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Characterizations of a spacetime admitting Ψ-conformal curvature tensor

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Abstract. In this paper, we introduce Ψ-conformal curvature tensor, a new tensor that generalizes the conformal curvature tensor. At first, we deduce a few fundamental geometrical properties of Ψ-conformal curvature tensor and pseudo Ψ-conharmonically symmetric manifolds and produce some interesting outcomes. Moreover, we study Ψ-conformally flat perfect fluid spacetimes. As a consequence, we establish a number of significant theorems about Minkowski spacetime, GRW-spacetime, projective collineation. Moreover, we show that if a Ψ-conformally flat spacetime admits a Ricci bi-conformal vector field, then it is either conformally flat or of Petrov type N. At last, we consider pseudo Ψ conformally symmetric spacetime admitting harmonic Ψ-conformal curvature tensor and prove that the semi-symmetric energy momentum tensor and Ricci semi-symmetry are equivalent and also, the Ricci collineation and matter collineation are equivalent.

1. Introduction

In general relativity (briefly, *GR*) and differential geometry, the Weyl conformal curvature tensor W performs a crucial role. In a Riemannian or a semi-Riemannian manifold, W is defined by[26]

$$\begin{aligned} W(U,V)G &= K(U,V)G - \frac{1}{n-2}[g(V,G)QU - g(U,G)QV \\ &\quad + S(V,G)U - S(U,G)V] \\ &\quad + \frac{R}{(n-1)(n-2)}[g(V,G)U - g(U,G)V], \end{aligned} \tag{1}$$

in which S, R and K are (0,2) type Ricci tensor, scalar curvature and the (1,3) type Riemannian curvature tensor, respectively.

The primary issue in differential geometry, among others, is the exploration of curvature properties. In this regard, S.S. Chern had stated in [8] " A fundamental notion is curvature, in its different forms ". As

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a result, the Riemann curvature tensor's discovery opens up a wide range of fascinating topics. Due to the aforementioned idea, we have developed a new curvature tensor in this study that we have named the Ψ-conformal curvature tensor.

In a semi-Riemannian manifold M^n, we define, a tensor C of type (0,4) by

$$\begin{aligned} C(U,V,G,H) &= \overline{K}(U,V,G,H) - \frac{\Psi}{n-2}[S(V,G)g(U,H) - S(U,G)g(V,H) \\ &\quad + S(U,H)g(V,G) - S(V,H)g(U,G)] \\ &\quad + \frac{\Psi R}{(n-2)(n-1)}[g(V,G)g(U,H) - g(U,G)g(V,H)], \end{aligned} \tag{2}$$

in which Ψ is the arbitrary scalar function and $\overline{K}$ stands for the (0,4)type Riemannian curvature tensor described by

$$\overline{K}(U,V,G,H) = g(K(U,V)G,H). \tag{3}$$

The preceding curvature tensor is named as a Ψ-conformal curvature tensor. The aforesaid curvature tensor, in particular, simplifies to a conformal curvature tensor for $\Psi = 1$. The curvature tensor and the Ψ-conformal curvature tensor are identical if $\Psi = 0$.

We are concentrating on symmetric spaces in this article due to their crucial relevance in differential geometry. In the late twenties E. Cartan [3], created these spaces.

Let ∇ stands for the Levi-Civita connection of (M^n, g), then the Riemannian manifold M^n is known as locally symmetric if $\nabla K = 0$ [3]. The aforementioned local symmetry criterion holds true for each point $U \in M^n$ where $F(U)$, the local geodesic symmetry is an isometry [20]. The class of manifolds of constant curvature is a very fundamental generalization of the class of Riemannian locally symmetric manifolds. Throughout the past few decades, numerous mathematicians have undermined the intriguing concept of locally symmetric manifolds in a variety of ways; for further information, see ([1], [4], [5], [22], [25]).

According to Chaki [5], for a non-zero 1-form D the $(M,g),(n>2)$, a non-flat Riemannian or a semi-Riemannian manifold is named pseudo symmetric if its curvature tensor $\overline{K}$ satisfies

$$\begin{aligned} (\nabla_X\overline{K})(U,V,G,H) &= 2D(X)\overline{K}(U,V,G,H) + D(U)\overline{K}(X,V,G,H) \\ &\quad + D(V)\overline{K}(U,X,G,H) + D(G)\overline{K}(U,V,X,H) \\ &\quad + D(H)\overline{K}(U,V,G,X), \end{aligned} \tag{4}$$

in which ρ stands for the vector field described by

$$g(U,\rho) = D(U), \tag{5}$$

for all U. If $D = 0$, then M reduces to a symmetric manifold in the Cartan sense. A pseudo symmetric manifold of dimension n is usually denoted by $(PS)_n$.

In [28], Zengin and Tasci initiated the investigation of pseudo-conharmonically symmetric manifolds. Recently, they investigated pseudo-conharmonically symmetric spacetimes [29].

Getting motivation from the foregoing investigations, we have studied a semi-Riemannian manifold $(M^n, g), (n > 2)$ whose Ψ-conformal curvature tensor C obeys

$$\begin{aligned} (\nabla_X C)(U,V,G,H) &= 2D(X)C(U,V,G,H) + D(U)C(X,V,G,H) \\ &\quad + D(V)C(U,X,G,H) + D(G)C(U,V,X,H) \\ &\quad + D(H)C(U,V,G,X). \end{aligned} \tag{6}$$

The above mentioned manifold shall be named a pseudo Ψ-conformally symmetric manifold and shall be denoted by $(PCS)_n$.

Lorentzian manifold is a special category of semi-Riemannian manifold equipped with a Lorentzian metric g. The Spacetime of general relativity is nothing but a time-oriented connected Lorentzian manifold

M^4 with the signature $(-,+,+,+)$. The study of the casual character of vectors of the Lorentzian manifold is started with the geometry of the Lorentz metric. The investigation of general relativity becomes a convenient choice for this casualty of the Lorentz manifold.

A perfect fluid (briefly, PF) which performs a significant role in general relativity does not have heat conduction terms and the stress term corresponding to viscosity [16]. Hence, for a PF the energy momentum tensor (briefly, EMT) T is written by

$$T(U,V) = (\sigma + p)B(U)B(V) + pg(U,V) \tag{7}$$

in which σ and p stand for the energy density and the isotropic pressure, respectively [20] and ρ is a unit timelike vector field ($g(\rho,\rho) = -1$) metrically equivalent to the 1-form B.

The Einstein's field equations without cosmological constant is written by

$$S(U,V) - \frac{R}{2}g(U,V) = kT(U,V) \tag{8}$$

where R and k are the scalar curvature and the gravitational constant, respectively.

A non-flat semi-Riemannian manifold satisfying the condition

$$(\nabla_{W_1}S)(U,V) = B_1(W_1)S(U,V), \tag{9}$$

is named a Ricci recurrent manifold [21] in which B_1 is a 1-form.

Mantica and Suh [17] have investigated Pseudo Z-symmetric spacetimes and Ozen[27] have studied m-Projectively flat spacetimes. A condition for which a pseudosymmetric spacetime would be a PF spacetime was recently discovered by Zhao et al.[30]). Moreover, in [10], we have studied ψ-conharmonically symmetric spacetime. As well, many authors have looked at the spacetime of general relativity in various methods; for additional information, see ([9], [15], [19]).

Motivated by the above investigations here we characterize the $(PCS)_n$ manifold and Ψ-conformally flat spacetimes.

The article is organized as:

The properties of Ψ-conformal curvature tensor are discussed in section 2. In section 3, we investigate the curvature properties of $(PCS)_n$ $(n > 2)$ manifold. Ψ-conharmonically flat spacetimes are investigated in section 4. Section 5 is devoted to investigate $(PCS)_4$ spacetimes.

2. Ψ-conformal curvature tensor

Let at every point of the manifold $\{e_j\}$ $(1 \leq j \leq n)$, be an orthonormal basis of the tangent space at each point of the manifold. Now from (2) we have

$$\sum_1^n C(V,G,e_j,e_j) = 0 = \sum_1^n C(e_j,e_j,V,G) \tag{10}$$

and

$$\begin{aligned}\sum_1^n C(e_j,e_j,G,H,e_i) &= \sum_1^n C(G,e_j,e_j,H)\\ &= (1-\Psi)S(G,H),\end{aligned} \tag{11}$$

where $R = \epsilon_j \sum_{j=1}^n S(e_j,e_j)$ (Here, we put $\epsilon_j = g(e_j, e_j)$, that is, $\epsilon_1 = -1, \epsilon_2 = \cdots = \epsilon_n = 1$).

From the equation(2) it can be deduced that

$$C(U, V, G, H) = -C(V, U, G, H), \tag{12}$$

$$C(U,V,G,H) = -C(U,V,H,G), \tag{13}$$

$$C(U,V,G,H) = C(G,H,U,V) \tag{14}$$

and

$$C(U,V,G,H) + C(V,G,U,H) + C(G,U,V,H) = 0. \tag{15}$$

Remark 2.1. *The conformal curvature tensor is a traceless tensor, but Ψ-conformal curvature tensor is not a traceless tensor.*

Proposition 2.2. *If a semi-Riemannian manifold M^n is Ψ-conformally flat, then the manifold becomes Ricci flat.*

Proof. If Ψ-conformal curvature tensor vanishes, then we acquire from (2) that

$$\begin{aligned}\overline{K}(U,V,G,H) &= \frac{\Psi}{n-2}[S(V,G)g(U,H) - S(U,G)g(V,H)\\ &+S(U,H)g(V,G) - S(V,H)g(U,G)]\\ &-\frac{\Psi R}{(n-2)(n-1)}[g(V,G)g(U,H) - g(U,G)g(V,H)].\end{aligned} \tag{16}$$

Contracting V and G in the foregoing equation yields

$$(1-\Psi)S(U,H) = 0. \tag{17}$$

Hence, we obtain

$$S(U,H) = 0, \tag{18}$$

Since, Ψ is an arbitrary scalar function, $\Psi \neq 1$.

This finishes the proof. □

Proposition 2.3. *Let the Ψ-conformal curvature tensor C be symmetric in the Cartan sense. Then the manifold M^n turns into a Ricci recurrent manifold.*

Proof. The Ψ-conformal curvature tensor C is written by

$$\begin{aligned}C(U,V)G &= K(U,V)G - \frac{\Psi}{n-2}[g(V,G)QU - g(U,G)QV\\ &+S(V,G)U - S(U,G)V]\\ &+\frac{\Psi R}{(n-2)(n-1)}[g(V,G)U - g(U,G)V],\end{aligned} \tag{19}$$

where Q is the Ricci operator defined by $g(QU,V) = S(U,V)$.

Covariant differentiation of (19) yields

$$\begin{aligned}(\nabla_{W_1}C)(U,V)G &= (\nabla_{W_1}K)(U,V)G\\ &-\frac{\Psi}{n-2}[(\nabla_{W_1}S)(V,G)U - (\nabla_{W_1}S)(U,G)V\\ &+g(V,G)(\nabla_{W_1}Q)U - g(U,G)(\nabla_{W_1}Q)V]\\ &-\frac{(W_1\Psi)}{n-2}[g(V,G)QU - g(U,G)V + S(V,G)U - S(U,G)V]\\ &+\frac{(W_1\Psi)R + \Psi(W_1R)}{(n-2)(n-1)}[g(V,G)U - g(U,G)V].\end{aligned} \tag{20}$$

By the assumption, the tensor C is symmetric in the Cartan sense, that is, $\nabla C = 0$. Therefore, the foregoing equation gives

$$\begin{aligned}(\nabla_{W_1}K)(U,V)G &= \frac{\Psi}{n-2}[(\nabla_{W_1}S)(V,G)U - (\nabla_{W_1}S)(U,G)V \\ &+g(V,G)(\nabla_{W_1}Q)U - g(U,G)(\nabla_{W_1}Q)V] \\ &+\frac{(W_1\Psi)}{n-2}[g(V,G)QU - g(U,G)V + S(V,G)U - S(U,G)V] \\ &-\frac{(W_1\Psi)R + \Psi(W_1R)}{(n-2)(n-1)}[g(V,G)U - g(U,G)V]. \end{aligned} \tag{21}$$

Contracting the previous equation, we acquire

$$(\nabla_{W_1}S)(V,G) = \frac{1}{1-\Psi}d\Psi(W_1)S(V,G). \tag{22}$$

Again contracting the equation (22), we obtain

$$dR(W_1) = \frac{R}{1-\Psi}d\Psi(W_1). \tag{23}$$

From the previous equation, we infer

$$(1-\Psi)(W_1 logR) = d\Psi(W_1). \tag{24}$$

Making use of (23) and (24) in (22), we obtain

$$(\nabla_{W_1}S)(V,G) = (W_1 logR)S(V,G). \tag{25}$$

This finishes the proof. □

3. $(PCS)_n$ ($n > 2$) manifolds

Proposition 3.1. *In a $(PCS)_n$ manifold, the Ψ-conformal curvature tensor satisfies the Second Bianchi Identity, that is,*

$$(\nabla_X C)(U,V,G,H) + (\nabla_G C)(U,V,H,X) + (\nabla_H C)(U,V,X,G) = 0. \tag{26}$$

Proof. Using (6) in the left hand side of the previous equation, we get the desired result. □

Proposition 3.2. *If a $(PCS)_n$ permits divergence-free Ψ-conformal curvature tensor, then the scalar curvature vanishes.*

Proof. From (2), taking a frame field over V and G we infer

$$C^\star(U,H) = (1-\Psi)S(U,H), \tag{27}$$

where $C^\star$ denotes the contracted Ψ-conformal curvature tensor.

Now contracting the equation(6) over X and H we acquire

$$\begin{aligned}(divC)(U,V)G &= D(C(U,V)G)) \\ &+D(U)C^\star(V,G) - D(V)C^\star(U,G). \end{aligned} \tag{28}$$

Making use of (2) and (27) in the foregoing equation, we obtain

$$\begin{aligned}(divC)(U,V)G &= D(K(U,V)G) - \frac{\Psi}{n-2}[D(QU)g(V,G) \\ &\quad -D(QV)g(U,G) + S(V,G)D(U) - S(U,G)D(V)] \\ &\quad +\frac{\Psi R}{(n-2)(n-1)}[g(V,G)D(U) - g(U,G)D(V)] \\ &\quad +D(U)(1-\Psi)S(V,G) \\ &\quad -D(V)(1-\Psi)S(U,G). \end{aligned} \tag{29}$$

If $divC = 0$, then contracting the previous equation, we infer

$$R[1 - \frac{2(1-n)}{n-2}\Psi]D(U) = 0. \tag{30}$$

Hence, either $R = 0$ or $\Psi = -\frac{(2-n)}{2(1-n)}$ = constant, a contradiction(since, Ψ is an arbitrary scalar function).
This finishes the proof. □

4. Ψ-Conformally flat spacetimes

To find out a model of the universe, Einstein applied the field equations of general relativity. The vast range of the universe displays isotropy and homogeneity, and the universe's matter (stars, nebulas, galaxies, and so on) can be comparable to a PF.

In this article, we explore the Ψ-Conformally flat spacetimes and pseudo Ψ-conformally symmetric spacetime. The conclusions reached for the pseudo Ψ-conformally symmetric manifolds apply equally to the Lorentzian situation. Now, we select the associated vector corresponding to the 1-form D is a unit timelike vector field, that is, $g(\rho, \rho) = -1$.

In this case, we take into account a PF spacetime with vanishing Ψ-conformal curvature tensor. In a Ψ-conformally flat PF spacetime (8) takes into the shape

$$kT(U,V) = 0. \tag{31}$$

Now using (31) and (7), we infer

$$k[(\sigma + p)B(U)B(V) + pg(U,V)] = 0. \tag{32}$$

Executing contraction over U and V, we acquire

$$3p - \sigma = 0. \tag{33}$$

Again, putting $U = V = \rho$ in (32), we obtain $\sigma = 0$ and hence using it (33), we provide $p = 0$. Therefore, the fluid is vaccum.

Thus, we write:

Theorem 4.1. *If a spacetime with vanishing Ψ-conformal curvature tensor satisfies EFE without cosmological constant, then the fluid is vaccum.*

Since a Ψ-conformally flat PF spacetime is Ricci flat, hence we obtain the scalar curvature $R = 0$. Thus, (16) infers that the spacetime has vanishing sectional curvature. Therefore, a Ψ-conformally flat PF spacetime and Minkowski spacetime are locally isometric ([13], p. 67).
Hence we write:

Theorem 4.2. *A Ψ-conformally flat PF spacetime is locally isometric to Minkowski spacetime.*

We know that

$$(divW)(U,V)G = \frac{n-3}{n-2}[\{(\nabla_U S)(V,G) - (\nabla_V S)(U,G)\} - \frac{1}{2(n-1)}\{g(V,G)dR(U) - g(U,G)dR(V)\}]. \tag{34}$$

Since in a Ψ-conformally flat PF-spacetime $S = 0$ and $R = 0$, from (34) we acquire $(divW)(U,V)G = 0$.

In [18], Mantica et al established that a PF-spacetime with R = constant and $(divW)(U,V)G = 0$ reduces to a GRW-spacetime.

Therefore, we state:

Theorem 4.3. *A Ψ-conformally flat PF spacetime is a GRW-spacetime.*

In a dust fluid spacetime [24], the T is described by

$$T(U,V) = \mu B_2(U)B_2(V), \tag{35}$$

in which B_2 stands for the velocity vector field of the flow, that is, $g(\rho,\rho) = -1$ and μ is the energy density of the dust-like matter.

Using the equation (8) and (35) we acquire

$$k\mu B_2(U)B_2(V) = 0. \tag{36}$$

Contracting the foregoing equation by taking a frame field, we get

$$k\mu = 0, \tag{37}$$

Therefore, the equation (35) yields

$$T(U,V) = 0. \tag{38}$$

Hence, the fluid is vacuum. This is not a physically significant scenario bearing in mind that the universe contains matter. Hence, we state

Theorem 4.4. *A dust fluid spacetime with vanishing Ψ-conformal curvature tensor satisfying EFE without cosmological constant does not exist.*

4.1. Projective collineation

If a continuous group of local diffeomorphism of M maps geodesics into geodesics, it is referred to as projective collineation (PC) [2] and its generator is referred to as a projective vector field. A vector field V is a P C if and only if

$$L_V\Gamma^i_{jk} = \delta^i_j q_k + \delta^i_k q_j,$$

in which L_V stands for the Lie derivative operator along V and $q_j = q_{,j}$ in which q is a 1-form. Therefore, locally q_j is an exact form. Specifically, if $L_V\Gamma^i_{jk} = 0$, then the PC turns into the affine collineation or affine motion. The maximum dimension of the projective algebra of M is $n^2 + n$ for which M is projectively flat. We know that the projective vector field V obeys

$$L_V K^l_{ijk} = \delta^l_k q_{i,j} - \delta^l_j q_{i,k}, \tag{39}$$

$$L_V K_{ij} = (1-n)q_{i,j}, \tag{40}$$

$$L_V P^l_{ijk} = 0, \tag{41}$$

in which K^l_{ijk}, K_{ij} and P^l_{ijk} are the components of the curvature tensor, Ricci tensor and projective curvature tensor, respectively.

Let us choose the PC in a Ψ-conformally flat spacetime. From equation (18) we see that the Ψ-conformally flat spacetime is Ricci flat and hence from (40), we acquire $q_{i,j} = 0$. If $q_i \neq 0$, then the PC is proper and for $n = 4$, the metric must be either a pp-wave [14] or flat. If $q_i = 0$,, then V generates an affine collineation and for the spacetime the metric is a pp-wave or decomposable, or V is a homothetic Killing vector field. Hence, we write the outcomes as:

Theorem 4.5. *Let a Ψ-conformally flat spacetime permits a projective collineation V . Then*

(i) the projective collineation is proper and, for $n = 4$, the metric is either a pp-wave or flat, provided $q_i \neq 0$.

(ii) V generates an affine collineation and the metric is a pp-wave or decomposable, or a homothetic Killing vector field, provided $q_i = 0$.

Definition 4.6. *On a Riemannian manifold a vector field X is named Ricci bi-conformal vector field [11] if it obeys the subsequent equations*

$$L_X g = \alpha g + \beta S \tag{42}$$

and

$$L_X S = \alpha S + \beta g \tag{43}$$

for non-zero smooth functions α and β.

Since $S = 0$, equation (42) implies $L_X g = \alpha g$ which entails that X is a conformal vector field. In [23], Sharma has established that " If a spacetime with divergence-free conformal curvature tensor permits a conformal Killing vector field, then the spacetime is either conformally flat or of Petrov type N."

Here $S = 0$, then obviously the divergence of the conformal curvature tensor vanishes. Therefore, we can write:

Theorem 4.7. *If a Ψ-conformally flat spacetime admits a Ricci bi-conformal vector field, then it is either conformally flat or of Petrov type N.*

5. $(PCS)_4$ spacetimes

Definition 5.1. *A semi-Riemannian manifold is called Ricci semi-symmetric if the Ricci tensor S fulfills*

$$K(U, V) \cdot S = 0,$$

for all $U, V \in \chi(M)$, where $K(U, V)$ acts as a derivation on the curvature tensor K.

Let the $(PCS)_4$ spacetime permits harmonic Ψ-conformal curvature tensor. Then by Proposition 3.2, we get the scalar curvature $R = 0$. Therefore, using equation (8), we acquire

$$S(U, V) = kT(U, V). \tag{44}$$

From the last relation, we infer $K \cdot S = K \cdot T$.

Theorem 5.2. *In a $(PCS)_4$ spacetime with harmonic Ψ-conformal curvature tensor, the semi-symmetric EMT and Ricci semi-symmetry are equivalent.*

Also, from equation (44), we obtain $\nabla S = \nabla T$. Since $\nabla T = 0$ entails $K \cdot T = 0$, hence we write the subsequent:

Theorem 5.3. *A $(PCS)_4$ spacetime with harmonic Ψ-conformal curvature tensor and covariant constant EMT is Ricci semi-symmetric.*

For a general relativistic spacetime the foregoing theorem has been established in [6].

In [12], De and Velimirovic established the subsequent outcomes:

Theorem 5.4. *In a PF spacetime let the EMT be semi-symmetric. Then the spacetime is characterized by the subsequent cases:*

(i) The PF behaves as a cosmological constant and the spacetime represents inflation. Also, it is named as a phantom barrier.

(ii) The PF will start to behave as exotic matter or, equivalently it represents the quintessence barrier.

Remark 5.5. *The above theorem holds in a $(PCS)_4$ spacetime with harmonic Ψ-conformal curvature tensor if the spacetime is Ricci semi-symmetric.*

We state that M permits a matter collineation if a provided symmetric vector field V (non trivial) of M leaves the matter tensor invariant ($L_V T_{ij} = 0$). Similarly, it is called Ricci collineation if $L_V S_{ij} = 0$ holds where S_{ij} are the components of the Ricci tensor S.

From equation (44), we can easily get

$$L_V S_{ij} = L_V T_{ij}. \tag{45}$$

Hence, we have

Theorem 5.6. *In a $(PCS)_4$ spacetime with harmonic Ψ-conformal curvature tensor the Ricci collineation and matter collineation are equivalent.*

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