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New characterizations of partial isometries in rings

Qingbing Xu[a], Long Wang[b,*], Junchao Wei[b]

[a]*Department of Basic courses, Chuzhou Polytechnic College, Chuzhou, Anhui 239000, P. R. China*
[b]*School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, P. R. China*

Abstract. In this paper, we study an element which is both group invertible and Moore-Penrose invertible to be partial isometry by discussing the existence of solutions in a definite set of some given constructive equations.

1. Introduction

Throughout this paper, R will denote an associative ring with unity. Recall that the group inverse [1] of $a \in R$ is the element $x \in R$ which satisfies

$$xax = x, \quad a = axa, \quad ax = xa.$$

The element x above is unique if it exists and is denoted by $a^\#$. The set of all group invertible elements of R is denoted by $R^\#$. In a ring R, an involution $* : R \to R$ is an anti-isomorphism which satisfies $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in R$. R is called a $*$-ring if R is a ring with an involution $*$. In what follows, R is a $*$-ring. An element $a \in R$ is said to be Moore-Penrose invertible if the following equations:

$$(1)\ axa = a,\ (2)\ xax = x,\ (3)\ (ax)^* = ax \text{ and } (4)\ (xa)^* = xa$$

have a common solution. Such a solution is unique if it exists, and is denoted by $a^\dagger$ as usual. The set of all Moore-Penrose invertible elements of R will be denoted by $R^\dagger$. The notion of Moore-Penrose inverse (or MP-inverse) has been investigated by many authors (see, for example, [1, 11, 13, 14]).

An element $a \in R$ is said to be *EP*, if $a \in R^\dagger \cap R^\#$ and $a^\dagger = a^\#$. The set of all *EP* elements of R will be denoted by R^{EP}. Mosić et al. in [4, Theorem 2.1] gave some necessary and sufficient conditions for an element a of R to be EP. Patrćio and Puystjens in [8, Proposition 2] proved that for a MP-invertible element $a \in R$, $a \in R^{EP}$ if and only if $aa^\dagger = a^\dagger a$. It is known by [15, Theorem 7.3] that $a \in R$ is EP if and only if a is group invertible and $(aa^\#)^* = aa^\#$. More results on *EP* elements can also be found in [2, 6, 9, 10, 12, 16]. An element $a \in R$ satisfying $aa^*a = a$ is called a partial isometry. It is proven that a is a partial isometry if and only if $a \in R^\dagger$ and $a^* = a^\dagger$. We write by R^{PI} to denote the set of all partial isometries of R. More results on partial isometries can also be found in [3, 5, 7, 17].

Motivated by the above results, this work is intended to provide some equivalent conditions for an element to be an EP element and partial isometry in rings with involution by using solutions of some equations.

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* Corresponding author
Email addresses: wanglseu@126.com (Qingbing Xu), wanglseu@126.com (Long Wang), jcwei@yzu.edu.cn (Junchao Wei)

2. Some characterizations of partial isometry elements

It is known that if a is a partial isometry, then we have $(a^{\dagger})^*a^{\dagger}(a^{\dagger})^* = aa^{\dagger}(a^{\dagger})^*$. Hence we can construct the following equation:

$$(a^{\dagger})^*x(a^{\dagger})^* = ax(a^{\dagger})^*. \tag{1}$$

Using the equation (1), we can characterize the partial isometry as follows.

Lemma 2.1. *Suppose $a \in R^{\#} \cap R^{\dagger}$, then $a^{\dagger} \in R^{\#}$ and $(a^{\dagger})^{\#} = (aa^{\#})^*a(aa^{\#})^*$.*

Proof. According to the definition of the group inverse, the result can be checked directly. □

Theorem 2.2. *Suppose $a \in R^{\#} \cap R^{\dagger}$, then $a \in R^{PI}$ if and only if Equation (1) has at least one solution in $\chi_a = \{a, a^{\#}, a^{\dagger}, a^*, (a^{\#})^*, (a^{\dagger})^*\}$.*

Proof. " ⇒ " Assume $a \in R^{PI}$, then $a^{\dagger} = a^*$. Hence $x = a^{\dagger}$ is a solution to the equation.

" ⇐ " (1) If $x = a$ is a solution, then $a^2(a^{\dagger})^* = (a^{\dagger})^*a(a^{\dagger})^*$. Post-multiply it by $a^*a^{\#}$, it gives that $a^2(a^{\dagger})^*a^*a^{\#} = (a^{\dagger})^*a(a^{\dagger})^*a^*a^{\#}$. Simplify it, we can obtain $a = (a^{\dagger})^*$ and $a \in R^{PI}$.

(2) If $x = a^{\#}$ is a solution, then one has $(a^{\dagger})^*a^{\#}(a^{\dagger})^* = aa^{\#}(a^{\dagger})^*$. Post-multiply it by a^*a, we have $(a^{\dagger})^*a^{\#}a = a$. Pre-multiply it by aa^*, it gives $aa^*a = a$, and consequently, $a \in R^{PI}$.

(3) If $x = a^{\dagger}$ is a solution, then $(a^{\dagger})^*a^{\dagger}(a^{\dagger})^* = aa^{\dagger}(a^{\dagger})^*$, that is, $(a^{\dagger})^*a^{\dagger}(a^{\dagger})^* = (a^{\dagger})^*$. Post-multiply it by a^*a, we have $(a^{\dagger})^* = a$, and consequently, we can obtain $a \in R^{PI}$.

(4) If $x = a^*$ is a solution, then $(a^{\dagger})^*a^*(a^{\dagger})^* = aa^*(a^{\dagger})^*$, which implies that $(a^{\dagger})^* = a$. That is, $a^{\dagger} = a^*$, and then, we have $a \in R^{PI}$.

(5) If $x = (a^{\#})^*$ is a solution, one deduces that $(a^{\dagger})^*(a^{\#})^*(a^{\dagger})^* = a(a^{\#})^*(a^{\dagger})^*$. It gives $a^{\dagger}a^{\#}a^{\dagger} = a^{\dagger}a^{\#}a^*$. Pre-multiply it by $a^{\dagger}a^3$, then we have $a^{\dagger} = a^*$, and consequently, $a \in R^{PI}$.

(6) If $x = (a^{\dagger})^*$ is a solution, then $((a^{\dagger})^*)^3 = a((a^{\dagger})^*)^2$. That is, $(a^{\dagger})^3 = (a^{\dagger})^2a^*$. Post-multiply it by $(a^{\#})^*$, it gives that

$$\begin{aligned}
(a^{\dagger})^3(a^{\#})^* &= (a^{\dagger})^2a^*(a^{\#})^* \\
&= (a^{\dagger})^2(a^{\#}a)^* \\
&= (a^{\dagger})^2(aa^{\dagger})(a^{\#}a)^* \\
&= (a^{\dagger})^2(a^{\#}aaa^{\dagger})^* \\
&= (a^{\dagger})^2(aa^{\dagger})^* \\
&= (a^{\dagger})^2.
\end{aligned}$$

Pre-multiply it by $((a^{\dagger})^{\#})^2$, we can obtain $a^{\dagger}(a^{\#})^* = a^{\dagger}(a^{\dagger})^{\#} = a^{\dagger}(aa^{\#})^*a(aa^{\#})^*$. That is, $a^{\#}(a^{\dagger})^* = aa^{\#}a^*aa^{\#}(a^{\dagger})^*$. Post-multiply it by a^*a, it gives that $a^{\#}a = aa^{\#}a^*a$. Pre-multiply it by a, then we have $a = aa^*a$, i.e. $a \in R^{PI}$. □

Modify Equation (1) to

$$(a^{\dagger})^*x(a^{\dagger})^* = xa(a^{\dagger})^*. \tag{2}$$

Theorem 2.3. *Suppose $a \in R^{\#} \cap R^{\dagger}$, then $a \in R^{PI}$ if and only if Equation (2) has at least one solution in $\chi_a = \{a, a^{\#}, a^{\dagger}, a^*, (a^{\#})^*, (a^{\dagger})^*\}$.*

Proof. " ⇒ " As $a \in R^{PI}$, we have $a^{\dagger} = a^*$. And then, we can obtain $x = a$ is a solution.

" ⇐ " (1) If $x = a$ is a solution, then $(a^{\dagger})^*a(a^{\dagger})^* = a^2(a^{\dagger})^*$. Post-multiply it by $a^*a^{\#}$, it gives that $(a^{\dagger})^* = a$, and hence $a \in R^{PI}$.

(2) If $x = a^{\#}$ is a solution, one has that $(a^{\dagger})^*a^{\#}(a^{\dagger})^* = a^{\#}a(a^{\dagger})^*$. Then post-multiply the equality by a^*a, and it follows $(a^{\dagger})^*a^{\#}a = a$. Note that the left side of this equality can be rewrited as $(a^{\dagger})^*a^{\#}a = [(a^{\dagger})^*a^{\dagger}a]a^{\#}a = (a^{\dagger})^*$. Hence, we can obtain $(a^{\dagger})^* = a$, and consequently, $a \in R^{PI}$.

(3) If $x = a^\dagger$ is a solution, then $(a^\dagger)^* a^\dagger (a^\dagger)^* = a^\dagger a(a^\dagger)^*$. Post-multiply it by $a^* aa^\#$, then we can obtain $(a^\dagger)^* a^\dagger (a^\dagger)^* a^* aa^\# = a^\dagger a(a^\dagger)^* a^* aa^\#$. That is, $(a^\dagger)^* a^\# = a^\dagger a$. Then, it implies that $(a^\#)^* a^\dagger = a^\dagger a$ and consequently, $a^\dagger aaa^\dagger = [(a^\#)^* a^\dagger]aa^\dagger = (a^\#)^* a^\dagger = a^\dagger a$. Pre-multiply it by $a^\# a$, it follows that $a^\# aa^\dagger a = a^\# aa^\dagger aaa^\dagger$ and $a^\# a = aa^\dagger$. Hence, we can obtain $a \in R^{EP}$. Note that $(a^\dagger)^* a^\# = a^\dagger a$. Pre-multiply it by a, by $a \in R^{EP}$, it gives that $(a^\dagger)^* = a$, and thus $a \in R^{PI}$.

(4) If $x = a^*$ is a solution, one concludes that $(a^\dagger)^* a^* (a^\dagger)^* = a^* a(a^\dagger)^*$, which forces that $(a^\dagger)^* = a^* a(a^\dagger)^*$. Post-multiply it by $a^* aa^\#$, it implies that $aa^\# = a^* a$. Pre-multiply it by a, it gives that $a = aa^* a$, and thus $a \in R^{PI}$.

(5) If $x = (a^\#)^*$ is a solution, then $(a^\dagger)^* (a^\#)^* (a^\dagger)^* = (a^\#)^* a(a^\dagger)^*$. Post-multiply it by a^*, this leads to $(a^\dagger)^* (a^\#)^* = (a^\#)^* a^2 a^\dagger$. Taking an involution of the above equality, we obtain that $a^\# a^\dagger = aa^\dagger a^* a^\#$. Post-multiply it by $a^\dagger a$, it gives that $a^\# a^\dagger a^\dagger a = aa^\dagger a^* a^\# a^\dagger a = aa^\dagger a^* a^\# = a^\# a^\dagger$. Pre-multiply it by a^2, then we have $aa^\dagger a^\dagger a = aa^\dagger$ and $aa^\dagger = a^\dagger aaa^\dagger$. This implies that $aR = a^\dagger aR \subset a^\dagger R$, and $a \in R^{EP}$ by [17, Lemma 2.3]. As $a^\# a^\dagger = aa^\dagger a^* a^\#$, pre-multiply it by a and post-multiply it by a^2, we have $a = aa^* a$.

(6) If $x = (a^\dagger)^*$ is a solution, we have $a^\dagger a^* a^\dagger = (a^\dagger)^3$. Post-multiply it by $a(aa^\#)^*$, it gives that $a^\dagger a^* = a^\dagger a^\dagger$. Pre-multiply it by $(aa^\#)^* a$, we can obtain $a^* = a^\dagger$. The proof is completed. □

3. More characterizations of partial isometry elements

In what follows, we will consider more characterizations of partial isometry elements in terms of the structure of the solution of the equations with two unknown elements. We modify Equation (1) to

$$(a^\dagger)^* x(a^\dagger)^* = ay(a^\dagger)^*. \tag{3}$$

Theorem 3.1. *The general solution of equation (3) is given as the form*

$$\begin{cases} x = p + u - a^\dagger auaa^\dagger \\ y = a^\dagger (a^\dagger)^* p + z - a^\dagger azaa^\dagger, \end{cases} \tag{4}$$

where p, u and z are any elements in rings.

Proof. By checking the equation directly, we obtain that the equation (4) is the solution of the equation (3). Indeed, taking $x = p + u - a^\dagger auaa^\dagger$ into the left side of the equation (3), it follows that the left is equal to $(a^\dagger)^* (p + u - a^\dagger auaa^\dagger)(a^\dagger)^* = (a^\dagger)^* p(a^\dagger)^*$. Moreover, putting $y = a^\dagger (a^\dagger)^* p + z - a^\dagger azaa^\dagger$ into the right side of the equation (3), then we obtain that the right is equal to $a(a^\dagger (a^\dagger)^* p + z - a^\dagger azaa^\dagger)(a^\dagger)^* = (a^\dagger)^* p(a^\dagger)^*$.

Next, we will prove that all solution of the equation (3) have the form of the equation (4). Now, let $x = x_0$ and $y = y_0$ be the solution of the equation (3). It means $(a^\dagger)^* x_0 (a^\dagger)^* = ay_0 (a^\dagger)^*$. Note that

$$a^* ay_0 aa^\dagger = a^* (ay_0 (a^\dagger)^*) a^* = a^* (a^\dagger)^* x_0 (a^\dagger)^* a^* = a^\dagger ax_0 aa^\dagger.$$

Then $x_0 = a^* ay_0 aa^\dagger + x_0 - a^* ay_0 aa^\dagger = a^* ay_0 aa^\dagger + x_0 - a^\dagger ax_0 aa^\dagger$. Set $p = a^* ay_0 aa^\dagger$ and $u = x_0$. Then, $x_0 = p + u - a^\dagger auaa^\dagger$. Moreover, since $a^\dagger (a^\dagger)^* p = a^\dagger \left(a^\dagger\right)^* \left(a^* ay_0 aa^\dagger\right) = a^\dagger aa^\dagger ay_0 aa^\dagger = a^\dagger ay_0 aa^\dagger$, we have $y_0 = a^\dagger ay_0 aa^\dagger + y_0 - a^\dagger ay_0 aa^\dagger = a^\dagger \left(a^\dagger\right)^* p + y_0 - a^\dagger ay_0 aa^\dagger$. Set $z = y_0$. Then $y_0 = a^\dagger (a^\dagger)^* p + z - a^\dagger azaa^\dagger$. The proof is completed. □

Next, some characterizations of partial isometry elements will be given in terms of the particular solutions of equations.

Corollary 3.2. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if the general solution of equation (3) is given as the form*

$$\begin{cases} x = p + u - a^\dagger auaa^\dagger \\ y = a^\dagger ap + z - a^\dagger azaa^\dagger, \end{cases} \tag{5}$$

where p, u and z are any elements in rings.

Proof. " ⇒ " Assume that $a \in R^{PI}$,then $a^\dagger = a^*$. It means that

$$y = a^\dagger(a^\dagger)^* p + z - a^\dagger azaa^\dagger = a^\dagger ap + z - a^\dagger azaa^\dagger.$$

" ⇐ " If (5) is the solution of the equation (3), then, by taking the solution (5) into the equation (3), the equation is still an equality. It follows that

$$\begin{aligned}(a^\dagger)^*(p + u - a^\dagger auaa^\dagger)(a^\dagger)^* &= a(a^\dagger ap + z - a^\dagger azaa^\dagger)(a^\dagger)^* \\ (a^\dagger)^* p(a^\dagger)^* + (a^\dagger)^* u(a^\dagger)^* - (a^\dagger)^* a^\dagger auaa^\dagger(a^\dagger)^* &= ap(a^\dagger)^* + az(a^\dagger)^* - azaa^\dagger(a^\dagger)^* \\ (a^\dagger)^* p(a^\dagger)^* + (a^\dagger)^* u(a^\dagger)^* - (a^\dagger)^* u(a^\dagger)^* &= ap(a^\dagger)^* + az(a^\dagger)^* - az(a^\dagger)^* \\ (a^\dagger)^* p(a^\dagger)^* &= ap(a^\dagger)^*.\end{aligned}$$

Since p is an any element in a ring, one can see that $(a^\dagger)^* a^\dagger(a^\dagger)^* = aa^\dagger(a^\dagger)^*$, by taking $p = a^\dagger$. This gives that $(a^\dagger)^* a^\dagger(a^\dagger)^* = (a^\dagger)^*$. Giving an involution of the above equality, it leads to $a^\dagger = a^\dagger(a^\dagger)^* a^\dagger$. Pre-multiply it by a, we have $aa^\dagger = aa^\dagger(a^\dagger)^* a^\dagger$, giving that $aa^\dagger = (a^\dagger)^* a^\dagger$. Furthermore, post-multiply it by a, and we obtain $aa^\dagger a = (a^\dagger)^* a^\dagger a$, that is $a = (a^\dagger)^*$. Again taking involution of the above equality, we obtain that $a^* = a^\dagger$ and $a \in R^{PI}$. □

Corollary 3.3. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI} \cap R^{EP}$ if and only if the general solution of the equation (3) is given as follows*

$$\begin{cases} x = p + u - a^\dagger auaa^\dagger \\ y = aa^\dagger p + z - a^\dagger azaa^\dagger, \end{cases} \tag{6}$$

where p, u and z are any elements in rings.

Proof. " ⇒ " As $a \in R^{PI} \cap R^{EP}$, then we have $a^\dagger = a^* = a^\#$. By Theorem 3.1, we can obtain the equation (6).
" ⇐ " Since (6) is the general solution of the equation (3), then we obtain that

$$(a^\dagger)^*(p + u - a^\dagger auaa^\dagger)(a^\dagger)^* = a(aa^\dagger p + z - a^\dagger azaa^\dagger)(a^\dagger)^*.$$

It leads to $(a^\dagger)^* p(a^\dagger)^* = a^2 a^\dagger p(a^\dagger)^*$. Taking $p = a^\#$, and thus $(a^\dagger)^* a^\#(a^\dagger)^* = aa^\#(a^\dagger)^*$. So, we can obtain $a \in R^{PI}$ (See the proof (2) of Theorem 2.3). □

To multiply the equality (3) by $a^\dagger$ on the left, we get the following equation

$$a^\dagger ax(a^\dagger)^* = a^\dagger(a^\dagger)^* y(a^\dagger)^*. \tag{7}$$

Theorem 3.4. *The general solution of equation (7) is given as the form*

$$\begin{cases} x = p + u - a^\dagger auaa^\dagger \\ y = a^* ap + z - a^\dagger azaa^\dagger, \end{cases} \tag{8}$$

where p, u and z are any elements in rings.

Proof. By checking the equation directly, we can obtain the equation (8) is the solution of the equation (7). Indeed, taking $x = p + u - a^\dagger auaa^\dagger$ into the left side of the equation (7), it follows that the left is equal to $a^\dagger a(p + u - a^\dagger auaa^\dagger)(a^\dagger)^* = a^\dagger ap(a^\dagger)^* + a^\dagger au(a^\dagger)^* - a^\dagger auaa^\dagger(a^\dagger)^* = a^\dagger ap(a^\dagger)^*$. Moreover, the right is $a^\dagger(a^\dagger)^*(a^* ap + z - a^\dagger azaa^\dagger)(a^\dagger)^* = a^\dagger ap(a^\dagger)^*$.

Now, set $x = x_0$ and $y = y_0$ to be the solution of equation(7). It means that

$$a^\dagger ax_0(a^\dagger)^* = a^\dagger(a^\dagger)^* y_0(a^\dagger)^*.$$

Note that $a^\dagger(a^\dagger)^* y_0 aa^\dagger = (a^\dagger(a^\dagger)^* y_0(a^\dagger)^*)a^* = a^\dagger ax_0(a^\dagger)^* a^* = a^\dagger ax_0 aa^\dagger$. Then, one can see that $x_0 = a^\dagger(a^\dagger)^* y_0 aa^\dagger + x_0 - a^\dagger(a^\dagger)^* y_0 aa^\dagger = a^\dagger(a^\dagger)^* y_0 aa^\dagger + x_0 - a^\dagger ax_0 aa^\dagger$. Here, we set $p = a^\dagger(a^\dagger)^* y_0 aa^\dagger$ and $u = x_0$. Then, $x_0 = p + u - a^\dagger auaa^\dagger$. Furthermore, note that $a^* ap = a^* a(a^\dagger(a^\dagger)^* y_0 aa^\dagger) = a^*(a^\dagger)^* y_0 aa^\dagger = a^\dagger ay_0 aa^\dagger$, this leads to $y_0 = a^* ap + y_0 - a^\dagger ay_0 aa^\dagger$. Set $z = y_0$. Then $y_0 = a^* ap + z - a^\dagger azaa^\dagger$. The proof is completed. □

Corollary 3.5. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{PI}$ if and only if the general solution of equation (7) can be given by (5).*

Proof. " $\Rightarrow$ " As $a \in R^{PI}$, then we have $a^* = a^{\dagger}$. By Theorem 3.4, we can obtain the equation (5) is the solution of the equation (7).

" $\Leftarrow$ " Assume that the general solution of equation (7) can be given by (5). Substitute the solution (5) into equation $a^{\dagger}ax(a^{\dagger})^* = a^{\dagger}(a^{\dagger})^*y(a^{\dagger})^*$, then we can obtain

$$\begin{aligned} a^{\dagger}a(p + u - a^{\dagger}auaa^{\dagger})(a^{\dagger})^* &= a^{\dagger}(a^{\dagger})^*(a^{\dagger}ap + z - a^{\dagger}azaa^{\dagger})(a^{\dagger})^* \\ a^{\dagger}ap(a^{\dagger})^* &= a^{\dagger}(a^{\dagger})^*a^{\dagger}ap(a^{\dagger})^* \\ a^{\dagger}ap(a^{\dagger})^* &= a^{\dagger}(a^{\dagger})^*p(a^{\dagger})^* \end{aligned}$$

Taking $p = a^{\dagger}$, one can see $a^{\dagger}aa^{\dagger}(a^{\dagger})^* = a^{\dagger}(a^{\dagger})^*a^{\dagger}(a^{\dagger})^*$, that is $a^{\dagger}(a^{\dagger})^* = a^{\dagger}(a^{\dagger})^*a^{\dagger}(a^{\dagger})^*$. Post-multiply it by a^*, which can be simplified as $a^{\dagger} = a^{\dagger}(a^{\dagger})^*a^{\dagger}$. Then pre-multiply it by a, we get $aa^{\dagger} = (a^{\dagger})^*a^{\dagger}$. Moreover, post-multiply it by a, it leads to $a = (a^{\dagger})^*$. We can obtain $a^* = a^{\dagger}$, and consequently, $a \in R^{PI}$. □

As we know, assume that $a \in R^{PI}$, then $a^* = a^{\dagger}$. It is immediate that $a^{\dagger} = a^{\dagger}aa^{\dagger} = a^{\dagger}(a^{\dagger})^*a^* = a^{\dagger}(a^{\dagger})^*a^{\dagger}$, so we can obtain the following equation

$$x = x(a^{\dagger})^*a^{\dagger}. \tag{9}$$

Theorem 3.6. *Let $a \in R^{\#} \cap R^{\dagger}$, then $a \in R^{PI}$ if and only if equation (9) has at least a solution in $\chi_a = \{a, a^{\#}, a^{\dagger}, a^*, (a^{\#})^*, (a^{\dagger})^*\}$.*

Proof. " $\Rightarrow$ " Obviously, $x = a^{\dagger}$ is a solution.

" $\Leftarrow$ " (1) If $x = a$ is a solution, it leads to $a = a(a^{\dagger})^*a^{\dagger}$. Post-multiply it by a, we get $a^2 = a(a^{\dagger})^*$. Pre-multiply it by $a^{\#}$, then $a = aa^{\#}(a^{\dagger})^*$. Again post-multiply it by a^*a, it gives $aa^*a = a$, and hence we can obtain $a \in R^{PI}$.

(2) If $x = a^*$ is a solution, then it follows that $a^* = a^*(a^{\dagger})^*a^{\dagger} = a^{\dagger}aa^{\dagger} = a^{\dagger}$, and consequently, $a \in R^{PI}$.

(3) If $x = a^{\dagger}$ is a solution, then we have $a^{\dagger} = a^{\dagger}(a^{\dagger})^*a^{\dagger}$. Pre-multiply it by a, one can see $aa^{\dagger} = (a^{\dagger})^*a^{\dagger}$. Post-multiply it by a, we can obtain $a = (a^{\dagger})^*$. Therefore, $a^* = a^{\dagger}$ and $a \in R^{PI}$.

(4) If $x = a^{\#}$ is a solution, then $a^{\#} = a^{\#}(a^{\dagger})^*a^{\dagger}$. Post-multiply it by aa^*a, we get $aa^{\#}a^*a = a^{\#}a$. Pre-multiply it by a, it leads to $aa^*a = a$. Thus, it gives $a \in R^{PI}$.

(5) If $x = (a^{\dagger})^*$ is a solution, then $(a^{\dagger})^* = (a^{\dagger})^*(a^{\dagger})^*a^{\dagger}$. It means that $a^{\dagger} = (a^{\dagger})^*a^{\dagger}a^{\dagger}$. Post-multiply it by $a(aa^{\#})^*$, we have $(aa^{\#})^* = (a^{\dagger})^*a^{\dagger}$. And pre-multiply it by a^*, it leads to $a^* = a^{\dagger}$ and $a \in R^{PI}$.

(6) If $x = (a^{\#})^*$ is a solution, then $(a^{\#})^* = (a^{\#})^*(a^{\dagger})^*a^{\dagger}$, we obtain that $a^{\#} = (a^{\dagger})^*a^{\dagger}a^{\#}$. Pre-multiply it by aa^* and simplify it, then it gives $aa^*a^{\#} = a^{\#}$. Post-multiply it by a^2, we get $aa^*a = a$, and hence $a \in R^{PI}$. □

Inspired by equation (9), we can rewrite it as follows

$$x = y(a^{\dagger})^*a^{\dagger} \tag{10}$$

Theorem 3.7. *The general solution of equation (10) is given as the form*

$$\begin{cases} x = p\left(a^{\dagger}\right)^* a^{\dagger} \\ y = p + z - zaa^{\dagger}, \end{cases} \tag{11}$$

where p and z are any elements in rings.

Proof. By checking the equation (10) directly, we can obtain the equation (11) is the solution of the equation (10). Indeed, we have

$$\begin{aligned} (p + z - zaa^{\dagger})(a^{\dagger})^*a^{\dagger} &= p(a^{\dagger})^*a^{\dagger} + z(a^{\dagger})^*a^{\dagger} - zaa^{\dagger}(a^{\dagger})^*a^{\dagger} \\ &= p(a^{\dagger})^*a^{\dagger} + z(a^{\dagger})^*a^{\dagger} - z(a^{\dagger})^*a^{\dagger} \\ &= p(a^{\dagger})^*a^{\dagger}. \end{aligned}$$

Now, let $x = x_0$ and $y = y_0$ be the solution of equation(10), then we can obtain

$$x_0 = y_0 \left(a^\dagger\right)^* a^\dagger = y_0 \left(a^\dagger a a^\dagger\right)^* a^\dagger = y_0 a a^\dagger \left(a^\dagger\right)^* a^\dagger$$

Write $p = y_0 a a^\dagger$. Then we have $x_0 = p(a^\dagger)^* a^\dagger$. Moreover, $y_0 = y_0 a a^\dagger + y_0 - y_0 a a^\dagger = p + z - z a a^\dagger$, where $z = y_0$. □

We know that, when $a \in R^{PI}$, then it gives $a^\dagger = a^\dagger a a^\dagger = a^\dagger (a^\dagger)^* a^\dagger$. Hence, we can obtain $a^\dagger \left(a^\dagger\right)^* a^\dagger$ is MP-invertible, and $(a^\dagger (a^\dagger)^* a^\dagger)^\dagger = a$. In what follows, we will consider the generalized invertibility of $a^\dagger (a^\dagger)^* a^\dagger$.

Lemma 3.8. *Let $a \in R^\dagger$. Then $a^\dagger \left(a^\dagger\right)^* a^\dagger \in R^\dagger$ and $\left(a^\dagger \left(a^\dagger\right)^* a^\dagger\right)^\dagger = a a^* a$.*

Proof. By checking the definition of MP-inverse directly, we can obtain the result. Indeed,

$$\begin{aligned} a^\dagger (a^\dagger)^* a^\dagger \cdot a a^* a &= a^\dagger (a^\dagger)^* a^* a = a^\dagger a a^\dagger a = a^\dagger a \\ a a^* a \cdot a^\dagger (a^\dagger)^* a^\dagger &= a a^* (a^\dagger)^* a^\dagger = a a^\dagger a a^\dagger = a a^\dagger \\ a a^* a \cdot a^\dagger (a^\dagger)^* a^\dagger \cdot a a^* a &= a a^\dagger \cdot a a^* a = a a^* a \\ a^\dagger (a^\dagger)^* a^\dagger \cdot a a^* a \cdot a^\dagger (a^\dagger)^* a^\dagger &= a^\dagger a \cdot a^\dagger (a^\dagger)^* a^\dagger = a^\dagger (a^\dagger)^* a^\dagger. \end{aligned}$$

The proof is completed. □

Corollary 3.9. *Let $a \in R^\dagger$. Then $a \in R^{EP}$ if and only if $a^\dagger (a^\dagger)^* a^\dagger \in R^{EP}$.*

Proof. Check the proof of Lemma 3.8, we have that

$$a^\dagger (a^\dagger)^* a^\dagger a a^* a = a a^* a a^\dagger (a^\dagger)^* a^\dagger \Leftrightarrow a^\dagger a = a a^\dagger.$$

It means that $a \in R^{EP}$ if and only if $a^\dagger (a^\dagger)^* a^\dagger \in R^{EP}$. □

Next, we will give a characterization of EP elements in terms of the invertible element in rings.

Theorem 3.10. *Let $a \in R^\dagger$. Then $a \in R^{EP}$ if and only if $u = a^\dagger (a^\dagger)^* a^\dagger + 1 - a^\dagger a \in R^{-1}$ and $u^{-1} = a a^* a + 1 - a^\dagger a$.*

Proof. " $\Rightarrow$ " By Lemma 3.8, it is easy to check that

$$(a^\dagger (a^\dagger)^* a^\dagger + 1 - a^\dagger a)(a a^* a + 1 - a^\dagger a) = a^\dagger (a^\dagger)^* a^\dagger a a^* a + 1 - a^\dagger a = 1.$$

Moreover, as $a \in R^{EP}$, we have $a a^\dagger = a^\dagger a$, and then,

$$(a a^* a + 1 - a^\dagger a)(a^\dagger (a^\dagger)^* a^\dagger + 1 - a^\dagger a) = a a^* a a^\dagger (a^\dagger)^* a^\dagger + 1 - a^\dagger a = a a^\dagger + 1 - a^\dagger a = 1.$$

" $\Leftarrow$ " If $u^{-1} u = 1$, then $(a a^* a + 1 - a^\dagger a)(a^\dagger (a^\dagger)^* a^\dagger + 1 - a^\dagger a) = 1$. This gives that $a a^* a a^\dagger (a^\dagger)^* a^\dagger + 1 - a^\dagger a = 1$. By Lemma 3.8, we have $a a^\dagger + 1 - a^\dagger a = 1$. That is $a a^\dagger = a^\dagger a$, and hence $a \in R^{EP}$. □

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References

[1] M.P. Drazin, Pseudo-inverses in associative rings and semigroups. Amer. Math. Monthly 1958;65:506-514.
[2] D. Drivaliaris, S. Karanasios and D. Pappas, Factorizations of EP operators. Linear Algebra Appl. 2008;429:1555-1567.
[3] D. Mosić and D.S. Djordjević, Further results on partial isometries and EP elements in rings with involution. Math. Comput. Model. 2011; 54:460-465.
[4] D. Mosić, D.S. Djordjević and J.J. Koliha, EP elements in rings. Linear Algebra Appl. 2009;431:527-535.
[5] D. Mosić and D.S. Djordjević, Partial isometries and EP elements in rings with involution. Electron. J. Linear Algebra 2009;18:761-722.
[6] D. Mosić and D.S. Djordjević, New characterizations of EP, generalized normal and generalized Hermitian elements in rings. Appl. Math. Comput. 2012;218:6702-6710.
[7] Y.C. Qu, J.C. Wei and H. Yao, Characterizations of normal elements in rings with involution. Acta. Math. Hungar. 2018;156(2):459-464.
[8] P. Patrício and R. Puystjens, Drazin-Moore-Penrose invertibility in rings. Linear Algebra Appl. 2004;389:159-173.
[9] R.E. Hartwig, Generalized inverses, EP elements and associates. Rev. Roumaine Math. Pures Appl. 1978;23:57-60.
[10] R.E. Hartwig and I.J. Katz, Products of EP elements in reflexive semigroups. Linear Algebra and Appl. 1976;14:11-19.
[11] R. Harte and M. Mbekhta, On generalized inverses in C^*-algebras. Studia Math. 1992;103:71-77.
[12] S. Karanasios, EP elements in rings and semigroup with involution and C^*-algebras. Serdica Math. J. 2015;41:83-116.
[13] J.J. Koliha, The Drazin and Moore-Penrose inverse in C^*-algebras. Math. Proc. R. Ir. Acad. 1999;99A:17-27.
[14] J.J. Koliha, D. Cvetković and D.S. Djordjević, Moore-Penrose inverse in rings with involution. Linear Algebra Appl. 2007;426:371-381.
[15] J.J. Koliha and P. Patrílcio, Elements of rings with equal spectral idempotents. J. Aust. Math. Soc. 2002;72:137-152.
[16] Y.G. Tian and H.X. Wang, Characterizations of EP matrices and weighted-EP matrices. Linear Algebra Appl. 2011;434:1295-1318.
[17] R.J. Zhao, H. Yao and J.C. Wei, Characterizations of partial isometries and two special kinds of EP elements. Czechoslovak Math. J. 2020;70(145):539-551.