},a,0,n}(y)\right]=0.$$

Now observe that

$$H_{\frac{1}{4z},\frac{1}{4z},\frac{1}{4z},a,0,n}(y)=(4\pi z)^{1/2}e^{ay}\sum_{m=0}^{[n/2]}\frac{n!}{(n-2m)!m!}z^m y^{n-2m},$$

which tends to zero as $y\to+\infty$.

Thus the result holds. □

Remark 2. The Gauss-Weierstrass semigroup also appears in [1, p. 521], amongst others.

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