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The forward order law for Moore-Penrose inverse of multiple matrix product

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Abstract. The relationship between the forward order law product $A_1^\dagger A_2^\dagger \cdots A_n^\dagger$ of the Moore-Penrose inverses of $A_1, A_2, \cdots, A_n$ and the seven common types of generalized inverse of $A_1A_2\cdots A_n$ will be studied in this paper. Especially, we will give the necessary and sufficient condition for the n terms forward order law

$$(A_1A_2\cdots A_n)^\dagger = A_1^\dagger A_2^\dagger \cdots A_n^\dagger.$$

1. Introduction

In this paper we use the following notations. $C^{m\times n}$ denotes the set of m by n matrices of complex entries, $C^m = C^{m\times 1}$, I_m denotes the identity matrix of order m, $O_{m\times n}$ is the m by n matrix with all zero entries (if no confusion occurs, we will drop the subscript). For a matrix $A \in C^{m\times n}$, $r(A)$ is the rank of A, A^* is the conjugate transpose of A, $R(A)$ and $N(A)$ are respectively the range space and the rank of the matrix A.

Let $A \in C^{m\times n}$ and consider the following four Penrose equations [20]:

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA. \tag{1}$$

For any matrix $A \in C^{m\times n}$, let $A\{i, j, \cdots, k\}$ denote the set of matrices $X \in C^{n\times m}$ which satisfy equations $(i), (j), \cdots, (k)$ from among equations $(1), (2), \cdots, (4)$ of (1.1). A matrix in $A\{i, j, \cdots, k\}$ is called an $\{i, j, \cdots, k\}$-inverse of A and denoted by $A^{(i,j,\cdots,k)}$. For example, an n by m matrix X of the set $A\{1\}$ is called a $\{1\}$-inverse of A and is denoted by $X = A^{(1)}$. The well-known seven common types of generalized inverse of A introduced from (1.1) are, respectively, the $\{1\}$-inverse, $\{1, 2\}$-inverse, $\{1, 3\}$-inverse, $\{1, 4\}$-inverse, $\{1, 2, 3\}$-inverse, $\{1, 2, 4\}$-inverse and $\{1, 2, 3, 4\}$-inverse, the last being the unique Moore-Penrose inverse of A and is denoted by $X = A^{(1,2,3,4)} = A^\dagger$. In particular, when A is nonsingular, then it is easily seen that $A^\dagger = A^{-1}$. We refer the reader to [1, 27] for basic results on generalized inverses.

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The concepts of generalized inverse were shown to be very useful in various applied mathematical settings. For example, applications to singular differential or difference equations, Markov chains, cryptography, iterative method or multibody system dynamics, and so on, which can be found in [1, 10, 16, 21, 22, 27]. In above applied mathematical settings, the large-scale scientific computing problems eventually translate to the least square problems. Using generalized inverse to give some fast and effective iterative solution algorithms for these least square problems has attracted considerable attention, and many interesting results have been obtained, see [1, 6, 10, 21, 27].

Suppose $A_i \in C^{m\times m}$, $i = 1, 2, \cdots, n$, and $b \in C^m$, the least squares problem is finding $x \in C^m$ that minimizes the norm:

$$\|(A_1A_2\cdots A_n)x - b\|_2 \tag{2}$$

is used in many practical scientific problems. Any solution x of the above LS can be expressed as $x = (A_1A_2\cdots A_n)^{(1,3)}b$. If the $(A_1A_2\cdots A_n)x = b$ is consistent, then the minimum norm solution x has the form $x = (A_1A_2\cdots A_n)^{(1,4)}b$. The unique minimal norm least square solution x of the above LS is $x = (A_1A_2\cdots A_n)^{\dagger}b$. One of the problems concerning the above LS is under what condition the reverse order law

$$A_n^{(i,j,\cdots,k)}A_{n-1}^{(i,j,\cdots,k)}\cdots A_1^{(i,j,\cdots,k)} = (A_1A_2\cdots A_n)^{(i,j,\cdots,k)} \tag{3}$$

hold. The other problem concerns with the above LS is under what condition the forward order law

$$A_1^{(i,j,\cdots,k)}A_2^{(i,j,\cdots,k)}\cdots A_n^{(i,j,\cdots,k)} = (A_1A_2\cdots A_n)^{(i,j,\cdots,k)} \tag{4}$$

hold.

If (1.3) or (1.4) is true, then according to the reverse order law (1.3) or the forward order law (1.4) and the iterative algorithm theory, we can naturally construct some ideal iterative sequence and then design some fast and effective iterative algorithms to solve (1.2). If (1.3) or (1.4) is not necessarily true, can we find the necessary and sufficient condition for (1.3) or (1.4) ? Then under certain conditions, some iterative algorithms are designed to solve (1.2) according to the reverse order law or the forward order law. Applying the reverse order law or the forward order law to design some fast and effective iterative algorithms to solve (1.2), will avoid multiple decompositions of the correlation matrices and keep it in each iteration. The structure of the iterative sequence reduces the amount of machine storage, maintains the convergence, stability of the algorithm, and improves the operation speed, see [1, 6, 8, 19, 21, 27].

The reverse order law for generalized inverse of multiple matrix products (1.3) yields a class of interesting problems that are fundamental in the theory of generalized inverse of matrices, see [1–5, 21, 27]. As one of the core problems in reverse order law, finding the necessary and sufficient condition for the reverse order law for generalized inverses of matrix products, is useful in both theoretical study and practical scientific computing, which has attracted considerable attention and many interesting results have been obtained, see [7, 9, 11–13, 15, 17, 24, 25].

The forward order law for generalized inverse of multiple matrix products (1.4), originally arose in studying the inverse of multiple matrix kronecker products. Let A_i, $i = 1, 2, \cdots, n$, be n nonsingular matrices, then the kronecker product $A_1 \bigotimes A_2 \bigotimes \cdots \bigotimes A_n$ is nonsingular too, and the inverse of $A_1 \bigotimes A_2 \bigotimes \cdots \bigotimes A_n$ satisfies the forward order law $A_1^{-1} \bigotimes A_2^{-1} \bigotimes \cdots \bigotimes A_n^{-1} = (A_1 \bigotimes A_2 \bigotimes \cdots \bigotimes A_n)^{-1}$. However, this so-called forward order law is not necessarily true for generalized inverse of multiple matrix products. An interesting problem is for given $\{i, j, \cdots, k\}$ and matrices A_i, $i = 1, 2, \cdots, n$, with $A_1A_2\cdots A_n$ meaningful, when

$$A_1^{(i,j,\cdots,k)}A_2^{(i,j,\cdots,k)}\cdots A_n^{(i,j,\cdots,k)} = (A_1A_2\cdots A_n)^{(i,j,\cdots,k)}$$

holds, or when

$$A_1\{i, j, \cdots, k\}A_2\{i, j, \cdots, k\}\cdots A_n\{i, j, \cdots, k\} \subseteq (A_1A_2\cdots A_n)\{i, j, \cdots, k\}$$

In 2007, Xiong and Zheng [29] gave the necessary and sufficient condition for the forward order law $A_1\{1\}A_2\{1\}\cdots A_n\{1\} \subseteq (A_1A_2\cdots A_n)\{1\}$. More equivalent conditions for the forward order law for generalized inverse of multiple matrix products have been derived, see [14, 23, 27, 28, 30, 31].

In this paper, we are interested on the relationship between $A_1^\dagger A_2^\dagger \cdots A_n^\dagger$ and $(A_1 \cdots A_n)^{(i,j,\cdots,k)}$. We will derive some necessary and sufficient conditions for $A_1^\dagger A_2^\dagger \cdots A_n^\dagger \in (A_1 \cdots A_n)\{i, j, \cdots, k\}$, where $\{i, j, \cdots, k\} \in \{\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$. In particular, we will give the necessary and sufficient condition for the n terms forward order law

$$(A_1 A_2 \cdots A_n)^\dagger = A_1^\dagger A_2^\dagger \cdots A_n^\dagger.$$

As the main tools in our discussion, we first present the following lemmas.

Lemma 1.1. [1, 27] *The Moore-Penrose inverse of matrix satisfy the following simple property:*

$$A^\dagger = A^*(AA^*)^\dagger = (A^*A)^\dagger A^* = A^*(A^*AA^*)^\dagger A^*. \tag{5}$$

Lemma 1.2. [21] *Let $A \in C^{m\times n}$ and $X \in C^{n\times m}$.Then*

(1)$X \in A\{1\} \Leftrightarrow AXA = A$;
(2)$X \in A\{1,2\} \Leftrightarrow AXA = A$ *and* $r(X) = r(A)$;
(3)$X \in A\{1,3\} \Leftrightarrow A^*AX = A^*$;
(4)$X \in A\{1,4\} \Leftrightarrow XAA^* = A^*$;
(5)$X \in A\{1,2,3\} \Leftrightarrow A^*AX = A^*$ *and* $r(X) = r(A)$;
(6)$X \in A\{1,2,4\} \Leftrightarrow XAA^* = A^*$ *and* $r(X) = r(A)$;
(7)$X = A^\dagger \Leftrightarrow A^*AX = XAA^* = A^*$ *and* $r(X) = r(A)$.

Lemma 1.3. [18] *Suppose matrices A, B, C and D satisfy the following conditions:*

$$R(B) \subseteq R(A) \text{ and } R(C^*) \subseteq R(A^*) \tag{6}$$

or

$$R(C) \subseteq R(D) \text{ and } R(B^*) \subseteq R(D^*). \tag{7}$$

Then

$$r\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r(A) + r(D - CA^\dagger B) \tag{8}$$

or

$$r\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r(D) + r(A - BD^\dagger C). \tag{9}$$

Lemma 1.4. [26] *Suppose $A_i \in C^{s_i\times l_i}$, $i = 1, 2, \cdots, n$ and $B_i \in C^{s_i\times l_{i+1}}$, $i = 1, 2, \cdots, n-1$, satisfy*

$$B_i = A_i X_i A_{i+1}, \ \ i = 1, 2, \cdots, n-1 \text{ for some } X_i. \tag{10}$$

Then

$$R(B_i) \subseteq R(A_i), \ \ R(B_i^*) \subseteq R(A_{i+1}^*), \ \ i = 1, 2, \cdots, n-1, \tag{11}$$

and the Moore-Penrose inverse of the $n \times n$ block matrix

$$J_n = \begin{pmatrix} O & O & \cdots & \cdots & O & A_n \\ O & O & \cdots & O & A_{n-1} & B_{n-1} \\ \vdots & \vdots & / & / & / & O \\ \vdots & O & / & / & / & \vdots \\ O & A_2 & B_2 & O & \cdots & O \\ A_1 & B_1 & O & O & \cdots & O \end{pmatrix},$$

may be repressed as

$$J_n^\dagger = \begin{pmatrix} F(1,n) & F(1,n-1) & \cdots & F(1,2) & F(1,1) \\ F(2,n) & F(2,n-1) & \cdots & F(2,2) & O \\ \vdots & \vdots & \diagup & \diagup & \vdots \\ F(n-1,n) & F(n-1,n-1) & O & \cdots & O \\ F(n,n) & O & O & \cdots & O \end{pmatrix}, \tag{12}$$

where

$$F(i,i) = A_i^\dagger,\ i = 1, 2, \cdots, n,$$

$$F(i,j) = (-1)^{j-i} A_i^\dagger B_i A_{i+1}^\dagger B_{i+1} \cdots A_{j-1}^\dagger B_{j-1} A_j^\dagger,\ 1 \le i \le j \le n.$$

2. The relationship between the generalized inverses of $A_1A_2\cdots A_n$ and $A_1^\dagger A_2^\dagger \cdots A_n^\dagger$

In this section, we will present the relationship between the forward order product $A_1^\dagger A_2^\dagger \cdots A_n^\dagger$ of the Moore-Penrose inverses of A_1, A_2, $\cdots$, A_n and the seven common types of generalized inverse of the product $A_1A_2\cdots A_n$.

Theorem 2.1. *Suppose $A_i \in C^{m\times m}$, $i = 1, 2, \cdots, n$ .Then the M-P inverse of the $(n+2)\times(n+2)$ block matrix*

$$M = \begin{pmatrix} O & O & O & \cdots & O & O & I_m \\ O & O & O & \cdots & O & A_n^*A_nA_n^* & A_n^* \\ O & O & O & \cdots & A_{n-1}^*A_{n-1}A_{n-1}^* & A_{n-1}^*A_n^* & O \\ \vdots & \vdots & \vdots & \diagup & \diagup & \diagup & \vdots \\ O & O & A_2^*A_2A_2^* & \diagup & \diagup & O & O \\ O & A_1^*A_1A_1^* & A_1^*A_2^* & \diagup & O & O & O \\ I_m & A_1^* & O & \cdots & O & O & O \end{pmatrix} \tag{13}$$

may be repressed as

$$M^\dagger = \begin{pmatrix} M(1,n+2) & M(1,n+1) & \cdots & M(1,2) & M(1,1) \\ M(2,n+2) & M(2,n+1) & \cdots & M(2,2) & O \\ \vdots & \vdots & \diagup & \diagup & \vdots \\ M(n+1,n+2) & M(n+1,n+1) & O & \cdots & O \\ M(n+2,n+2) & O & O & \cdots & O \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} & M(1,1) = M(n+2,n+2) = I_m, \\ & M(i,i) = (A_{i-1}^*A_{i-1}A_{i-1}^*)^\dagger,\ \ i = 2, 3, \cdots, n+1, \\ & M(1,j) = (-1)^{j-1} A_1^*(A_1^*A_1A_1^*)^\dagger A_1^*A_2^*(A_2^*A_2A_2^*)^\dagger \cdots A_{j-2}^*A_{j-1}^*(A_{j-1}^*A_{j-1}A_{j-1}^*)^\dagger, \\ & \ \ j = 2, 3, \cdots, n+1 \\ & M(i,n+2) = (-1)^{n+2-i}(A_{i-1}^*A_{i-1}A_{i-1}^*)^\dagger A_{i-1}^*A_i^*(A_i^*A_iA_i^*)^\dagger \cdots A_{n-1}^*A_n^*(A_n^*A_nA_n^*)^\dagger A_n^*, \\ & \ \ i = 2, 3, \cdots, n+1 \\ & M(i,j) = (-1)^{j-i}(A_{i-1}^*A_{i-1}A_{i-1}^*)^\dagger A_{i-1}^*A_i^*(A_i^*A_iA_i^*)^\dagger \cdots A_{j-2}^*A_{j-1}^*(A_{j-1}^*A_{j-1}A_{j-1}^*)^\dagger, \\ & \ \ 2 \le i \le j \le n. \end{aligned}$$

In particular,

$$\begin{aligned} & M(1,n+2) = PM^{\dagger}Q \\ = \ & (-1)^{n+2-1}(I_m)^{\dagger}A_1^*(A_1^*A_1A_1^*)^{\dagger}A_1^*A_2^*(A_2^*A_2A_2^*)^{\dagger}A_2^*A_3^*\cdots A_{n-1}^*A_n^*(A_n^*A_nA_n^*)^{\dagger}A_n^*(I_m)^{\dagger} \\ = \ & (-1)^{n+1}A_1^{\dagger}A_2^{\dagger}\cdots A_n^{\dagger}, \end{aligned} \tag{15}$$

where $P = (I_m,\ O,\ \cdots,\ O)$ *and* $Q = (I_m,\ O,\ \cdots,\ O)^*$.

Proof. Combining the formula (2.1) with Lemma 1.1 , we have

$$A_1^* = I_mA_1^{\dagger}A_1A_1^* = I_m(A_1^*A_1)^{\dagger}A_1^*A_1A_1^*, \text{ and } R(A_1^*) \subseteq R(I_m),\ \ R(A_1) \subseteq R(A_1A_1^*A_1). \tag{16}$$

$$\begin{aligned} & A_i^*A_{i+1}^* = A_i^*A_iA_i^{\dagger}A_{i+1}^{\dagger}A_{i+1}A_{i+1}^* = A_i^*A_iA_i^*(A_iA_i^*)^{\dagger}(A_{i+1}^*A_{i+1})^{\dagger}A_{i+1}^*A_{i+1}A_{i+1}^*, \text{ and} \\ & R(A_i^*A_{i+1}^*) \subseteq R(A_i^*A_iA_i^*),\ \ R(A_{i+1}A_i) \subseteq R(A_{i+1}A_{i+1}^*A_{i+1}),\ \ i = 1,2,\cdots,n-1. \end{aligned} \tag{17}$$

$$A_n^* = A_n^*A_nA_n^{\dagger}I_m = A_n^*A_nA_n^*(A_nA_n^*)^{\dagger}I_m, \text{ and } R(A_n^*) \subseteq R(A_n^*A_nA_n^*),\ \ R(A_n) \subseteq R(I_m). \tag{18}$$

From the formulas (2.4)-(2.6) and the formulas (1.10)-(1.12) in Lemma 1.4, we have the results in Theorem 2.1.

In particular, from Lemma 1.1, we have

$$A_i^{\dagger} = A_i^*(A_i^*A_iA_i^*)^{\dagger}A_i^*,\quad i = 1,2,\cdots,n,$$

then the last equality in (2.3) holds. ■

We know that for any matrix $S \in C^{m\times n}$,

$$r(S^*SS^*) = r(S^*S) = r(S^*) = r(S). \tag{19}$$

By the formula (2.7) and the structure of M in (2.1), we at once see that it has the following simple properties, which will be used in the sequel.

Theorem 2.2. *Let* M, P *and* Q *be given as in Theorem 2.1 and let* $A = A_1A_2\cdots A_n$. *Then*

$$r(M) = 2m + r(A_1) + r(A_2) + \cdots + r(A_n), \tag{20}$$

$$R(Q) \subseteq R(M) \text{ and } R(P^*) \subseteq R(M^*), \tag{21}$$

$$R(QA) \subseteq R(M) \text{ and } R(P^*A^*) \subseteq R(M^*). \tag{22}$$

Proof. Let

$$D_1 = \begin{pmatrix} I_m & -A_1^* & O & \cdots & O \\ O & I_m & O & \cdots & O \\ O & O & I_m & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & I_m \end{pmatrix},\ D_2 = \begin{pmatrix} I_m & O & O & \cdots & O \\ O & I_m & -(A_1A_1^*)^{\dagger}A_2^* & \cdots & O \\ O & O & I_m & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & I_m \end{pmatrix}, \cdots,$$

$$D_n = \begin{pmatrix} I_m & \cdots & O & O & O \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ O & \cdots & I_m & -(A_nA_{n-1})^{\dagger} & O \\ O & \cdots & O & I_m & O \\ O & \cdots & O & O & I_m \end{pmatrix}, D_{n+1} = \begin{pmatrix} I_m & \cdots & O & O & O \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ O & \cdots & I_m & O & O \\ O & \cdots & O & I_m & -(A_nA_n^*)^{\dagger} \\ O & \cdots & O & O & I_m \end{pmatrix},$$

$$D_{n+2} = \begin{pmatrix} O \\ O \\ \vdots \\ O \\ I_m \end{pmatrix}, \tag{23}$$

and let

$$T_1=\begin{pmatrix} I_m & O & O & \cdots & O\\ -A_n^* & I_m & O & \cdots & O\\ O & O & I_m & \cdots & O\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ O & O & O & \cdots & I_m\end{pmatrix},\ T_2=\begin{pmatrix} I_m & O & O & \cdots & O\\ O & I_m & O & \cdots & O\\ O & -A_{n-1}^*(A_n^*A_n)^\dagger & I_m & \cdots & O\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ O & O & O & \cdots & I_m\end{pmatrix},\cdots,$$

$$T_n=\begin{pmatrix} I_m & \cdots & O & O & O\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ O & \cdots & I_m & O & O\\ O & \cdots & -A_1^*(A_2^*A_2)^\dagger & I_m & O\\ O & \cdots & O & O & I_m\end{pmatrix},T_{n+1}=\begin{pmatrix} I_m & \cdots & O & O & O\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ O & \cdots & I_m & O & O\\ O & \cdots & O & I_m & O\\ O & \cdots & O & -(A_1^*A_1)^\dagger & I_m\end{pmatrix},$$

$$T_{n+2}=\left(O,O,\cdots,O,I_m\right). \tag{24}$$

From the formulas (2.1) and (2.11), we have

$$MD_1\cdots D_{n+1}=\begin{pmatrix} O & O & \cdots & O & I_m\\ O & O & \cdots & A_n^*A_nA_n^* & O\\ \vdots & \vdots & \iddots & \vdots & \vdots\\ O & A_1^*A_1A_1^* & \cdots & O & O\\ I_m & O & \cdots & O & O\end{pmatrix} \text{ and } MD_1\cdots D_{n+1}D_{n+2}=Q. \tag{25}$$

Since D_i, $i=1,2,\cdots,n+1$ are nonsingular, then combining the formula (2.7) with (2.13), we have

$$r(M)=r(MD_1D_2\cdots D_{n+1})=2m+r(A_1)+r(A_2)+\cdots+r(A_n), \tag{26}$$

and

$$R(QA)\subseteq R(Q)=R(MD_1D_2\cdots D_{n+1}D_{n+2})\subseteq R(M). \tag{27}$$

On the other hand, from the formulas (2.1) and (2.12), we have

$$T_{n+1}\cdots T_1M=\begin{pmatrix} O & O & \cdots & O & I_m\\ O & O & \cdots & A_n^*A_nA_n^* & O\\ \vdots & \vdots & \iddots & \vdots & \vdots\\ O & A_1^*A_1A_1^* & \cdots & O & O\\ I_m & O & \cdots & O & O\end{pmatrix} \text{ and } T_{n+2}T_{n+1}\cdots T_2T_1M=Q^*=P. \tag{28}$$

From the formula (2.16), we have

$$R(P^*A^*)\subseteq R(P^*)=R((T_{n+2}T_{n+1}\cdots T_2T_1M)^*)=R(M^*T_1^*T_2^*\cdots T_{n+1}^*T_{n+2}^*)\subseteq R(M^*). \tag{29}$$

Combining the formulas (2.14), (2.15) with (2.17), we have the results in Theorem 2.2. ■

From Theorem 2.1 and Theorem 2.2, the necessary and sufficient condition can be derived for $X=A_1^\dagger A_2^\dagger\cdots A_n^\dagger$ to be a {1}-inverse, {1, 2}-inverse, {1, 3}-inverse, {1, 4}-inverse, {1, 2, 3}-inverse, {1, 2, 4}-inverse or the Moore-Penrose inverse of $A=A_1A_2\cdots A_n$.

Theorem 2.3. *Suppose $A=A_1A_2\cdots A_n$ and $X=A_1^\dagger A_2^\dagger\cdots A_n^\dagger$, where $A_i\in C^{m\times m}$, $i=1,2,\cdots,n$. M, P and Q are given by Theorem 2.1. Then X is a inner inverse of A, that is, $X\in A\{1\}$ if and only if $A_1,A_2,\cdots,A_n$ and A satisfy the following rank equality:*

$$r\begin{pmatrix}(-1)^nA & E_1\\ E_2 & N\end{pmatrix}=2m+r(A_1)+r(A_2)+\cdots+r(A_n)-r(A), \tag{30}$$

where $E_1 = (O, \cdots, O, I_m)$, $E_2 = (O, \cdots, O, I_m)^*$ *and*

$$N = \begin{pmatrix} O & O & \cdots & O & A_n^*A_nA_n^* & A_n^* \\ O & O & \cdots & A_{n-1}^*A_{n-1}A_{n-1}^* & A_{n-1}^*A_n^* & O \\ \vdots & \vdots & \diagup & \diagup & \diagup & \vdots \\ O & A_2^*A_2A_2^* & \diagup & \diagup & O & O \\ A_1^*A_1A_1^* & A_1^*A_2^* & \diagup & O & O & O \\ A_1^* & O & \cdots & O & O & O \end{pmatrix}.$$

Proof. From the formulas (2.1)-(2.3) in Theorem 2.1, we have

$$X = A_1^\dagger A_2^\dagger \cdots A_n^\dagger = (-1)^{n+1} PM^\dagger Q \tag{31}$$

and

$$M = \begin{pmatrix} O & E_1 \\ E_2 & N \end{pmatrix}. \tag{32}$$

By Lemma 1.2 (1), we know that $X \in A\{1\}$ if and only if

$$r(A - AXA) = 0.$$

Since

$$r(A - AXA) = r(A - (-1)^{n+1}APM^\dagger QA) = r((-1)^{n+1}A - APM^\dagger QA), \tag{33}$$

we get that $X \in A\{1\}$ if and only if

$$r((-1)^{n+1}A - APM^\dagger QA) = 0.$$

Combining Lemma 1.3 with (2.9) and (2.10) in Theorem 2.2, we have

$$\begin{aligned} & r((-1)^{n+1}A - APM^\dagger QA) \\ = \; & r\begin{pmatrix} (-1)^{n+1}A & AP \\ QA & M \end{pmatrix} - r(M) \\ = \; & r\begin{pmatrix} (-1)^{n+1}A & O \\ O & M - (-1)^{n+1}QAP \end{pmatrix} - r(M) \\ = \; & r(M + (-1)^n QAP) + r(A) - r(M). \end{aligned} \tag{34}$$

From the structures of M, P and Q shown in (2.3) and the results in (2.20), we have

$$r(M + (-1)^n QAP) = r[\begin{pmatrix} O & E_1 \\ E_2 & N \end{pmatrix} + r\begin{pmatrix} (-1)^n A & O \\ O & O \end{pmatrix}] = r\begin{pmatrix} (-1)^n A & E_1 \\ E_2 & N \end{pmatrix} \tag{35}$$

Substituting (2.23) and (2.8) into (2.22), and combining (2.21), we arrive at (2.18). ∎

Theorem 2.4. *Suppose M, P and Q are given by Theorem 2.1, A and X are given by Theorem 2.3. Then X is a reflexive inner inverse of A, that is, $X \in A\{1, 2\}$ if and only if $A_1, A_2, \cdots, A_n$ and A satisfy (2.18) and the following rank equality:*

$$r(N) = r(A) + r(A_1) + r(A_2) + \cdots + r(A_n), \tag{36}$$

where N is given as in (2.18).

Proof. By Lemma 1.2 (2), we know that $X \in A\{1,2\}$ if and only if

$$r(A - AXA) = 0 \text{ and } r(X) = r(A). \tag{37}$$

The result in Theorem 2.3 shows that the first rank equality in (2.25) is equivalent to (2.18). We will now prove the second rank equality in (2.25) is equivalent to (2.24).

From (2.19), we easily see that

$$r(X) = r((-1)^{n+1}PM^{\dagger}Q) = r(-PM^{\dagger}Q). \tag{38}$$

By Lemma 1.3 and Theorem 2.2, we have

$$\begin{aligned} r(X) &= r((-1)^{n+1}PM^{\dagger}Q) = r\begin{pmatrix} M & Q \\ P & O \end{pmatrix} - r(M) \\ &= r\begin{pmatrix} O & E_1 & I_m \\ E_2 & N & O \\ I_m & O & O \end{pmatrix} - r(M) = r\begin{pmatrix} O & O & I_m \\ O & N & O \\ I_m & O & O \end{pmatrix} - r(M) = 2m + r(N) - r(M). \end{aligned} \tag{39}$$

Combining (2.26) and (2.27), the second rank equality in (2.25) will lead to (2.24). ■

Theorem 2.5. *Suppose M, P and Q are given by Theorem 2.1, A and X are given by Theorem 2.3. Then X is a least squares inner inverse of A, that is, $X \in A\{1,3\}$ if and only if $A_1, A_2, \cdots, A_n$ and A satisfy the following rank equality:*

$$r\begin{pmatrix} (-1)^n A^*A & A^*E_1 \\ E_2 & N \end{pmatrix} = m + r(A_1) + r(A_2) + \cdots + r(A_n), \tag{40}$$

where E_1, E_2 and N are given as in (2.18).

Proof. According to Lemma 1.2 (3) and (2.19), $X \in A\{1,3\}$ if and only if

$$r(A^* - A^*AX) = 0.$$

Since

$$r(A^* - A^*AX) = r(A^* - (-1)^{n+1}A^*APM^{\dagger}Q) = r((-1)^{n+1}A^* - A^*APM^{\dagger}Q), \tag{41}$$

we get that $X \in A\{1,3\}$ if and only if

$$r((-1)^{n+1}A^* - A^*APM^{\dagger}Q) = 0.$$

According to Lemma 1.3 and the formulas (2.9) and (2.10) in Theorem 2.2, we have

$$\begin{aligned} r(A^* - A^*AX) &= r((-1)^{n+1}A^* - A^*APM^{\dagger}Q) = r\begin{pmatrix} M & Q \\ A^*AP & (-1)^{n+1}A^* \end{pmatrix} - r(M) \\ &= r\begin{pmatrix} O & E_1 & I_m \\ E_2 & N & O \\ A^*A & O & (-1)^{n+1}A^* \end{pmatrix} - r(M) = r\begin{pmatrix} O & O & I_m \\ E_2 & N & O \\ A^*A & (-1)^n A^*E_1 & O \end{pmatrix} - r(M) \\ &= r\begin{pmatrix} A^*A & (-1)^n A^*E_1 \\ E_2 & N \end{pmatrix} + m - r(M) = r\begin{pmatrix} (-1)^n A^*A & A^*E_1 \\ E_2 & N \end{pmatrix} + m - r(M). \end{aligned} \tag{42}$$

Combining (2.8), (2.29) with (2.30), we have the results in Theorem 2.5. ■

The next conclusion can be derived form the formulas (3), (4) in Lemma 1.2 and the results in Theorem 2.5.

Theorem 2.6. *Suppose M, P and Q are given by Theorem 2.1, A and X are given by Theorem 2.3. Then X is a minimum norm inner inverse of A, that is, $X \in A\{1,4\}$ if and only if $A_1, A_2, \cdots, A_n$ and A satisfy the following rank equality:*

$$r\begin{pmatrix} (-1)^n AA^* & E_1 \\ E_2A^* & N \end{pmatrix} = m + r(A_1) + r(A_2) + \cdots + r(A_n), \tag{43}$$

where E_1, E_2 and N is given as in (2.18).

The next three theorems can be seen form the formulas (3)-(6) in Lemma 1.2 and the results in Theorem 2.4 − Theorem 2.6.

Theorem 2.7. *Suppose M, P and Q are given by Theorem 2.1, A and X are given by Theorem 2.3. Then $X \in A\{1,2,3\}$ if and only if $A_1, A_2, \cdots, A_n$ and A satisfy the rank equalities in (2.24) and (2.28).*

Theorem 2.8. *Suppose M, P and Q are given by Theorem 2.1, A and X are given by Theorem 2.3. Then $X \in A\{1,2,4\}$ if and only if $A_1, A_2, \cdots, A_n$ and A satisfy the rank equalities in (2.24) and (2.31).*

Theorem 2.9. *Suppose M, P and Q are given by Theorem 2.1, A and X are given by Theorem 2.3. Then $X \in A\{1,3,4\}$ if and only if $A_1, A_2, \cdots, A_n$ and A satisfy the rank equalities in (2.28) and (2.31).*

According to the formula (7) in Lemma 1.2, we have $X = A_1^\dagger A_2^\dagger \cdots A_n^\dagger = (A_1A_2\cdots A_n)^\dagger = A^\dagger$ if and only if the following three rank equalities hold:

$r(X) = r(A)$ and $r(A^* - A^*AX) = 0$ and $r(A^* - XAA^*) = 0$.

Thus, from Theorem 2.4 − Theorem 2.6 we immediately obtain the following key result.

Theorem 2.10. *Suppose M, P and Q are given by Theorem 2.1, A and X are given by Theorem 2.3. Then X is the Moore-Penrose inverse of A, that is, the forward order law in (2.32) holds, if and only if $A_1, A_2, \cdots, A_n$ and A satisfy the three rank equalities in (2.24), (2.28) and (2.31).*

3. The forward order law for Moore-Penrose inverse of $A_1A_2\cdots A_n$

In addition to the result in Theorem 2.10, we can also deduce another rank equality as a necessary and sufficient condition for the forward order law in (2.32) to hold.

Theorem 3.1. *Suppose M, P and Q are given by Theorem 2.1, A and X are given by Theorem 2.3. Then X is the Moore-Penrose inverse of A, that is, the forward order law*

$$A_1^\dagger A_2^\dagger \cdots A_n^\dagger = (A_1A_2\cdots A_n)^\dagger \tag{44}$$

holds if and only if $A_1, A_2, \cdots, A_n$ and A satisfy the following equality:

$$r\begin{pmatrix} (-1)^n A^*AA^* & A^*E_1 \\ E_2A^* & N \end{pmatrix} = r(A_1) + r(A_2) + \cdots + r(A_n) + r(A), \tag{45}$$

where E_1, E_2 and N are given as in (2.18).

Proof. From (2.3) in Theorem 2.1, we know that $X = A_1^\dagger A_2^\dagger \cdots A_n^\dagger = (A_1A_2\cdots A_n)^\dagger = A^\dagger$ holds if and only if $A_1, A_2, \cdots, A_n$ and A satisfy the following equality:

$$r(A^\dagger - X) = r(A^\dagger - (-1)^{n+1}PM^\dagger Q) = r((-1)^{n+1}A^\dagger - PM^\dagger Q) = 0. \tag{46}$$

Now using the matrices in (3.3), we construct a 3×3 block matrix as follows:

$$G = \begin{pmatrix} M & O & Q \\ O & (-1)^n A^*AA^* & A^* \\ P & A^* & O \end{pmatrix}.$$

It follows from Theorem 2.2 that

$$R\begin{pmatrix} Q \\ A^* \end{pmatrix} \subseteq R\begin{pmatrix} M & O \\ O & (-1)^n A^*AA^* \end{pmatrix},$$

$$R(\begin{pmatrix} P, & A^* \end{pmatrix}^*) \subseteq R\begin{pmatrix} M^* & O \\ O & (-1)^n AA^*A \end{pmatrix}.$$

Hence by the rank formulas in Lemma 1.3, we have

$$\begin{aligned} r(G) &= r\begin{pmatrix} M & O \\ O & (-1)^n A^*AA^* \end{pmatrix} + r(-\begin{pmatrix} P, & A^* \end{pmatrix}\begin{pmatrix} M & O \\ O & (-1)^n A^*AA^* \end{pmatrix}^\dagger \begin{pmatrix} Q \\ A^* \end{pmatrix}) \\ &= r(M) + r(A^*AA^*) + r(PM^\dagger Q - (-1)^{n+1}A^*(A^*AA^*)^\dagger A^*). \end{aligned} \tag{47}$$

Combining the formulas (2.7), (3.3), (3.4) with Lemma 1.1 and Theorem 2.2, we have

$$A^\dagger = A^*(A^*AA^*)^\dagger A^* \ \textit{and} \ r(A^*AA^*) = r(A^*A) = r(A^*) = r(A)$$

and

$$\begin{aligned} r(G) &= r(M) + r(A) + r[(-1)^{n+1}A^\dagger - PM^\dagger Q] \\ &= 2m + r(A_1) + r(A_2) + \cdots + r(A_n) + r(A) + r(A^\dagger - X). \end{aligned} \tag{48}$$

On the other hand, substituting the complete expression of M in (2.20) and then calculating the rank of G will produce the following result

$$\begin{aligned} r(G) &= r\begin{pmatrix} O & E_1 & O & I_m \\ E_2 & N & O & O \\ O & O & (-1)^n A^*AA^* & A^* \\ I_m & O & A^* & O \end{pmatrix} = r\begin{pmatrix} O & O & O & I_m \\ E_2 & N & O & O \\ O & -A^*E_1 & (-1)^n A^*AA^* & A^* \\ I_m & O & A^* & O \end{pmatrix} \\ &= r\begin{pmatrix} O & O & O & I_m \\ O & N & -E_2A^* & O \\ O & -A^*E_1 & (-1)^n A^*AA^* & A^* \\ I_m & O & A^* & O \end{pmatrix} = r\begin{pmatrix} O & O & O & I_m \\ O & N & -E_2A^* & O \\ O & -A^*E_1 & (-1)^n A^*AA^* & O \\ I_m & O & O & O \end{pmatrix} \\ &= r\begin{pmatrix} N & -E_2A^* \\ -A^*E_1 & (-1)^n A^*AA^* \end{pmatrix} + 2m = r\begin{pmatrix} (-1)^n A^*AA^* & A^*E_1 \\ E_2A^* & N \end{pmatrix} + 2m. \end{aligned} \tag{49}$$

Combining (3.3), (3.4),(3.5) with (3.6) will yield the results in Theorem 3.1. ■

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References

[1] A. Ben-Israel and T. N. E. Greville. Generalized Inverses: Theory and Applications. *John Wiley, New York.*, 2003.
[2] D. S. Cvetković-Ilić. Reverse order laws for $\{1,3,4\}$-generalized inverses in C^*-algebras. *Appl. Math. Letters.*, 2011; 24(2): 210-213.
[3] D. S. Cvetković-Ilić. New conditions for the reverse order laws for $\{1,3\}$ and $\{1,4\}$-generalized inverses. *Electronic Journal of Linear Algebra*, 2012; 23: 231-242.
[4] D. S. Cvetković-Ilić and R. Harte. Reverse order laws in C^*-algebras. *Linear Algebra Appl.*, 2011; 434: 1388-1394.
[5] D. S. Cvetković-Ilić and J. Milosevic. Reverse order laws for $\{1,3\}$-generalized inverses. *Linear and Multilinear Algebra*, 2019; 67: 613-624.
[6] S. L. Campbell and C. D. Meyer. Generalized Inverse of Linear Transformations. *Dover, NewYork*, 1979.

[7] D. S. Cvetković-IIić and J. Nikolov. Reverse order laws for $\{1,2,3\}$-generalized inverses. *Appl. Math. Comp.,* 2014; 234 (15): 114-117.
[8] D. S. Cvetković-IIić and J. Nikolov. Reverse order laws for reflexive generalized inverse of operators. *Linear and Multilinear Algebra,* 2015; 63(6): 1167-1175.
[9] D. S. Cvetković-Ilić and V. Pavlovi ć. A comment on some recent results concerning the reverse order law for $\{1,3,4\}$-inverses. *Appl. Math. Comp.,* 2010; 217: 105-109.
[10] D. S. Cvetković-Ilić and Y. Wei. Algebraic Properties of Generalized Inverses, Series: Developments in Mathematics, Vol. 52, *John Springer,* 2017
[11] A. R. Depierro and M. Wei. Reverse order laws for recive generalized inverse of products of matrices. *Linear Algebra Appl.,* 1996; 277: 299-311.
[12] D. S. Djordjevic. Futher results on the reverse order law for generalized inverses. *SIAM J. Matrix. Anal. Appl.,* 2007; 29: 1242-1246.
[13] T. N. E. Greville. Note on the generalized inverses of a matrix products. *SIAM Review,* 1966; 8: 518-521.
[14] X. J. Liu, S. Huang and D. S. Cvetkovic-Ilic. Mixed-tipe reverse-order law for $\{1,3\} - inverses$ over Hilbert spaces. *Applied Mathematics and Computation,* 2012; 218: 8570-8577.
[15] X. J. Liu, S. W and D. S. Cvetkovic-Ilic. New results on reverse order law for $\{1,2,3\}$ and $\{1,2,4\}$-inverses of bounded operators. *Mathematics of Computation,* 2013; 82 (283): 1597-1607.
[16] C. D. Meyer and J. M. Shoaf. Updating finite Markov chains by using technique of group inversion. *J. Statist Comput. Simulation.,* 1980; 11: 163-181.
[17] J. Nikolov and D. S. Cvetković-Ilić. Reverse order laws for the weighted generalized inverses. *Appl. Math. Letters.,* 2011; 24: 2140-2145.
[18] G. Marsaglia and G. P. Styan. Equalities and inequalities for ranks of matrices. *Linear and Multilinear Algebra,* 1974; 2: 269-292.
[19] V. Pavlović and D. S. Cvetković-Ilić. Applications of completions of operator matrices to reverse order law for $\{1\}$-inverses of operators on Hilbert spaces. *Linear Algebra Appl.,* 2015; 484: 219-236.
[20] R. Penrose. A generalized inverse for matrix. *Proc. Cambridge Philos, Soc.,* 1955; 51: 406-413.
[21] C. R. Rao and S. K. Mitra. Generalized Inverses of Matrices and its Applications. *Wiley, New York;* 1971.
[22] B. Simeon, C. Fuhrer and P. Rentrop. The Drazin inverse in multibody system dynamics. *Numer. Math.,* 1993; 64: 521-539.
[23] P. Stanimirovic and M. Tasic. Computing generalized inverses using *LU* factorrization of Matrix product. *Int. J. Comp. Math.,* 2008; 85: 1865-1878.
[24] W. Sun and Y. Wei. Inverse order rule for weighted generalized inverse. *SIAM. J. Matrix Anal.,* 1998; 19: 772-775.
[25] Y. Tian. The Moore-Penrose inverse order of a triple matrix product. *Math. Pract. Theory.,* 1992; 1: 64-70.
[26] Y. Tian. Reverse order laws for generalized inverses of multiple matrix products. *Linear Algebra Appl.,* 1994; 211: 85-100.
[27] G. R. Wang, Y. M. Wei and S. Z. Qiao. Generalized Inverses: Theory and Computations. *Science Press, Beijing.,* 2004.
[28] H. T. Werner. *G*-inverse of Matrix Products Date Analysis Statistical Inference. *Eul-Verlag, Bergisch-Gladbach,* 1992; 531-546.
[29] Z. Xiong and B. Zheng. Forward order law for the generalized inverses of multiple matrix products. *J. Appl. Math and Computing.,* 2007; 25 (1-2): 415-424.
[30] Z. Xiong and Z. Liu. The forward order law for least square g-inverse of multiple matrix products. *Mathematics,* 2019; 7 (3): Article 277: 1-10.
[31] Z. Xiong and Z. Liu. The forward order laws for $\{1,2,3\}$- and $\{1,2,4\}$-inverses of multiple matrix products. *Complex Analysis and Operator Theory,* 2019; 13 (8): 3579-3594.