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On the solvability of a semiperiodic boundary value problem for a pseudohyperbolic equation

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Abstract. The solvability of the boundary value problem for pseudohyperbolic equations of the third order is investigated. For the problem under study, an algorithm for finding an approximate solution is proposed and sufficient conditions for unique solvability are established.

1. Introduction

On $\Omega = [0, X] \times [0, Y]$ we consider the semiperiodic boundary value problem

$$\frac{\partial^3 u}{\partial x^2 \partial y} = A(x, y)\frac{\partial^2 u}{\partial x^2} + C(x, y)\frac{\partial^2 u}{\partial y^2} + f(x, y), \quad (x, y) \in \Omega, \tag{1}$$

$$u(x, 0) = u(x, Y), \quad x \in [0, X], \tag{2}$$

$$u(0, y) = \varphi(y), \quad y \in [0, Y], \tag{3}$$

$$\frac{\partial u(0, y)}{\partial x} = \psi(y), \quad y \in [0, Y], \tag{4}$$

where $(n \times n)$ - matrix functions $A(x, y), C(x, y)$, n-vector functions $f(x, y)$ are continuous on Ω, n-vector functions $\varphi(y)$, $\psi(y)$ are continuously differentiable on $[0, Y]$, here

$$||u(x, y)|| = \max_{i=\overline{1,n}} |u_i(x, y)|, \quad ||A(x, y)|| = \max_{i=\overline{1,n}} \sum_{j=1}^{n} |a_{ij}(x, y)|.$$

Let $C(\Omega, R^n)$ be the spaces of functions $u : \Omega \to R^n$, which are continuous on Ω, with the rate $||u||_0 = \max ||u(x, y)||$.

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Boundary value problems for hyperbolic equations of the third order have been investigated by many authors [1-5]. The interest in this type of equations is explained both by the theoretical significance of the results obtained and by their important practical applications [6]. Hyperbolic equations with two independent variables of the third and higher order are used as mathematical models of various processes: unsteady rectilinear flow of an incompressible fluid of the second order [7 ,8]; Navier-Stokes-Oldroyd fluid flows [9]; vibrations of elastic-viscous thread [10,11]; vibrations of the rod in the presence of relaxation and aftereffect of the simplest type [12]; the phenomenon of flutter of a cantilever wing [13, 14] and others.

In this paper, a semi-periodic boundary value problem for pseudohyperbolic equations of the third order is reduced to an equivalent problem of a family of boundary value problems for ordinary differential equations and functional relations. When solving a family of boundary value problems for ordinary differential equations, the parametrization method is used. [15-19] Application of this approach allowed to establish the coefficients of the unique solvability of the semiperidic problem for pseudhyperbolic equations and to propose new algorithms for finding the approximate solution.

The function $u(x, y) \in C(\Omega, R^n)$, with partial derivatives $\dfrac{\partial^2 u(x,y)}{\partial y^2} \in C(\Omega, R^n)$, $\dfrac{\partial^2 u(x,y)}{\partial x^2} \in C(\Omega, R^n)$, $\dfrac{\partial^3 u(x,y)}{\partial x^2 \partial y} \in C(\Omega, R^n)$ is called the classical solution to the problem (1)-(4), if it satisfies the system (1) with all $(x, y) \in \Omega$, and boundary conditions(2)–(4).

2. Main result

To find the solution, we introduce the function $g(x, y) = \frac{\partial u(x,y)}{\partial x}$, $w(x, y) = \frac{\partial^2 u(x,y)}{\partial y^2}$ and the problem (1)-(4) we write in the form

$$\frac{\partial^2 g}{\partial x \partial y} = A(x, y)\frac{\partial g}{\partial x} + C(x, y)w + f(x, y), \quad (x, y) \in \Omega, \tag{5}$$

$$g(x, 0) = g(x, Y), \quad x \in [0, X], \tag{6}$$

$$g(0, y) = \psi(y), \quad y \in [0, Y], \tag{7}$$

$$u(x, y) = \varphi(y) + \int_0^x g(\xi_1, y) d\xi_1, \tag{8}$$

$$w(x, y) = \varphi''(y) + \int_0^x \frac{\partial^2 g(\xi_1, y)}{\partial y^2} d\xi_1. \tag{9}$$

We reintroduce the notation $z(x, y) = \frac{\partial g(x,y)}{\partial x}$ and the problem (5)-(9)reduced to a family of periodic boundary value problems for a system of ordinary differential equations of the form

$$\frac{\partial z}{\partial y} = A(x, y)z + C(x, y)w + f(x, y), \quad (x, y) \in \Omega, \tag{10}$$

$$z(x,0) = z(x,Y), \quad x \in [0,X], \tag{11}$$

$$g(x,y) = \psi(y) + \int_0^x z(\xi,y)d\xi, \quad y \in [0,Y], \tag{12}$$

$$u(x,y) = \varphi(y) + \psi(y)x + \int_0^x \int_0^\xi z(\xi_1,y)d\xi_1 d\xi, \tag{13}$$

$$w(x,y) = \varphi''(y) + \psi''(y)x + \int_0^x \int_0^\xi \frac{\partial^2 z(\xi_1,y)}{\partial y^2} d\xi_1 d\xi, \tag{14}$$

To solve problem (10)-(14) for the step $h > 0 : Nh = Y$ we partition $[0,Y) = \bigcup_{r=1}^{N}[(r-1)h, rh), N = 1,2,...$ [2]. In this case, the area Ω is divided into N parts. By $u_r(x,y)$, $\omega_r(x,y)$, $v_r(x,y)$, $g_r(x,y)$ we denote, respectively, the restrictions of the functions $v(x,y), g(x,y), u(x,y), w(x,y)$ on $\Omega_r = [0,X] \times [(r-1)h, rh)$, $\quad r = \overline{1,N}$. By $\lambda_r(x)$ we denote the value of the function $z_r(x,y)$ at $y = (r-1)h$, i.e. $\lambda_r(x) = z_r(x,(r-1)h)$ and denote $v_r(x,y) = z_r(x,y) - \lambda_r(x), r = \overline{1,N}$. We obtain an equivalent boundary value problem for the unknown functions $\lambda_r(x)$:

$$\frac{\partial v_r}{\partial y} = A(x,y)v_r + A(x,y)\lambda_r(x) + C(x,y)w_r + f(x,y), \quad (x,y) \in \Omega_r, \quad r = \overline{1,N}, \tag{15}$$

$$v_r(x,(r-1)h) = 0, \quad x \in [0,X], \quad r = \overline{1,N}, \tag{16}$$

$$\lambda_1(x) - \lambda_N(x) - \lim_{y \to Y-0} v_N(x,y) = 0, \quad x \in [0,X], \tag{17}$$

$$\lambda_s(x) + \lim_{t \to sh-0} v_s(x,y) - \lambda_{s+1}(x) = 0, \quad x \in [0,X], \quad s = \overline{1,N-1}. \tag{18}$$

$$g(x,y) = \psi(y) + \int_0^x v_r(\xi,y)d\xi + \int_0^x \lambda_r(\xi,y)d\xi, \tag{19}$$

$$u_r(x,y) = \varphi(y) + \psi(y)x + \int_0^x \int_0^\xi v_r(\xi_1,y)d\xi_1 d\xi + \int_0^x \int_0^\xi \lambda_r(\xi_1)d\xi_1 d\xi, \tag{20}$$

$$w_r(x,y)=\varphi''(y)+\psi''(y)x+\int\limits_0^x\int\limits_0^{\xi}\frac{\partial^2 v_r(\xi_1,y)}{\partial y^2}d\xi_1 d\xi, \tag{21}$$

where $(x,y)\in\Omega_r$, $r=\overline{1,N}$, (18)- the condition of gluing functions in the internal lines of the partition.

Problem (15),(16) for fixed $\lambda_r(x), w_r(x,y)$ is a one-parameter family of Cauchy problems for systems of ordinary differential equations, where $x\in[0,Y]$, which is equivalent to the integral equation

$$v_r(x,y)=\int\limits_{(r-1)h}^{y}A(x,\tau)v_r(x,\tau)d\tau+\int\limits_{(r-1)h}^{y}A(x,\tau)d\tau\cdot\lambda_r(x)+\int\limits_{(r-1)h}^{y}(C(x,\tau)w_r+f(x,\tau))d\tau, \tag{22}$$

Instead of $v_r(x,\tau)$ we substitute the corresponding right-hand side of (22) and repeating this process ν ($\nu=1,2,...$) times we obtain

$$v_r(x,y)=D_{\nu r}(x,y)\lambda_r(x)+F_{\nu r}(x,y,w_r)+G_{\nu r}(x,y,v_r),\quad r=\overline{1,N}, \tag{23}$$

where

$$D_{\nu r}(x,y)=\sum_{j=0}^{\nu-1}\int\limits_{(r-1)h}^{y}A(x,\tau_1)d\tau_1\ldots\int\limits_{(r-1)h}^{\tau_j}A(x,\tau_{j+1})d\tau_{j+1}\ldots d\tau_1,$$

$$\begin{aligned}F_{\nu r}(x,y,w_r) &= \int\limits_{(r-1)h}^{y}\Big[C(x,\tau_1)w_r(x,\tau_1)+f(x,\tau_1)\Big]d\tau_1\\ &+\sum_{j=1}^{\nu-1}\int\limits_{(r-1)h}^{y}A(x,\tau_1)\ldots\int\limits_{(r-1)h}^{\tau_{j-1}}A(x,\tau_j)\int\limits_{(r-1)h}^{\tau_j}\Big[C(x,\tau_{j+1})w_r(x,\tau_{j+1})+f(x,\tau_{j+1})\Big]d\tau_{j+1}d\tau_j\ldots d\tau_1,\end{aligned}$$

$$G_{\nu r}(x,y,v_r)=\int\limits_{(r-1)h}^{y}A(x,\tau_1)\ldots\int\limits_{(r-1)h}^{\tau_{\nu-2}}A(x,\tau_{\nu-1})\int\limits_{(r-1)h}^{\tau_{\nu-1}}A(x,\tau_\nu)v_r(x,\tau_\nu)d\tau_\nu d\tau_{\nu-1}\ldots d\tau_1,$$

$\tau_0=y, r=\overline{1,N}$. Passing to the limit as $y\to rh-0$ in (23) we have

$$\lim_{y\to rh-0}v_r(x,y)=D_{\nu r}(x,rh)\lambda_r(x)+F_{\nu r}(x,rh,w_r)+G_{\nu r}(x,rh,v_r),$$

$x\in[0,\omega], r=\overline{1,N}$. Substituting in (17),(18) instead of $\lim\limits_{y\to rh-0}v_r(x,y), r=\overline{1,N}$, the corresponding to them right-hand sides for the unknown functions $\lambda_r(x), r=\overline{1,N}$, we obtain the system of functional equations:

$$Q_\nu(x,h)\lambda(x)=-F_\nu(x,h,w)-G_\nu(x,h,v), \tag{24}$$

where

$$Q_\nu(x,h) = \begin{bmatrix} I & 0 & \dots & 0 & -[I + D_{\nu N}(x, Nh)] \\ I + D_{\nu 1}(x,h) & -I & \dots & 0 & 0 \\ 0 & I + D_{\nu 2}(x, 2h) & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I + D_{\nu,N-1}(x,(N-1)h) & -I \end{bmatrix},$$

$$F_\nu(x,h,w) = (-F_{\nu N}(x, Nh, w_N), F_{\nu 1}(x,h,w_1), \dots, F_{\nu,N-1}(x,(N-1)h, w_{N-1})),$$

$$G_\nu(x,h,v) = (-G_{\nu N}(x, Nh, v_N), G_{\nu 1}(x,h,v_1), \dots, G_{\nu,N-1}(x,(N-1)h, v_{N-1})),$$

and I is the unit matrix of dimension of n.

For finding a system of three functions $\{\lambda_r(x), v_r(x,y), w_r(x,y), r = \overline{1,N}$, we have a closed system consisting of equations (24), (23) and (21).

Assuming the invertibility of the matrix $Q_\nu(x,h)$ for all $x \in [0,X]$, from equation (24), where $v_r(x,y) = 0$, $w_r(x,y) = \varphi''(t)$, we find $\lambda^{(0)}(x) = (\lambda_1^{(0)}(x), \lambda_2^{(0)}(x), \dots, \lambda_N^{(0)}(x))'$:

$$\lambda^{(0)}(x) = -[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,\ddot{\varphi}) + G_\nu(x,h,0)\Big\}.$$

Using equation (23), at $\lambda_r(x) = \lambda_r^{(0)}(x)$ we find the functions $\{v_r^{(0)}(x,y)\}$, $r = \overline{1,N}$, i.e.

$$v_r^{(0)}(x,y) = D_{\nu r}(x,y)\lambda_r^{(0)}(x) + F_{\nu r}(x,y,\dot{\varphi}) + G_{\nu r}(x,y,0)$$

. The functions $g_r^{(0)}(x,y), u_r^{(0)}(x,y), w_r^{(0)}(x,y), r = \overline{1,N}$, are defined from the relations

$$g_r^{(0)}(x,y) = \psi(y) + \int_0^x v_r^{(0)}(\xi,y)d\xi + \int_0^x \lambda_r^{(0)}(\xi,y)d\xi,$$

$$u_r^{(0)}(x,y) = \varphi(y) + \psi(y)x + \int_0^x\int_0^\xi v_r^{(0)}(\xi_1,y)d\xi_1 d\xi + \int_0^x\int_0^\xi \lambda_r^{(0)}(\xi_1)d\xi_1 d\xi.$$

$$w_r^{(0)}(x,y) = \varphi''(y) + \psi''(y)x + \int_0^x\int_0^\xi \frac{\partial v_r^{(0)}(\xi_1,y)}{\partial y^2}d\xi_1 d\xi,$$

where $(x,y) \in \Omega_r, r = \overline{1,N}$.

For the initial approximation of problem (15)-(21) we take the system $\{\lambda_r^{(0)}(x), v_r^{(0)}(x,t), w_r^{(0)}(x,y), r = \overline{1,N}$ and construct successive approximations on the following algorithm :

Step 1. A) Assuming that $w_r(x,y) = w_r^{(0)}(x,y), r = \overline{1,N}$, we find the first approximations of $\lambda_r(x), \widetilde{v}_r(x,y), r = \overline{1,N}$, by solving the problem (15)-(18). Taking $\lambda_r^{(1,0)}(x) = \lambda_r^{(0)}(x)$, $\quad v_r^{(1,0)}(x,y) = v_r^{(0)}(x,y)$, we find the system of couples $\{\lambda_r^{(1)}(x), v_r^{(1)}(x,y)\}, r = \overline{1,N}$, as the limit of the sequence $\lambda_r^{(1,m)}(x)$, $v_r^{(1,m)}(x,y)$, defined in the following way:

Step 1.1. Assuming the invertibility of the matrix $Q_\nu(x,h), x \in [0,X]$, from equation (24), where $v_r(x,y) = v_r^{(1,0)}(x,y)$, we find $\lambda^{(1,1)}(x) = (\lambda_1^{(1,1)}(x), \lambda_2^{(1,1)}(x), \dots, \lambda_N^{(1,1)}(x))'$:

$$\lambda^{(1,1)}(x) = -[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(0)} + G_\nu(x,h,v^{(1,0)})\Big\}.$$

Substituting the found $\lambda_r^{(1,1)}(x)$, $r=\overline{1,N}$, in (23) we find

$$v_r^{(1,1)}(x,y)=D_{\nu r}(x,y)\lambda_r^{(1,1)}(x)+F_{\nu r}(x,y,w^{(0)})+G_{\nu r}(x,y,v^{(1,0)}).$$

Step 1.2. From equation (24), where $v_r(x,y)=v_r^{(1,1)}(x,y)$, we define

$$\lambda^{(1,2)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(0)})+G_\nu(x,h,v^{(1,1)})\Big\}.$$

Using as expression (20) again, we find the functions $\{v_r^{(1,2)}(x,y)\}$, $r=\overline{1,N}$,

$$v_r^{(1,2)}(x,y)=D_{\nu r}(x,y)\lambda_r^{(1,2)}(x)+F_{\nu r}(x,y,w^{(0)})+G_{\nu r}(x,y,v^{(1,1)}).$$

On step $(1,m)$ we obtain the system of couples $\{\lambda_r^{(1,m)}(x),v_r^{(1,m)}(x,y)\}, r=\overline{1,N}$.

Suppose that the solution of problem (15)-(18) is a sequence of systems of couples

$$g_r^{(1,m)}(x,y)\},\{\lambda_r^{(1,m)}(x),v_r^{(1,m)}(x,y)\}$$

which are defined for $x\in[0,X]$, $(x,y)\in\Omega_r$ respectively, and converge as $m\to\infty$ to continuous functions $\lambda_r^{(1)}(x)$, $v_r^{(1)}(x,y)$, $r=\overline{1,N}$.

B) The functions $g_r^{(1)}(x,y),w_r^{(1)}(x,y),u_r^{(1)}(x,y),r=\overline{1,N}$, are defined from the relations

$$g_r^{(1)}(x,y)=\psi(y)+\int\limits_0^x v_r^{(1)}(\xi,y)d\xi+\int\limits_0^x \lambda_r^{(1)}(\xi,y)d\xi,$$

$$u_r^{(1)}(x,y)=\varphi(y)+\psi(y)x+\int\limits_0^x\int\limits_0^\xi v_r^{(1)}(\xi_1,y)d\xi_1 d\xi+\int\limits_0^x\int\limits_0^\xi \lambda_r^{(1)}(\xi_1)d\xi_1 d\xi,$$

$$w_r^{(1)}(x,y)=\varphi^{''}(y)+\psi^{''}(y)x+\int\limits_0^x\int\limits_0^\xi \frac{\partial v_r^{(1)}(\xi_1,y)}{\partial y}d\xi_1 d\xi,$$

where $(x,y)\in\Omega_r, r=\overline{1,N}$.

Step 2. A) Assuming that $w_r(x,y)=w_r^{(1)}(x,y), r=\overline{1,N}$, we find the second approximations of $\lambda_r(x),v_r(x,y)$, $r=\overline{1,N}$, by solving problem (15)-(18). Taking

$$\lambda_r^{(2,0)}(x)=\lambda_r^{(1)}(x),\quad v_r^{(2,0)}(x,y)=v_r^{(1)}(x,y),$$

we find the system of couples $\{\lambda_r^{(2)}(x),v_r^{(2)}(x,y)\}, r=\overline{1,N}$, as the limit of the sequence $\lambda_r^{(2,m)}(x)$, $v_r^{(2,m)}(x,y)$, defined in the following way:

Step 2.1. Assuming the invertibility of the matrix $Q_\nu(x,h)$ from equation (24), where $v_r(x,y)=v_r^{(2,0)}(x,y)$, we find $\lambda^{(2,1)}(x)=(\lambda_1^{(2,1)}(x),\lambda_2^{(2,1)}(x),...,\lambda_N^{(2,1)}(x))'$:

$$\lambda^{(2,1)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(1)})+G_\nu(x,h,v^{(2,0)})\Big\}.$$

Substituting the found $\lambda_r^{(2,1)}(x)$, $r=\overline{1,N}$, in (23) we find

$$v_r^{(2,1)}(x,y)=D_{\nu r}(x,y)\lambda_r^{(2,1)}(x)+F_{\nu r}(x,y,w^{(1)})+G_{\nu r}(x,y,v^{(2,0)}).$$

Step 2.2. From equation(24), where $v_r(x,y) = v_r^{(2,1)}(x,y)$, we define

$$\lambda^{(2,2)}(x) = -[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,w^{(1)}) + G_\nu(x,h,v^{(2,1)})\Big\}.$$

Using the expression (23), we find the functions $\{v_r^{(2,2)}(x,y)\}$, $r = \overline{1,N}$:

$$v_r^{(2,2)}(x,y) = D_{\nu r}(x,y)\lambda_r^{(2,2)}(x) + F_{\nu r}(x,y,w^{(1)}) + G_{\nu r}(x,y,v^{(2,1)}).$$

On step $(2,m)$ we obtain the system of couples $\{\lambda_r^{(2,m)}(x), v_r^{(2,m)}(x,y)\}$, where $(x,y) \in \Omega_r, r = \overline{1,N}$.

Suppose that the solution of problem (15)-(18) is a sequence of systems of couples $\{\lambda_r^{(2,m)}(x), v_r^{(2,m)}(x,y)\}$ which as $m \to \infty$ converges to $\{\lambda_r^{(2)}(x), v_r^{(2)}(x,y)\}$, $r = \overline{1,N}$.

B) The functions $g_r^{(2)}(x,y), u_r^{(2)}(x,y), w_r^{(2)}(x,y), r = \overline{1,N}$, are defined from the relations

$$g_r^{(2)}(x,y) = \psi(y) + \int_0^x v_r^{(2)}(\xi,y)d\xi + \int_0^x \lambda_r^{(2)}(\xi,y)d\xi,$$

$$u_r^{(2)}(x,y) = \varphi(y) + \psi(y)x + \int_0^x\int_0^\xi v_r^{(2)}(\xi_1,y)d\xi_1 d\xi + \int_0^x\int_0^\xi \lambda_r^{(2)}(\xi_1)d\xi_1 d\xi,$$

$$w_r^{(2)}(x,y) = \varphi''(y) + \psi''(y)x + \int_0^x\int_0^\xi \frac{\partial v_r^{(2)}(\xi_1,y)}{\partial y}d\xi_1 d\xi,$$

where $(x,y) \in \Omega_r, r = \overline{1,N}$. Continuing the process, at the k-th step we obtain the system $\{\lambda_r^{(k)}(x)$, $v_r^{(k)}(x,y)$, $w_r^{(k)}(x,y)$, $u_r^{(k)}(x,y)\}$, $g_r^{(k)}(x,y)\}$, $r = \overline{1,N}$.

The conditions of the following statement ensure the feasibility and convergence of the proposed algorithm, as well as the unique solvability of problem (15)-(21).

Theorem 1. *Let for some $0 \le \mu < 1$, $h > 0 : Nh = Y, N = 1,2,...,$ and $\nu, \nu \in \mathbb{N}$, $(nN \times nN)$ the matrix $Q_\nu(x,h)$ be invertible for all $x \in [0,X]$ let the following inequalities be satisfied*

1) $\|[Q_\nu(x,h)]^{-1}\| \le \gamma_\nu(x,h)$; 2) $q_\nu(x,h) = \Big\{1 + \gamma_\nu(x,h)\sum_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}\Big\}\frac{(\alpha(x)h)^\nu}{\nu!} \le \mu$.

Then there exists a unique solution (λ_r^*, v_r^*) to problem (15)-(21) and the following estimates are valid

$$\max\Big\{\max_{r=\overline{1,N}}\|\lambda_r^*(x) - \lambda_r^{(k)}(x)\| + \max_{r=\overline{1,N}}\sup_{y\in[(r-1)h,rh)}\|v_r^*(x,y) - v_r^{(k)}(x,y)\|, \max_{r=\overline{1,N}}\sup_{y\in[(r-1)h,rh)}\Big\|\frac{\partial^2 v_r^*(x,y)}{\partial y^2} - \frac{\partial^2 v_r^{(k)}(x,y)}{\partial y^2}\Big\|,$$

$$\int_0^x\Big(\max_{r=\overline{1,N}}\|\lambda_r^*(x_1) - \lambda_r^{(k)}(x_1)\| + \max_{r=\overline{1,N}}\sup_{y\in[(r-1)h,rh)}\|v_r^*(x_1,y) - v_r^{(k)}(x_1,y)\|\Big)dx_1\Big\} \le$$

$$\le \frac{\beta_\nu(x,h)\int_0^x\beta_\nu(\xi,h)d\xi\Big)^{k-1}}{(k-1)!}\Big(e^{\int_0^x\beta_\nu(\xi,h)d\xi}\int_0^x\max\Big\{\chi_\nu(\xi,h),\phi_\nu(\xi,h)\Big\}d\xi\max\Big\{[\max_{y\in[0,T]}\|\varphi''(y)\| + \|f\|_0],[\max_{t\in[0,T]}\|\varphi'''(y)\| + \|f'\|_0]\Big\},$$

$$\max_{r=\overline{1,N}}\sup_{y\in[(r-1)h,rh]}\|w_r^*(x,y) - w_r^{(k)}(x,y)\| \le \int_0^x\max\Big\{\max_{r=\overline{1,N}}\|\lambda_r^*(\xi) - \lambda_r^{(k)}(\xi)\| + \max_{r=\overline{1,N}}\sup_{y\in[(r-1)h,rh)}\|v_r^*(\xi,t) - v_r^{(k)}(\xi,y)\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[(r-1)h,rh)} \Big\|\frac{\partial^2 v_r^*(\xi,y)}{\partial y^2} - \frac{\partial^2 v_r^{(k)}(\xi,y)}{\partial y^2}\Big\|\Big\}d\xi, \quad k=1,2,...$$

where $\alpha(x) = \max_{y\in[0,T]} \|A(x,y)\|$, $\sigma(x) = \max_{y\in[0,Y]} \|C(x,y)\|$, $\delta_\nu(x,h) = \Big\{1+\gamma_\nu(x,h)\sum_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}\Big\}h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}$

$$\theta_\nu(x,h) = \Big\{1+\gamma_\nu(x,h)\sum_{j=0}^{\nu}\frac{(\alpha(x)h)^j}{j!}\Big\}h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!},$$

$$\rho_\nu(x,h) = \int_0^x\int_0^\xi\Big[\alpha'(\xi_1)\max\{\sigma(\xi_1),1\}\gamma_\nu(\xi_1,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(\xi_1)h)^j}{j!} + \max\{\sigma'(\xi_1),1\}\Big]d\xi_1 d\xi$$

$$\beta_\nu(x,h) = \max\Big\{\Big[\frac{\delta_\nu(x,h)\sigma(x)}{1-q_\nu(x,h)} + \Big[\frac{\delta_\nu(x,h)}{1-q_\nu(x,h)}\frac{(\alpha(x)h)^\nu}{\nu!} + h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\Big]\gamma_\nu(x,h)\Big]\sigma(x),$$

$$\int_0^x\Big[\alpha(\xi)\Big(1+\delta_\nu(\xi,h)\Big)+1\Big](\xi)\sigma(\xi)d\xi\Big\},$$

$$\chi_\nu(x,h) = \Big[\frac{\delta_\nu(x,h)}{1-q_\nu(x,h)}\Big[1+\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\Big] + \gamma_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\Big]\sigma(x)\rho_\nu(x,h)+$$

$$+\Big[\frac{1}{1-q_\nu(x,h)}\Big[1+\gamma_\nu(x,h)\sum_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!} + \gamma_\nu(x,h)q_\nu(x,h)\Big] + \gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\Big]\delta_\nu(x,h),$$

$$\phi_\nu(x,h) = \int_0^x\int_0^\xi\Big\{\alpha(\xi_1)(1+\delta_\nu(\xi_1,h))+1\Big\}\sigma(\xi_1)\rho_\nu(\xi_1,h)d\xi_1 d\xi.$$

The proof of Theorem 1 is similar to the proof of Theorem 1 from [1]. By virtue of the equivalence of problems (1)-(4) and (15)-(21) from Theorem 1 follows

Theorem 2. *Let the assumptions of Theorem 1 be satisfied. Then problem (1)-(4) as a unique solution* $u^*(x,y)$ *and the evaluation is performed.*

$$\max\Big(\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\frac{\partial u_r^*(x,y)}{\partial x} - \frac{\partial u_r^{(k)}(x,y)}{\partial x}\|,$$

$$\max_{r=\overline{1,N}} \sup_{y\in[(r-1)h,rh)} \|\frac{\partial u_r^*(x,y)}{\partial y} - \frac{\partial u_r^{(k)}(x,y)}{\partial y}\|, \max_{r=\overline{1,N}} \sup_{y\in[(r-1)h,rh)} \|u_r^*(x,y) - u_r^{(k)}(x,y)\|\Big) \le$$

$$\le \beta_\nu(x,h)\sum_{j=k-1}^{\infty}\frac{1}{j!}\Big(\int_0^x\beta(\xi,h)d\xi\Big)^j\int_0^x\max\Big\{\chi(\xi,h),\phi(\xi,h),\int_0^{\xi_1,h}\chi(\xi_1)d\xi_1\Big\}d\xi\times$$

$$\times\max\Big\{\max_{y\in[0,T]}\|\varphi''(y)\| + \|f\|_0, \max_{y\in[0,Y]}\|\varphi'''(y)\| + \|f'\|_0\Big\}.$$

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