Filomat 37:3 (2023), 863–870
https://doi.org/10.2298/FIL2303863T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Least-squares solutions of the generalized reduced biquaternion matrix equations

Yong Tian[a], Xin Liu[a], Yang Zhang[b]

[a]*Macau Institute of Systems Engineering, Faculty of Innovation Engineering, Macau University of Science and Technology, Avenida Wai Long, TaiPa, Macau, 999078, P.R.China*
[b]*Department of Mathematics, University of Manitoba, Winnipeg, MB, R3T 2N2, Canada*

Abstract. In this paper, we introduce the definition of the generalized reduced biquaternions and propose a real representation of a generalized reduced biquaternion matrix. By using the real matrix representation, we discuss the least-squares problems of the classic generalized reduced biquaternion matrix equation $AXC = B$. The least-squares solution to the above matrix equation is formulated by a least-squares real solution of its corresponding real matrix equation. Furthermore, two numerical examples are given to illustrate our results.

1. Introduction

Let $\mathbb{R}$ be the real number field, and $0 \neq u, v \in \mathbb{R}$. We define the generalized reduced biquaternion algebra $\mathbf{Q}_{GR}$ as a commutative 4-dimensional Clifford algebra satisfying:

$$\mathbf{Q}_{GR} = \{q = q_1 + q_2 i + q_3 j + q_4 k : q_1, q_2, q_3, q_4 \in \mathbb{R}\}, \tag{1}$$

where

$$i^2 = u, j^2 = v, k^2 = ijk = uv,$$
$$ij = ji = k, jk = kj = vi, ki = ik = uj.$$

When $u = -1, v = 1$, $\mathbf{Q}_{GR}$ is the reduced biquaternion algebra $\mathbf{Q}_R$, which was first introduced in [21]. As a special case of generalized reduced biquaternions, the reduced biquaternions has been extensively studied and applied to many problems in various areas (see, for example, [4–7, 17–20, 23]). In [5], they studied the functions of reduced biquaternion variables and obtained the generalized Cauchy-Riemann conditions. [17] proposed a simplified reduced biquaternion polar form which is successfully applied for processing color images. In [18], they developed several algorithms for calculating the eigenvalues, eigenvectors and the singular value decompositions of reduced biquaternion matrices. As applications, they applied the

2020 *Mathematics Subject Classification*. Primary 15B33; Secondary 15A24.
Keywords. Generalized (reduced) biquaternions; Real representation; Least-squares problem; Matrix equation.
Received: 13 January 2021; Accepted: 21 February 2022
Communicated by Dragana Cvetković Ilić
This research is supported by Macao Science and Technology Development Fund (No. 0013/2021/ITP), the grants from the National Natural Science Foundation of China (11571220), and the Natural Sciences and Engineering Research Council of Canada (NSERC) (RGPIN 2020-06746), the joint research and Development fund of Wuyi University, Hong Kong and Macao (2019WGALH20), the MUST Faculty Research Grants (FRG-22-073-FIE).
Email addresses: zytianyong@163.com (Yong Tian), xiliu@must.edu.mo (Xin Liu), yang.zhang@umanitoba.ca (Yang Zhang)

results into the processing of color images in the digital media. Two types of multistate Hopfield neural networks based on reduced biquaternions were investigated in [7]. Moreover, [9, 10] discussed some algebraic properties of reduced biquaternion matrices as well as the generalized Sylvester/Stein matrix equations by means of real/complex representations. As efficient methods, the real/complex representation methods have been widely used in the study of many kinds of quaternions. This is one of standard and popular ways to investigate the fundamental properties of different kinds of quaternions, like the Hamilton quaternions, split quaternions, biquaternions, the generalized quaternions, and so on (see, for example, [8–16, 22, 24]). Motivated by the above works, we aim to deal with the following least-squares problem by the real representation method.

In this paper, we discuss the least-squares problem for matrix equation $AXC = B$ over the reduced biquaternions, that is, given $A \in \mathbf{Q}_{GR}^{m\times n}, B \in \mathbf{Q}_{GR}^{m\times q}, C \in \mathbf{Q}_{GR}^{p\times q}$, find $X \in \mathbf{Q}_{GR}^{n\times p}$ such that

$$\|AXC - B\|_F = \min_{X_0\in\mathbf{Q}_{GR}^{n\times p}} \|AX_0C - B\|_F,$$

where the Frobenius norm $\|\cdot\|_F$ is defined in next section.

2. Main results

In this section, we first propose a new real representation of a generalized reduced biquaternion matrix, and then we use this real representation to solve our least-squares problem.

For a given generalized reduced biquaternion matrix $A = A_1 + A_2 i + A_3 j + A_4 k,\ A_1, \ldots, A_4 \in \mathbb{R}^{m\times n}$, we define the real representation A^R of A as

$$A^R = \begin{bmatrix} A_1 & uA_2 & vA_3 & uvA_4 \\ A_2 & A_1 & vA_4 & vA_3 \\ A_3 & uA_4 & A_1 & uA_2 \\ A_4 & A_3 & A_2 & A_1 \end{bmatrix}. \tag{2}$$

The above real representation has the following properties:

Proposition 2.1. *Let $A, B \in \mathbf{Q}_{GR}^{m\times n}, C \in \mathbf{Q}_{GR}^{n\times p}, k \in \mathbb{R}$. Then*

$$(A + B)^R = A^R + B^R,\ (AC)^R = A^R C^R,\ (kB)^R = kB^R, \tag{3}$$

$$R_m{}^{-1} A^R R_n = A^R,\ Q_m{}^{-1} A^R Q_n = A^R,\ S_m{}^{-1} A^R S_n = A^R, \tag{4}$$

where

$$R_n = \begin{bmatrix} 0 & uI_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & uI_n \\ 0 & 0 & I_n & 0 \end{bmatrix},\ Q_n = \begin{bmatrix} 0 & 0 & vI_n & 0 \\ 0 & 0 & 0 & vI_n \\ I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \end{bmatrix},\ S_n = \begin{bmatrix} 0 & 0 & 0 & uvI_n \\ 0 & 0 & vI_n & 0 \\ 0 & uI_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{bmatrix},$$

I_n is the identity matrix of order n, and 0's stand for zero matrices with appropriate sizes. In particular, when $u = -1, v = 1$,

$$A^R = \begin{bmatrix} A_1 & -A_2 & A_3 & -A_4 \\ A_2 & A_1 & A_4 & A_3 \\ A_3 & -A_4 & A_1 & -A_2 \\ A_4 & A_3 & A_2 & A_1 \end{bmatrix}. \tag{5}$$

is the real representation of the reduced biquaternion matrix A. Now using this real representation, we can define the Frobenius norm of the generalized reduced biquaternion matrix A as

$$\|A\|_F \equiv \frac{1}{2}\|A^R\|_F. \tag{6}$$

To solve the mentioned least-squares problem, we need the following useful result.

Lemma 2.2. *Let $A \in \mathbf{Q}_{GR}^{m\times n}, B \in \mathbf{Q}_{GR}^{m\times q}, C \in \mathbf{Q}_{GR}^{p\times q}$. Then*

$$\min_{X_0\in\mathbf{Q}_{GR}^{n\times p}} \|AX_0C - B\|_F = \frac{1}{2}\min_{Y_0\in\mathbb{R}^{4n\times 4p}} \|A^RY_0C^R - B^R\|_F.$$

Proof. Assume that X, Y are the least-squares solutions to the generalized reduced biquaternion matrix equations

$$AXC = B \tag{7}$$

and

$$A^RYC^R = B^R, \tag{8}$$

separately, i.e.,

$$\|AXC - B\|_F = \min_{X_0\in\mathbf{Q}_{GR}^{n\times p}} \|AX_0C - B\|_F.$$

$$\|A^RYC^R - B^R\|_F = \min_{Y_0\in\mathbb{R}^{4n\times 4p}} \|A^RY_0C^R - B^R\|_F.$$

It follows from (3) and (6) that

$$\min_{X_0\in\mathbf{Q}_{GR}^{n\times p}} \|AX_0C - B\|_F = \frac{1}{2}\min_{X_0\in\mathbf{Q}_{GR}^{n\times p}} \|A^RX_0{}^RC^R - B^R\|_F \geq \frac{1}{2}\min_{Y_0\in\mathbb{R}^{4n\times 4p}} \|A^RY_0C^R - B^R\|_F. \tag{9}$$

Conversely, for Y, by (4), we have

$$\|A^RYC^R - B^R\|_F = \|(R_m{}^{-1}A^RR_n)Y(R_p{}^{-1}C^RR_q) - (R_m{}^{-1}B^RR_q)\|_F,$$
$$\|A^RYC^R - B^R\|_F = \|(Q_m{}^{-1}A^RQ_n)Y(Q_p{}^{-1}C^RQ_q) - (Q_m{}^{-1}B^RQ_q)\|_F,$$
$$\|A^RYC^R - B^R\|_F = \|(S_m{}^{-1}A^RS_n)Y(S_p{}^{-1}C^RS_q) - (S_m{}^{-1}B^RS_q)\|_F.$$

Simplifying the right hand-sides of the above three equations gives

$$\|A^RYC^R - B^R\|_F = \|A^R(R_nYR_p{}^{-1})C^R - B^R\|_F,$$
$$\|A^RYC^R - B^R\|_F = \|A^R(Q_nYQ_p{}^{-1})C^R - B^R\|_F,$$
$$\|A^RYC^R - B^R\|_F = \|A^R(S_nYS_p{}^{-1})C^R - B^R\|_F.$$

Now we construct a new matrix as

$$\mathcal{Y} = \frac{1}{4}(Y + R_nYR_p{}^{-1} + Q_nYQ_p{}^{-1} + S_nYS_p{}^{-1}). \tag{10}$$

Then

$$\begin{aligned}\|A^RYC^R - B^R\|_F &\leq \|A^R\mathcal{Y}C^R - B^R\|_F \\ &\leq \frac{1}{4}(\|A^RYC^R - B^R\|_F + \|A^R(R_nYR_p{}^{-1})C^R - B^R\|_F \\ &+ \|A^R(Q_nYQ_p{}^{-1})C^R - B^R\|_F + \|A^R(S_nYS_p{}^{-1})C^R - B^R\|_F) \\ &= \|A^RYC^R - B^R\|_F,\end{aligned}$$

which implies

$$\|A^RYC^R - B^R\|_F = \|A^R\mathcal{Y}C^R - B^R\|_F = \min_{Y_0\in\mathbb{R}^{4n\times 4p}} \|A^RY_0C^R - B^R\|_F. \tag{11}$$

That is, $\mathcal{Y}$ is also a least-squares solution to (8).

Next we prove there exists $\mathcal{X}$ such that $\mathcal{X}^R = \mathcal{Y}$. Assume that

$$Y = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{bmatrix} \in \mathbb{R}^{4n\times 4p},\ Z_{st} \in \mathbb{R}^{n\times p}, s,t = 1,2,3,4.$$

and then replace it in (10), which produces another representation for $\mathcal{Y}$:

$$\mathcal{Y} = \begin{bmatrix} \widehat{Z}_1 & u\widehat{Z}_2 & v\widehat{Z}_3 & uv\widehat{Z}_4 \\ \widehat{Z}_2 & \widehat{Z}_1 & v\widehat{Z}_4 & v\widehat{Z}_3 \\ \widehat{Z}_3 & u\widehat{Z}_4 & \widehat{Z}_1 & u\widehat{Z}_2 \\ \widehat{Z}_4 & \widehat{Z}_3 & \widehat{Z}_2 & \widehat{Z}_1 \end{bmatrix},$$

with

$$\widehat{Z}_1 = \tfrac{1}{4}(Z_{11} + Z_{22} + Z_{33} + Z_{44}), \qquad \widehat{Z}_2 = \tfrac{1}{4}(\tfrac{1}{u}Z_{12} + Z_{21} + \tfrac{1}{u}Z_{34} + Z_{43}),$$

$$\widehat{Z}_3 = \tfrac{1}{4}(\tfrac{1}{v}Z_{13} + \tfrac{1}{v}Z_{24} + Z_{31} + Z_{42}), \quad \widehat{Z}_4 = \tfrac{1}{4}(\tfrac{1}{uv}Z_{14} + \tfrac{1}{v}Z_{23} + \tfrac{1}{u}Z_{32} + Z_{41}).$$

Now, we construct a generalized reduced biquaternion matrix $\mathcal{X}$ by $\mathcal{Y}$:

$$\mathcal{X} = \widehat{Z_1} + \widehat{Z_2}i + \widehat{Z_3}j + \widehat{Z_4}k = \frac{1}{4}\begin{bmatrix} I_n & I_ni & I_nj & I_nk \end{bmatrix}\mathcal{Y}\begin{bmatrix} I_p \\ \frac{1}{u}I_pi \\ \frac{1}{v}I_pj \\ \frac{1}{uv}I_pk \end{bmatrix}.$$

Clearly $\mathcal{X}^R = \mathcal{Y}$. Hence, by (11),

$$\begin{aligned} \frac{1}{2}\min_{Y_0\in\mathbb{R}^{4n\times 4p}} \|A^RY_0C^R - B^R\|_F = \frac{1}{2}\|A^R\mathcal{Y}C^R - B^R\|_F &= \frac{1}{2}\|A^R\mathcal{X}^RC^R - B^R\|_F \\ &= \|A\mathcal{X}C - B\|_F \\ &\geq \min_{X_0\in\mathbf{Q}_{GR}^{n\times p}} \|AX_0C - B\|_F. \end{aligned} \tag{12}$$

Combing (9) and (12), we have

$$\frac{1}{2}\min_{Y_0\in\mathbb{R}^{4n\times 4p}} \|A^RY_0C^R - B^R\|_F = \min_{X_0\in\mathbf{Q}_{GR}^{n\times p}} \|AX_0C - B\|_F.$$

□

Next we solve the least-squares problem by using real representation method.

Theorem 2.3. *Let* $A \in \mathbf{Q}_{GR}^{m\times n}, B \in \mathbf{Q}_{GR}^{m\times q}, C \in \mathbf{Q}_{GR}^{p\times q}$.

(a) If $X \in \mathbf{Q}_{GR}^{n\times p}$ *is a least-squares solution to the matrix equation* (7)*, then* $Y = X^R$ *is a least-squares solution to the matrix equation* (8)*.*

(b) If $Y \in \mathbb{R}^{4n\times 4p}$ *is a least-squares solution to the matrix equation (8), then*

$$X = \frac{1}{16}\begin{bmatrix} I_n & I_ni & I_nj & I_nk \end{bmatrix}(Y + Q_nYQ_p^{-1} + R_nYR_p^{-1} + S_nYS_p^{-1})\begin{bmatrix} I_p \\ \frac{1}{u}I_pi \\ \frac{1}{v}I_pj \\ \frac{1}{uv}I_pk \end{bmatrix} \tag{13}$$

is a least-squares solution to the matrix equation (7).

Proof. Assume that X is a least-squares solution to (7), i.e.,

$$\|AXC - B\|_F = \min_{X_0 \in \mathbf{Q}_{GR}^{n\times p}} \|AX_0C - B\|_F.$$

It follows from (3) and Lemma 2.2 that

$$\|A^R X^R C^R - B^R\|_F = 2\|AXC - B\|_F = 2 \min_{X_0 \in \mathbf{Q}_{GR}^{n\times p}} \|AX_0C - B\|_F = \min_{Y_0 \in \mathbb{R}^{4n\times 4p}} \|A^R Y_0 C^R - B^R\|_F.$$

Thus, $Y = X^R$ is a least-squares solution to (8), i.e., (a) follows.

Suppose Y is a solution to (8). Then $\|A^R Y C^R - B^R\|_F = \min_{Y_0 \in \mathbb{R}^{4n\times 4p}} \|A^R Y_0 C^R - B^R\|_F$. Similar to the proof of Lemma 2.2, we can prove

$$\begin{aligned} \|A^R Y C^R - B^R\|_F &= \|A^R (Q_n Y Q_p^{-1}) C^R - B^R\|_F, \\ \|A^R Y C^R - B^R\|_F &= \|A^R (R_n Y R_p^{-1}) C^R - B^R\|_F, \\ \|A^R Y C^R - B^R\|_F &= \|A^R (S_n Y S_p^{-1}) C^R - B^R\|_F. \end{aligned} \tag{14}$$

Thus, it is easy to verify that $Q_n Y Q_p^{-1}$, $R_n Y R_p^{-1}$, and $S_n Y S_p^{-1}$ are also solutions to (8). If we set

$$\mathcal{Y} = \frac{1}{4}(Y + Q_n Y Q_p^{-1} + R_n Y R_p^{-1} + S_n Y S_p^{-1}). \tag{15}$$

Then we have

$$\begin{aligned} \|A^R Y C^R - B^R\|_F &\leq \|A^R \mathcal{Y} C^R - B^R\|_F \\ &\leq \tfrac{1}{4}(\|A^R Y C^R - B^R\|_F + \|A^R (Q_n Y Q_p^{-1}) C^R - B^R\|_F \\ &\quad + \|A^R (R_n Y R_p^{-1}) C^R - B^R\|_F + \|A^R (S_n Y S_p^{-1}) C^R - B^R\|_F) \\ &= \|A^R Y C^R - B^R\|_F. \end{aligned}$$

Therefore, $\|A^R Y C^R - B^R\|_F = \|A^R \mathcal{Y} C^R - B^R\|_F$, that is, $\mathcal{Y}$ is also a solution to (8).
Let

$$Y = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{bmatrix} \in \mathbb{R}^{4n\times 4p},\ Z_{st} \in \mathbb{R}^{n\times p},\ s, t = 1, 2, 3, 4, \tag{16}$$

and submit it in (15), we obtain

$$\mathcal{Y} = \begin{bmatrix} \widehat{Z}_1 & u\widehat{Z}_2 & v\widehat{Z}_3 & uv\widehat{Z}_4 \\ \widehat{Z}_2 & \widehat{Z}_1 & v\widehat{Z}_4 & v\widehat{Z}_3 \\ \widehat{Z}_3 & u\widehat{Z}_4 & \widehat{Z}_1 & u\widehat{Z}_2 \\ \widehat{Z}_4 & \widehat{Z}_3 & \widehat{Z}_2 & \widehat{Z}_1 \end{bmatrix},$$

with

$$\begin{aligned} \widehat{Z}_1 &= \tfrac{1}{4}(Z_{11} + Z_{22} + Z_{33} + Z_{44}), & \widehat{Z}_2 &= \tfrac{1}{4}(\tfrac{1}{u}Z_{12} + Z_{21} + \tfrac{1}{u}Z_{34} + Z_{43}), \\ \widehat{Z}_3 &= \tfrac{1}{4}(\tfrac{1}{v}Z_{13} + \tfrac{1}{v}Z_{24} + Z_{31} + Z_{42}), & \widehat{Z}_4 &= \tfrac{1}{4}(\tfrac{1}{uv}Z_{14} + \tfrac{1}{v}Z_{23} + \tfrac{1}{u}Z_{32} + Z_{41}). \end{aligned}$$

Now, we construct a generalized reduced biquaternion matrix X by $\mathcal{Y}$ as follows:

$$X = \widehat{Z}_1 + \widehat{Z}_2 i + \widehat{Z}_3 j + \widehat{Z}_4 k = \frac{1}{4}\begin{bmatrix} I_n & I_n i & I_n j & I_n k \end{bmatrix} \mathcal{Y} \begin{bmatrix} I_p \\ \frac{1}{u} I_p i \\ \frac{1}{v} I_p j \\ \frac{1}{uv} I_p k \end{bmatrix}. \tag{17}$$

Clearly, $X^R = \mathcal{Y}$. This means that $X^R = \mathcal{Y}$ is a solution to (8), i.e.,

$$\|A^R X^R C^R - B^R\|_F = \min_{Y_0 \in \mathbb{R}^{4n\times 4p}} \|A^R Y_0 C^R - B^R\|_F. \tag{18}$$

It follows from Lemma 2.2 and (18) that

$$\|AXC - B\|_F = \frac{1}{2}\|A^R X^R C^R - B^R\|_F = \min_{X_0 \in \mathbf{Q}_{GR}^{n\times p}} \|AX_0C - B\|_F.$$

Hence X given by (17) is a solution to (7). □

In the special case: $u = -1$ and $v = 1$, by Theorem 2.3, we have the following corollary for the least-squares solutions to the matrix equation (7) over the reduced biquaternions.

Corollary 2.4. *Let $A \in \mathbf{Q}_R^{m\times n}, B \in \mathbf{Q}_R^{m\times q}, C \in \mathbf{Q}_R^{p\times q}$. Then*

(a) If $X \in \mathbf{Q}_R^{n\times p}$ is a least-squares solution to the reduced biquaternion matrix equation

$$AXC = B, \tag{19}$$

then $Y = X^R \in \mathbb{R}^{4n\times 4p}$ is a least-squares solution to the real matrix equation

$$A^R Y C^R = B^R. \tag{20}$$

(b) If $Y \in \mathbb{R}^{4n\times 4p}$ is a least-squares solution to the real matrix equation (20), then

$$X = \frac{1}{16}\begin{bmatrix} I_n & I_n i & I_n j & I_n k \end{bmatrix}(Y + Q_n Y Q_p^{-1} + R_n Y R_p^{-1} + S_n Y S_p^{-1})\begin{bmatrix} I_p \\ -I_p i \\ I_p j \\ -I_p k \end{bmatrix}$$

is a least-squares solution to the reduced biquaternion matrix equation (19), where

$$R_t = \begin{bmatrix} 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \end{bmatrix}, \; Q_t = \begin{bmatrix} 0 & 0 & I_t & 0 \\ 0 & 0 & 0 & I_t \\ I_t & 0 & 0 & 0 \\ 0 & I_t & 0 & 0 \end{bmatrix}, \; S_t = \begin{bmatrix} 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \\ 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{bmatrix}, \; t = n, p.$$

Example 2.5. *Given the generalized biquaternion matrices*

$$A = \begin{bmatrix} i & 1+j \\ -1+j & -k \end{bmatrix}, \; B = \begin{bmatrix} -2+4i+3k \\ 2-2i+j-2k \end{bmatrix}.$$

Find the least-squares solution of the generalized biquaternion matrix equation

$$AX = B \tag{21}$$

with $u = 1, v = 1$.

By Theorem 2.3, we consider the real matrix equation $A^R Y = B^R$ with

$$A^R = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \; B^R = \begin{bmatrix} -2 & 4 & 0 & 3 \\ 2 & -2 & 1 & -2 \\ 4 & -2 & 3 & 0 \\ -2 & 2 & -2 & 1 \\ 0 & 3 & -2 & 4 \\ 1 & -2 & 2 & -2 \\ 3 & 0 & 4 & -2 \\ -2 & 1 & -2 & 2 \end{bmatrix}.$$

Since rank (A^R) = *rank* $(A^R, B^R) = 8$, *the real matrix equation*

$$A^R Y = B^R$$

has a unique least-squares solution

$$Y = \begin{bmatrix} 7 & -6 & 6 & -4 \\ 0 & 0 & 4 & -3 \\ -6 & 7 & -4 & 6 \\ 0 & 0 & -3 & 4 \\ 6 & -4 & 7 & -6 \\ 4 & -3 & 0 & 0 \\ -4 & 6 & -6 & 7 \\ -3 & 4 & 0 & 0 \end{bmatrix}.$$

By direct computation, we obtain

$$X = \frac{1}{16}\begin{bmatrix} I_2 & I_2 i & I_2 j & I_2 k \end{bmatrix}(Y + Q_2 Y Q_2^{-1} + R_2 Y R_2^{-1} + S_2 Y S_2^{-1})\begin{bmatrix} I_2 \\ I_2 i \\ I_2 j \\ I_2 k \end{bmatrix}$$

$$= \begin{bmatrix} 7 - 6i + 6j - 4k & 4j - 3k \end{bmatrix}^T$$

is the least-squares solution to the generalized reduced biquaternion matrix equation $AX = B$.

Example 2.6. *Find the least-squares solution of the reduced biquaternion matrix equation* (21) *with* $u = -1, v = 1$.

By Corollary 2.4, we consider the corresponding real matrix equation

$$A^R Y = B^R, \tag{22}$$

with

$$A^R = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad B^R = \begin{bmatrix} -2 & -4 & 0 & -3 \\ 2 & 2 & 1 & 2 \\ 4 & -2 & 3 & 0 \\ -2 & 2 & -2 & 1 \\ 0 & -3 & -2 & -4 \\ 1 & 2 & 2 & 2 \\ 3 & 0 & 4 & -2 \\ -2 & 1 & -2 & 2 \end{bmatrix}.$$

Since rank (A^R) = *rank* $(A^R, B^R) = 8$, *the matrix equation* (22) *has a unique solution*

$$Y = \begin{bmatrix} 1 & -6 & 0 & -4 \\ 4 & 0 & 0 & -3 \\ 6 & 1 & 4 & 0 \\ 0 & 4 & 3 & 0 \\ 0 & -4 & 1 & -6 \\ 0 & -3 & 4 & 0 \\ 4 & 0 & 6 & 1 \\ 3 & 0 & 0 & 4 \end{bmatrix}.$$

By direct computation, we have that

$$X = \frac{1}{16}\begin{bmatrix} I_2 & I_2 i & I_2 j & I_2 k \end{bmatrix}(Y + Q_2 Y Q_2^{-1} + R_2 Y R_2^{-1} + S_2 Y S_2^{-1})\begin{bmatrix} I_2 \\ -I_2 i \\ I_2 j \\ -I_2 k \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 6i + 4k & 4 + 3k \end{bmatrix}^T$$

is the least-squares solution to the reduced biquaternion matrix equation $AX = B$.

References

[1] W. Rudin, Real and Complex Analysis, (3rd edition), McGraw-Hill, New York, 1986.

[2] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.

[3] P. Erdös, S. Shelah, Separability properties of almost-disjoint families of sets, Israel Journal of Mathematics 12 (1972) 207–214.

[4] F. Catoni, R. Cannata, E. Nichelatti, P. Zampetti, Hypercomplex numbers and functions of hypercomplex variable: a matrix study, Adv. Appl. Clifford Algebras 15 (2005) 183–213.

[5] F. Catoni, R. Cannata, P. Zampetti, An introduction to commutative quaternions, Adv. Appl. Clifford Algebras 16 (2006) 1–28.

[6] L. Guo, M. Zhu, X. Ge, Reduced biquaternion canonical transform, convolution and correlation, Signal Process 91 (2011) 2147–2153.

[7] T. Isokawa, H. Nishimura, N. Matsui, Commutative quaternion and multistate hopfield neural networks, In Proc. Int. Joint Conf. Neural Netw. (2010) 1281–21286.

[8] T. S. Jiang, Z. Z. Zhang, Z. W. Jiang, Algebraic techniques for Schrödinger equations in split quaternionic mechanics, Comput. Math. Appl. 75 (7) (2018) 2217–2222.

[9] H. H. Kösal, M. Akyiğit, M. Tosun, Consimilarity of commutative quaternion matrices, Miskolc Math. Notes 16 (2) (2015) 965–977.

[10] H. H. Kösal, M. Tosun, Commutative quaternion matrices, Adv. Appl. Clifford Algebras 24 (2014) 769–779.

[11] X. Liu, Y. Zhang, Consistency of split quaternion matrix equations $AX^{\star} - XB = CY + D$ and $X - AX^{\star}B = CY + D$, Adv. Appl. Clifford Algebras 29 (2019) No. 64.

[12] X. Liu, Z. He, The split quaternion matrix equation $AX = B$, Banach Journal of Mathematical Analysis 14 (2020) 228–248.

[13] X. Liu, Q. W. Wang, Y. Zhang, Consistency of quaternion matrix equations $AX^{\star} - XB = C$ and $X - AX^{\star}B = C$, Electronic of Linear Algebra 35 (2019) 394–407.

[14] X. Liu, Y. Zhang, Least-squares solutions $X = \pm X^{\eta *}$ to split quaternion matrix equation $AXA^{\eta *} = B$, Mathematical Methods in the Applied Sciences 43 (2019) 2189–2201.

[15] K. E. Özen, M. Tosun, On the matrix algebra of elliptic biquaternions, Mathematical Methods in the Applied Sciences 43 (6) (2020) 2984–2998.

[16] K. E. Özen, M. Tosun, p-trigonometric approach to elliptic biquaternions, Advances in Applied Clifford Algebras 28 (3) (2018) 1–16.

[17] S. C. Pei, J. H. Chang, J. J. Ding, Commutative reduced biquaternions and their fourier transform for signal and image processing applications, IEEE Transactions on Signal Processing. 52 (2004) 2012–2031.

[18] S. C. Pei, J. H. Chang, J. J. Ding, M. Y. Chen, Eigenvalues and singular value decompositions of reduced biquaternion matrices, IEEE Trans. Circ. Syst. 55 (2008) 2673–2685.

[19] D. A. Pinotsis, Segre Quaternions, Spectral Analysis and a Four-Dimensional Laplace Equation, in Progress in Analysis and its Applications, M. Ruzhansky and J. Wirth, eds., World Scientific, Singapore, 2010, pp. 240.

[20] H. D. Schtte, J. Wenzel, Hypercomplex numbers in digital signal processing, Proc IEEE Int Symp Circuits Syst. 2 (1990) 1557–1560.

[21] C. Segre, The real representations of complex elements and extension to bicomplex systems, Math. Ann. 40 (1892) 413–467.

[22] Q. W. Wang, X. X. Yang, S. F. Yuan, The least square solution with the least norm to a system of quaternion matrix equations, Iran J. Sci. Technol. Trans. Sci., 42 (2018) 1317–1325.

[23] S. F. Yuan, Y. Tian, M. Z. Li, On Hermitian solutions of the reduced biquaternion matrix equation $(AXB, CXD) = (E, G)$, Linear and Multilinear Algebra 68 (2020), 1355–1373.

[24] C. E. Yu, X. Liu and Y. Zhang, The generalized quaternion matrix equation $AXB + CX^{*}D = E$, Mathematical Methods in the Applied Sciences, 43 (2020) 8506–8517.