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A refinement of the Cauchy-Schwarz inequality accompanied by new numerical radius upper bounds

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Abstract. This present work aims to ameliorate the celebrated Cauchy-Schwarz inequality and provide several new consequences associated with the numerical radius upper bounds of Hilbert space operators. More precisely, for arbitrary $a, b \in H$ and $\alpha \geq 0$, we show that

$$|\langle a, b\rangle|^2 \leq \frac{1}{\alpha+1} \|a\| \, \|b\| \, |\langle a, b\rangle| + \frac{\alpha}{\alpha+1} \|a\|^2 \|b\|^2$$

$$\leq \|a\|^2 \|b\|^2.$$

As a consequence, we provide several new upper bounds for the numerical radius that refine and generalize some of Kittaneh's results in [A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix. Studia Math. 2003;158:11–17] and [Cauchy-Schwarz type inequalities and applications to numerical radius inequalities. Math. Inequal. Appl. 2020;23:1117–1125], respectively. In particular, for arbitrary $A, B \in B(H)$ and $\alpha \geq 0$, we show the following sharp upper bound

$$w^2(B^*A) \leq \frac{1}{2\alpha+2} \left\| |A|^2 + |B|^2 \right\| w(B^*A) + \frac{\alpha}{2\alpha+2} \left\| |A|^4 + |B|^4 \right\|,$$

with equality holds when $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is also worth mentioning here that some specific values of $\alpha \geq 0$ provide more accurate estimates for the numerical radius. Finally, some related upper bounds are also provided.

1. Introduction

Let $B(H)$ denotes the C^*−algebra of all bounded linear operators on the complex Hilbert space H, with inner product $\langle \cdot, \cdot \rangle$. The numerical range of $A \in B(H)$, denoted by $W(A)$, is the image of the unit sphere of H under the mapping $x \mapsto \langle Ax, x\rangle$. A relevant concept is a numerical radius, which is the supermum of the absolute values of all numbers in $W(A)$, that is

$$w(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

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It is well known that $w(\cdot)$ forms a norm on $B(H)$ which is equivalent with the usual operator norm

$$\|A\| = \sup_{\|x\|=1} \langle Ax, Ax\rangle^{\frac{1}{2}},$$

for $A \in B(H)$. More precisely, the following sharp two-sided inequality holds

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|, \tag{1}$$

where the sharpness holds when $A^2 = 0$ and A is normal for the first and second inequalities respectively. Another important fact for the numerical radius is the power inequality which asserts that

$$w(A^n) \leq w^n(A),$$

for any $A \in B(H)$ and $n \in \mathbb{N}$.

In [3], Kittaneh substantially provided an improvement for the upper bound in (1) by showing that if $A \in B(H)$, then

$$w(A) \leq \frac{1}{2}\left\||A| + |A^*|\right\|. \tag{2}$$

Other improvement for the inequality (1) has given by the same author as follows

$$w^2(A) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|, \tag{3}$$

which in turn has further refined in [6] by Kittaneh and Moradi as follows

$$w^2(A) \leq \frac{1}{6}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{3}w(A)\left\||A| + |A^*|\right\|. \tag{4}$$

Another important facts about the numerical radius upper bounds that of our interest are due to Dragomir in [4], which assert that for $A, B \in B(H)$ and $r \geq 1$ then

$$w^r(B^*A) \leq \frac{1}{2}\left\||A|^{2r} + |B|^{2r}\right\|. \tag{5}$$

The last inequality has further refined by the same author for the case $r = 2$ by showing that

$$w^2(B^*A) \leq \frac{1}{6}\left\||A|^4 + |A^*|^4\right\| + \frac{1}{3}w(B^*A)\left\||A|^2 + |A^*|^2\right\|. \tag{6}$$

Other upper bounds for the numerical radius of bounded linear operators can be found in [8].

Inspired by the aforementioned results, this present work aims to establish new numerical radius upper bounds of Hilbert space operators by providing a new refinement for the celebrated Cauchy-Schwarz inequality. In particular, our results refine inequalities (3) and (5) for the case $r = 2$ and give inequalities (4) and (6) as special cases. Finally, the obtained upper bounds have compared with the previously known bounds to demonstrate their reliability.

2. The Main Results

In this section, we will establish our main results on numerical radius upper bounds depending on a new refinement of the celebrated Cauchy-Schwarz inequality. For this purpose, we first introduce some important results involved in our subsequent discussion. The first result is a consequence of the spectral theorem along with Jenseńs inequality (see[1]).

Lemma 2.1. *Let $A \in B(H)$ be a positive operator and $x \in H$ be any unit vector. Then, for $r \geq 1$, we have*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle .$$

The next result concerns with special type of a non-negative convex functions and can be found in [2].

Lemma 2.2. *Let f be a non-negative convex function on $[0, \infty)$ and $A, B \in B(H)$ be positive operators. Then*

$$\left\| f\left(\frac{A+B}{2}\right)\right\| \leq \left\|\frac{f(A)+f(B)}{2}\right\|.$$

The following inequality is a special case of the mixed Schwarz inequality and can be found in [1].

Lemma 2.3. *Let $A \in B(H)$. Then for any $x, y \in H$,*

$$\left|\langle Ax, y\rangle\right|^2 \leq \langle |A|\, x, x\rangle \langle |A^*|\, y, y\rangle .$$

The last result is the Buzano extension of Cauchy-Schwarz's inequality (see [5]).

Lemma 2.4. *Let $a, b, e \in H$ with $\|e\| = 1$. Then*

$$|\langle a, e\rangle \langle e, b\rangle| \leq \frac{1}{2}\Big(\|a\|\,\|b\| + |\langle a, b\rangle|\Big).$$

Now, we are in a position to present our first result that concerned about a refined version of the celebrated Cauchy-Schwarz inequality. In light of this result, we next provide several new upper bounds for the numerical radius of Hilbert space operators.

Lemma 2.5. *Let $a, b \in H$. Then for any $\alpha \geq 0$,*

$$|\langle a, b\rangle|^2 \leq \frac{1}{\alpha+1}\,\|a\|\,\|b\|\,|\langle a, b\rangle| + \frac{\alpha}{\alpha+1}\|a\|^2\|b\|^2 \leq \|a\|^2\|b\|^2. \tag{7}$$

Proof. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\langle a, b\rangle|^2 &\leq \|a\|\,\|b\|\,|\langle a, b\rangle| \\ &\leq \|a\|\,\|b\|\,|\langle a, b\rangle| + \alpha\left(\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2\right).\end{aligned}$$

Therefore,

$$\begin{aligned}|\langle a, b\rangle|^2 &\leq \frac{1}{\alpha+1}\,\|a\|\,\|b\|\,|\langle a, b\rangle| + \frac{\alpha}{\alpha+1}\|a\|^2\|b\|^2 \\ &\leq \frac{1}{\alpha+1}\|a\|^2\|b\|^2 + \frac{\alpha}{\alpha+1}\|a\|^2\|b\|^2 \\ &= \|a\|^2\|b\|^2.\end{aligned}$$

□

Utilizing the inequality (7), we can state the following refinement of inequality (5).

Theorem 2.6. *Let $A, B \in B(H)$. Then for any $\alpha \geq 0$,*

$$w^2(B^*A) \leq \frac{1}{2\alpha+2}\left\||A|^2 + |B|^2\right\| w\left(B^*A\right) + \frac{\alpha}{2\alpha+2}\left\||A|^4 + |B|^4\right\|. \tag{8}$$

Proof. Let $x \in H$ be any unit vector. By setting $a = Ax$ and $b = Bx$ in Lemma 2.5, we have

$$\begin{aligned}
|\langle Ax, Bx\rangle|^2 &\leq \frac{1}{\alpha+1}\|Ax\|\,\|Bx\|\,|\langle Ax, Bx\rangle| + \frac{\alpha}{\alpha+1}\|Ax\|^2\|Bx\|^2 \\
&= \frac{1}{\alpha+1}|\langle B^*Ax, x\rangle| \sqrt{\left\langle |A|^2x, x\right\rangle}\sqrt{\left\langle |B|^2x, x\right\rangle} + \frac{\alpha}{\alpha+1}\left\langle |A|^2x, x\right\rangle \left\langle |B|^2x, x\right\rangle \\
&\leq \frac{1}{2\alpha+2}|\langle B^*Ax, x\rangle| \left\langle \left(|A|^2+|B|^2\right)x, x\right\rangle + \frac{\alpha}{2\alpha+2}\left(\left\langle |A|^2x, x\right\rangle^2 + \left\langle |B|^2x, x\right\rangle^2\right) \text{ (by AM–GM inequality)} \\
&\leq \frac{1}{2\alpha+2}|\langle B^*Ax, x\rangle| \left\langle \left(|A|^2+|B|^2\right)x, x\right\rangle + \frac{\alpha}{2\alpha+2}\left\langle \left(|A|^4+|B|^4\right)x, x\right\rangle \quad \text{(by Lemma 2.1)} \\
&\leq \frac{1}{2\alpha+2} w(B^*A)\left\||A|^2+|B|^2\right\| + \frac{\alpha}{2\alpha+2}\left\||A|^4+|B|^4\right\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
w^2(B^*A) &= \sup\left\{|\langle B^*Ax, x\rangle|^2 : x \in H, \|x\| = 1\right\} \\
&\leq \frac{1}{2\alpha+2}\left\||A|^2+|B|^2\right\| w(B^*A) + \frac{\alpha}{2\alpha+2}\left\||A|^4+|B|^4\right\|.
\end{aligned}$$

□

As a consequence of (8), we have the following new refinement of inequality (5) for the case $r = 2$.

Corollary 2.7. *Let $A, B \in B(H)$. Then for any $\alpha \geq 0$,*

$$\begin{aligned}
w^2(B^*A) &\leq \frac{1}{2\alpha+2}\left\||A|^2+|B|^2\right\| w(B^*A) + \frac{\alpha}{2\alpha+2}\left\||A|^4+|B|^4\right\| \\
&\leq \frac{1}{2}\left\||A|^4+|B|^4\right\|.
\end{aligned}$$

Proof.

$$\begin{aligned}
w^2(B^*A) &\leq \frac{1}{2\alpha+2}\left\||A|^2+|B|^2\right\| w(B^*A) + \frac{\alpha}{2\alpha+2}\left\||A|^4+|B|^4\right\| \quad \text{(by Theorem 2.6)} \\
&\leq \frac{1}{4\alpha+4}\left\||A|^2+|B|^2\right\|^2 + \frac{\alpha}{2\alpha+2}\left\||A|^4+|B|^4\right\| \quad \text{(by (5))} \\
&= \frac{1}{4\alpha+4}\left\|\left(|A|^2+|B|^2\right)^2\right\| + \frac{\alpha}{2\alpha+2}\left\||A|^4+|B|^4\right\| \\
&\leq \frac{1}{2\alpha+2}\left\||A|^4+|B|^4\right\| + \frac{\alpha}{2\alpha+2}\left\||A|^4+|B|^4\right\| \quad \text{(by Lemma 2.2)} \\
&= \frac{1}{2}\left\||A|^4+|B|^4\right\|.
\end{aligned}$$

□

Remark 2.8. *It is noteworthy that $\alpha = \frac{1}{2}$ in* (8) *leads to the inequality* (6) *as a special case. Also, for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, the inequality* (6) *gives the upper bound $w^2(B^*A) \leq \frac{1}{3}$ whereas inequality* (8) *gives $w^2(B^*A) \leq \frac{2}{7}$ (when $\alpha = \frac{1}{6}$). Thus, the upper bound of* (8) *gives a more accurate estimate than the upper bound of* (6) *for specific values of α.*

The next theorem provides a new upper bound that will be utilized to refine inequality (3).

Theorem 2.9. *Let $A \in B(H)$ and $p \in [0,1]$. Then*

$$w^2(A) \leq \frac{p}{2}\left\||A|^2 + |A^*|^2\right\| + \frac{1-p}{2} w(A)\left\||A| + |A^*|\right\|. \tag{9}$$

Proof. Let $x \in H$ be any unit vector. Then, we have

$$\begin{aligned} |\langle Ax, x\rangle|^2 &= p|\langle Ax, x\rangle|^2 + (1-p)|\langle Ax, x\rangle|^2 \\ &\leq p\langle |A|x, x\rangle\langle |A^*|x, x\rangle + (1-p)|\langle Ax, x\rangle| \sqrt{\langle |A|x, x\rangle}\sqrt{\langle |A^*|x, x\rangle} \quad \text{(by Lemma 2.3)} \\ &\leq \frac{p}{2}\left\langle\left(|A|^2 + |A^*|^2\right)x, x\right\rangle + \frac{1-p}{2}|\langle Ax, x\rangle|\langle(|A| + |A^*|)x, x\rangle \quad \text{(by AM–GM inequality)} \\ &\leq \frac{p}{2}\left\||A|^2 + |A^*|^2\right\| + \frac{1-p}{2} w(A)\left\||A| + |A^*|\right\|. \end{aligned}$$

Thus,

$$\begin{aligned} w^2(A) &= \sup\left\{|\langle Ax, x\rangle|^2 : x \in H, \|x\| = 1\right\} \\ &\leq \frac{p}{2}\left\||A|^2 + |A^*|^2\right\| + \frac{1-p}{2} w(A)\left\||A| + |A^*|\right\|. \end{aligned}$$

□

Corollary 2.10. *Let $A \in B(H)$ and $p \in [0,1]$. Then*

$$\begin{aligned} w^2(A) &\leq \frac{p}{2}\left\||A|^2 + |A^*|^2\right\| + \frac{1-p}{2} w(A)\left\||A| + |A^*|\right\| \\ &\leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|. \end{aligned}$$

Proof.

$$\begin{aligned} w^2(A) &\leq \frac{p}{2}\left\||A|^2 + |A^*|^2\right\| + \frac{1-p}{2} w(A)\left\||A| + |A^*|\right\| \quad \text{(by Theorem 2.9)} \\ &\leq \frac{p}{2}\left\||A|^2 + |A^*|^2\right\| + \frac{1-p}{4}\left\||A| + |A^*|\right\|^2 \quad \text{(by (2))} \\ &\leq \frac{p}{2}\left\||A|^2 + |A^*|^2\right\| + \frac{1-p}{2}\left\||A|^2 + |A^*|^2\right\| \quad \text{(by Lemma (2.2))} \\ &= \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|. \end{aligned}$$

□

Remark 2.11. *The inequality* (4) *is a special case of* (9) *when $p = \frac{1}{3}$. In addition, for $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, the inequality* (4) *gives the upper bound $w^2(A) \leq \frac{1}{3}$ whereas inequality* (9) *gives $w^2(A) \leq \frac{7}{24}$ (when $p = \frac{1}{6}$). Thus, the new upper bound* (9) *provides a more accurate estimate than the one in* (4) *for particular values of p.*

The next lemma will be employed to prove a new upper bound in Theorem 2.13.

Lemma 2.12. *Let $a, b, e \in H$ with $\|e\| = 1$ and $\alpha \geq 0$. Then*

$$|\langle a, e\rangle\langle e, b\rangle|^2 \leq \frac{2\alpha + 1}{2\alpha + 2}\|a\|^2\|b\|^2 + \frac{1}{2\alpha + 2}\|a\|\|b\||\langle a, b\rangle|.$$

Proof.

$$\begin{aligned}
|\langle a,e\rangle\langle e,b\rangle|^2 &\le \frac{1}{4}(\|a\|\|b\|+|\langle a,b\rangle|)^2\\
&\le \frac{1}{2}\left(\|a\|^2\|b\|^2+|\langle a,b\rangle|^2\right) \quad \text{(by convexity of } f(t)=t^2)\\
&\le \frac{1}{2}\left(\|a\|^2\|b\|^2+\frac{1}{1+\alpha}\|a\|\|b\||\langle a,b\rangle|+\frac{\alpha}{1+\alpha}\|a\|^2\|b\|^2\right) \quad \text{(by Lemma 2.5)}\\
&= \frac{2\alpha+1}{2\alpha+2}\|a\|^2\|b\|^2+\frac{1}{2\alpha+2}\|a\|\|b\||\langle a,b\rangle|.
\end{aligned}$$

□

Theorem 2.13. *Let $A \in B(H)$ and $\alpha \ge 0$. Then*

$$w^4(A) \le \frac{2\alpha+1}{4\alpha+4}\left\||A|^4+|A^*|^4\right\|+\frac{1}{4\alpha+4}\left\||A|^2+|A^*|^2\right\|w\left(A^2\right).$$

Proof. Let $x \in H$ be any unit vector and let $e=x$, $a=Ax$ and $b=A^*x$ in Lemma 2.12. Then we have

$$\begin{aligned}
|\langle Ax,x\rangle|^4 &\le \frac{2\alpha+1}{2\alpha+2}\|Ax\|^2\|A^*x\|^2+\frac{1}{2\alpha+2}\|Ax\|\|A^*x\|\left|\left\langle A^2x,x\right\rangle\right|\\
&= \frac{2\alpha+1}{2\alpha+2}\left\langle |A|^2x,x\right\rangle\left\langle |A^*|^2x,x\right\rangle+\frac{1}{2\alpha+2}\sqrt{\left\langle |A|^2x,x\right\rangle}\sqrt{\left\langle |A^*|^2x,x\right\rangle}\left|\left\langle A^2x,x\right\rangle\right|\\
&\le \frac{2\alpha+1}{4\alpha+4}\left\langle\left(|A|^4+|A^*|^4\right)x,x\right\rangle+\frac{1}{4\alpha+4}\left\langle\left(|A|^2+|A^*|^2\right)x,x\right\rangle\left|\left\langle A^2x,x\right\rangle\right|.
\end{aligned}$$

By taking the supremum over all unit vectors $x \in H$, we get the desired bound. □

Corollary 2.14. *Let $A \in B(H)$ and $\alpha \ge 0$. Then*

$$\begin{aligned}
w^4(A) &\le \frac{2\alpha+1}{4\alpha+4}\left\||A|^4+|A^*|^4\right\|+\frac{1}{4\alpha+4}\left\||A|^2+|A^*|^2\right\|w(A^2)\\
&\le \frac{1}{2}\left\||A|^4+|A^*|^4\right\|.
\end{aligned}$$

Proof.

$$\begin{aligned}
w^4(A) &\le \frac{2\alpha+1}{4\alpha+4}\left\||A|^4+|A^*|^4\right\|+\frac{1}{4\alpha+4}\left\||A|^2+|A^*|^2\right\|w(A^2) \quad \text{(by Theorem 2.13)}\\
&\le \frac{2\alpha+1}{4\alpha+4}\left\||A|^4+|A^*|^4\right\|+\frac{1}{8\alpha+8}\left\||A|^2+|A^*|^2\right\|^2 \quad \text{(by (5))}\\
&= \frac{2\alpha+1}{4\alpha+4}\left\||A|^4+|A^*|^4\right\|+\frac{1}{8\alpha+8}\left\|\left(|A|^2+|A^*|^2\right)^2\right\|\\
&\le \frac{2\alpha+1}{4\alpha+4}\left\||A|^4+|A^*|^4\right\|+\frac{1}{4\alpha+4}\left\||A|^4+|A^*|^4\right\| \quad \text{(by Lemma 2.2)}\\
&= \frac{1}{2}\left\||A|^4+|A^*|^4\right\|.
\end{aligned}$$

□

The next result which is obtained in [[7], Remark 3.2] by Bani-Domi and Kittaneh is a direct consequence of Theorem 2.13 by letting $\alpha = 1$.

Corollary 2.15. *Let $A \in B(H)$. Then*

$$w^4(A) \le \frac{3}{8}\left\||A|^4+|A^*|^4\right\|+\frac{1}{8}\left\||A|^2+|A^*|^2\right\|w\left(A^2\right).$$

Finally, we remark that the inequalities in Theorem 2.9 and Theorem 2.13 become equalities if A is normal.

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