Filomat 37:30 (2023), 10301–10321
https://doi.org/10.2298/FIL2330301J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some one-variable identities on generalized matrix functions which imply determinant

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Abstract. In this paper we prove that if a generalized matrix function satisfies some one-variable identities for some classes of n-by-n matrices over $\mathbb{C}$, then it is a scalar multiple of the determinant.

1. Introduction

Throughout the paper denote by $M_n(\mathbb{C})$ the set of all n-by-n matrices over $\mathbb{C}$ and let $\mathbb{S}_n$ be the symmetric group of degree n. Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be an arbitrary function. The *generalized matrix function* associated with G and χ is the function $d_\chi^G : M_n(\mathbb{C}) \to \mathbb{C}$ given by

$$d_\chi^G(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $A = (a_{ij}) \in M_n(\mathbb{C})$. The determinant and the permanent are two famous generalized matrix functions. In fact, if $G = \mathbb{S}_n$ and $\chi = \varepsilon$ is the alternating character of G, then $d_\chi^G = \det$ is the determinant and if $G = \mathbb{S}_n$ and $\chi = 1_G$ is the principal character of G, then $d_\chi^G = \text{per}$ is the permanent. Clearly if $\chi, \varphi : G \to \mathbb{C}$ are two functions and $\lambda \in \mathbb{C}$, then $d_{\chi+\lambda\varphi}^G = d_\chi^G + \lambda d_\varphi^G$, and if $\hat{\chi}$ is the unique extension of χ to $\mathbb{S}_n$ which vanishes outside of G, then $d_\chi^G = d_{\hat{\chi}}^{\mathbb{S}_n}$. We refer the reader to the books [4] and [5] for some deep information about generalized matrix functions.

Let us introduce some notations and preliminaries which will be used throughout. For each $\sigma \in \mathbb{S}_n$, let

$$\text{Fix}(\sigma) = \{i : 1 \leq i \leq n, \sigma(i) = i\}$$

be the set of fixed points of σ and $l(\sigma) = n - |\text{Fix}(\sigma)|$ be the length of σ. Obviously $\sigma = 1$ if and only if $l(\sigma) = 0$, and also $l(\sigma) \neq 1$ for all $\sigma \in \mathbb{S}_n$. It is important to note that the composition of permutations in $\mathbb{S}_n$ means left-to-right, that is, $(\sigma\tau)(i) = \tau(\sigma(i))$, for any $\sigma, \tau \in \mathbb{S}_n$. It is also known that each $1 \neq \sigma \in \mathbb{S}_n$ can be uniquely written as a product of (nontrivial) disjoint cycles. The number of (nontrivial) disjoint cycles in the decomposition of σ is denoted by $c(\sigma)$. We denote the set of involutions of $\mathbb{S}_n$ by $\mathbb{T}_n$, that is,

2020 *Mathematics Subject Classification*. 15A15, 12E05, 20C15.
Keywords. Generalized matrix functions; Determinant; Polynomials.
Received: 18 April 2023; Accepted: 30 June 2023
Communicated by Dijana Mosić
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$\mathbb{T}_n = \{\sigma \in \mathbb{S}_n : \sigma^2 = 1\}$.

Let $E_{rs} = (\delta_{ir}\delta_{sj}) \in M_n(\mathbb{C})$ be the standard matrix units, that is, the matrix which has 1 in the (r, s)-th entry and 0 elsewhere. Also for each $\sigma \in \mathbb{S}_n$, let $A_\sigma = (\delta_{\sigma(i)j}) \in M_n(\mathbb{C})$ be the permutation matrix induced by σ. It can be easily verified that for any $\sigma, \tau \in \mathbb{S}_n$:
(1) $A_\sigma = I_n$ if and only if $\sigma = 1$;
(2) $A_{\sigma\tau} = A_\sigma A_\tau$;
(3) $\det A_\sigma = \mathrm{sgn}(\sigma)$;
(4) A_σ is diagonalizable;
(5) if σ has order m, then each eigenvalue of A_σ is an m-th root of unity;
(6) $A_\sigma^{-1} = A_{\sigma^{-1}} = A_\sigma^t$;
(7) A_σ is a symmetric matrix if and only if $\sigma^2 = 1$;
(8) $E_{rs}A_\sigma = E_{r\sigma(s)}$ and $A_\sigma E_{rs} = E_{\sigma^{-1}(r)s}$.

Let us consider the following question:

Question: If $G \leqslant \mathbb{S}_n$, $\chi : G \to \mathbb{C}$ is a function, and C is a class of matrices in $M_n(\mathbb{C})$ such that the two-variable identity

$$d_\chi^G(AB) = d_\chi^G(A)d_\chi^G(B),$$

holds for all $A, B \in C$, then what is the relationship between d_χ^G and det?

This question has been extensively studied for several classes C. Exercise 2 of Chapter 8 in [5] says that if χ is an irreducible character of $G = \mathbb{S}_n$ and $C = M_n(\mathbb{C})$, then $d_\chi^G = \det$. The authors in [2] showed that $d_\chi^G = \det$ if χ is nonzero and C is the set of all nonsingular matrices in $M_n(\mathbb{C})$, and $d_\chi^G = \chi(1)\det$ if C is the set of all singular matrices in $M_n(\mathbb{C})$. It was proved later in [6] that if χ is a character of G and C is the set of all symmetric matrices in $M_n(\mathbb{C})$, then $d_\chi^G = \det$. Recently, we proved in [3], among other things, that $d_\chi^G = \det$ or $d_\chi^G = \mathrm{per}$ if χ is nonzero and C is the set of all permutation matrices in $M_n(\mathbb{C})$, $d_\chi^G = \det$ if χ is nonzero and C is the set of all nonsingular symmetric matrices in $M_n(\mathbb{C})$, and $d_\chi^G = \chi(1)\det$ if C is the set of all singular symmetric matrices in $M_n(\mathbb{C})$.

The main purpose of this paper is the study of generalized matrix functions that satisfy some one-variable identities related to the above question. More precisely, let $G \leqslant \mathbb{S}_n$, $\chi : G \to \mathbb{C}$ be a function, and $C, C', \mathcal{D}, \mathcal{D}'$ be the set of all nonsingular matrices, all singular matrices, all nonsingular symmetric matrices, all singular symmetric matrices in $M_n(\mathbb{C})$, respectively. We prove that the following are equivalent:
(1) $d_\chi^G(A) = \chi(1)\det(A)$ for all $A \in C \cup C'$ (for all $A \in \mathcal{D} \cup \mathcal{D}'$) and $\chi(1) \in \{0, 1\}$;
(2) $d_\chi^G(A^2) = d_\chi^G(A)^2$ for all $A \in C$ (for all $A \in \mathcal{D}$);
(3) $d_\chi^G(I_n) = d_\chi^G(A)d_\chi^G(A^{-1})$ for all $A \in C$ (for all $A \in \mathcal{D}$);
(4) $d_\chi^G(AA^t) = d_\chi^G(A)d_\chi^G(A^t)$ for all $A \in C$ (for all $A \in \mathcal{D}$).

Also we prove that the following are equivalent:
(1) $d_\chi^G(A) = \chi(1)\det(A)$ for all $A \in C \cup C'$ (for all $A \in \mathcal{D} \cup \mathcal{D}'$);
(2) $d_\chi^G(A^2) = d_\chi^G(A)^2$ for all $A \in C'$ (for all $A \in \mathcal{D}'$);
(3) $d_\chi^G(AA^t) = d_\chi^G(A)d_\chi^G(A^t)$ for all $A \in C'$ (for all $A \in \mathcal{D}'$).

It should be remarked that the proofs of the above results will be much more simple if it is assumed that χ is a character of G but they became much more difficult if the character condition on χ is removed, as it will be seen later.

2. Main Results

To state our results, we need three key lemmas. In [3], a binary relation on $\mathbb{S}_n$ was defined as follows:

$$\sigma \sim \tau \Longleftrightarrow \prod_{i=1}^{n} a_{i\sigma(i)} = \prod_{i=1}^{n} a_{i\tau(i)}, \text{ for any symmetric matrix } A = (a_{ij}) \in M_n(\mathbb{C}),$$

where $\sigma, \tau \in \mathbb{S}_n$. It is clear that $\sim$ is an equivalence relation on $\mathbb{S}_n$. The equivalence class of $\sigma \in \mathbb{S}_n$ is denoted by $[\sigma]$.

The first lemma is:

Lemma 2.1. *Let $\sigma, \tau \in \mathbb{S}_n - \{1\}$, where $\sigma = \sigma_1 \ldots \sigma_s$ is the decomposition of σ into disjoint cycles. Then*
(i) $[\sigma] = \{\sigma_1^{n_1} \ldots \sigma_s^{n_s} : n_1, \ldots, n_s \in \{-1, 1\}\}$;
(ii) *if $\tau(i) \in \{\sigma(i), \sigma^{-1}(i)\}$ for any $1 \le i \le n$, and $c(\tau) \le c(\sigma)$, then $\tau \in [\sigma]$.*

Proof. These are parts (vii) and (viii) of Lemma 2.5 in [3]. □

The second lemma is:

Lemma 2.2. *Let $\sigma, \tau \in \mathbb{S}_n$ such that $\tau(i) \in \{i, \sigma(i), \sigma^{-1}(i)\}$ for any $1 \le i \le n$. Then one of the following holds:*
(i) $l(\tau) < l(\sigma)$;
(ii) $l(\tau) = l(\sigma)$ *and* $c(\tau) > c(\sigma)$;
(iii) $\tau \in [\sigma]$.

Proof. By hypothesis $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\tau)$. If $\mathrm{Fix}(\sigma) \subset \mathrm{Fix}(\tau)$, then $l(\tau) < l(\sigma)$ and one has (i). If $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\tau)$, then $l(\tau) = l(\sigma)$ and $\tau(i) \in \{\sigma(i), \sigma^{-1}(i)\}$ for any $1 \le i \le n$. Now if $c(\tau) > c(\sigma)$, then one has (ii), and if $c(\tau) \le c(\sigma)$, then by Lemma 2.1 one has (iii). This completes the proof. □

It should be remarked that if $\tau \in [\sigma]$, then $l(\tau) = l(\sigma)$ and $c(\tau) = c(\sigma)$.

Finally the third lemma is:

Lemma 2.3. *Let $\sigma, \tau \in \mathbb{S}_n$, where σ is a cycle and k is an integer. If $\tau(i) \in \{\sigma^k(i), \sigma^{k+1}(i)\}$ for any $1 \le i \le n$, then $\tau \in \{\sigma^k, \sigma^{k+1}\}$.*

Proof. Notice that by hypothesis $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\tau)$. It suffices to show that if $\tau(a) = \sigma^k(a)$ for some $a \notin \mathrm{Fix}(\sigma)$, then $\tau = \sigma^k$. Assuming

$$\sigma = (a\ \sigma(a) \ldots \sigma^{m-1}(a)),$$

for some integer $m \ge 2$, and assuming by way of contradiction that $\tau \ne \sigma^k$, there exists the least integer $1 \le r \le m-1$ such that $\tau(\sigma^r(a)) \ne \sigma^k(\sigma^r(a))$. It can be shown by induction that $\tau(\sigma^j(a)) = \sigma^{k+1}(\sigma^j(a))$, for any $r \le j \le m-1$. This is true by hypothesis for $j = r$. Now assuming $r < j \le m-1$ and that $\tau(\sigma^{j-1}(a)) = \sigma^{k+1}(\sigma^{j-1}(a))$, one has

$$\sigma^k(\sigma^j(a)) = \sigma^{k+j}(a) = \tau(\sigma^{j-1}(a)) \ne \tau(\sigma^j(a)),$$

and so by hypothesis $\tau(\sigma^j(a)) = \sigma^{k+1}(\sigma^j(a))$. In particular,

$$\tau(\sigma^{m-1}(a)) = \sigma^{k+1}(\sigma^{m-1}(a)) = \sigma^{k+m}(a) = \sigma^k(a) = \tau(a),$$

implying that $\sigma^{m-1}(a) = a$, a contradiction. □

We are now ready to state our first theorem.

Theorem 2.4. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a nonzero function. Then $d_\chi^G = \det$ if and only if $d_\chi^G(I_n) = d_\chi^G(A)d_\chi^G(A^{-1})$ for all nonsingular matrices $A \in M_n(\mathbb{C})$.*

Proof. By hypothesis

$$\chi(1) = d_\chi^G(I_n) = d_\chi^G(I_n)d_\chi^G(I_n) = \chi(1)^2,$$

so either $\chi(1) = 0$ or $\chi(1) = 1$. If $\chi(1) = 1$, then by hypothesis $d_\chi^G(A) \neq 0$ for all nonsingular matrices $A \in M_n(\mathbb{C})$ and so the result follows by applying Theorem 2.1 of [1] or of [2].

Now if $\chi(1) = 0$, then

$$0 = d_\chi^G(I_n) = d_\chi^G(A)d_\chi^G(A^{-1}) = d_\varphi^{\mathbb{S}_n}(A)d_\varphi^{\mathbb{S}_n}(A^{-1}),$$

for all nonsingular matrices $A \in M_n(\mathbb{C})$, where $\varphi = \hat{\chi}$. We claim that $\varphi = 0$ and so $\chi = 0$, a clear contradiction. Suppose by way of contradiction that there is some $\sigma \in \mathbb{S}_n$ so that $l(\sigma)$ is as minimum as possible and $\varphi(\sigma) \neq 0$. Thus $l(\sigma) \geq 2$, for $\varphi(1) = 0$, and if $\tau \in \mathbb{S}_n$ with $l(\tau) < l(\sigma)$, then $\varphi(\tau) = 0$.

Let $m \geq 2$ be the order of σ. For any $1 < x \in \mathbb{R}$, the matrix $A(x) = I_n - xA_\sigma$ is nonsingular because

$$\lambda I_n = A(x)B(x),$$

where

$$B(x) = \sum_{i=0}^{m-1} x^i A_{\sigma^i}, \quad \lambda = 1 - x^m,$$

and so by hypothesis

$$d_\varphi^{\mathbb{S}_n}(A(x))d_\varphi^{\mathbb{S}_n}(B(x)) = \lambda^n d_\varphi^{\mathbb{S}_n}(A(x))d_\varphi^{\mathbb{S}_n}(\lambda^{-1}B(x)) = 0.$$

We show that $d_\varphi^{\mathbb{S}_n}(A(x)) \neq 0$ and hence $d_\varphi^{\mathbb{S}_n}(B(x)) = 0$. Let $A(x) = (a_{ij}(x))$ and let $\tau \in \mathbb{S}_n$ be such that $\prod_{i=1}^n a_{i\tau(i)}(x) \neq 0$. Hence $\tau(i) \in \{i, \sigma(i)\}$ for any $1 \leq i \leq n$ and so $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\tau)$. If $\mathrm{Fix}(\sigma) \subset \mathrm{Fix}(\tau)$, then $l(\tau) < l(\sigma)$ and so by the choice of σ we have $\varphi(\tau) = 0$. Thus if $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\tau)$, then $\tau = \sigma$ and therefore

$$d_\varphi^{\mathbb{S}_n}(A(x)) = \varphi(\sigma)\prod_{i=1}^n a_{i\sigma(i)}(x) = \varphi(\sigma)(1-x)^{|\mathrm{Fix}(\sigma)|}(-x)^{l(\sigma)} \neq 0.$$

Now if $B(x) = (b_{ij}(x))$ and $k \in \mathrm{Fix}(\sigma)$, then for any $1 \leq l \leq n$

$$b_{kl}(x) = b_{lk}(x) = \delta_{kl}(\sum_{i=0}^{m-1} x^i). \qquad (1)$$

Note that each nonzero non-diagonal entry $b_{ij}(x)$ of $B(x)$ is the sum of some distinct elements of the set $\{x, x^2, \ldots, x^{m-1}\}$, that is,

$$b_{ij}(x) = x^{s_1} + x^{s_2} + \cdots + x^{s_k}, \qquad (2)$$

for some $1 \leq s_1 < s_2 < \cdots < s_k \leq m-1$, where

$$\sigma^{s_1}(i) = \sigma^{s_2}(i) = \cdots = \sigma^{s_k}(i) = j.$$

Suppose now that $\tau \in \mathbb{S}_n$ is chosen so that $\prod_{i=1}^n b_{i\tau(i)}(x) \neq 0$. Then $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\tau)$ by (1). If $\mathrm{Fix}(\sigma) \subset \mathrm{Fix}(\tau)$, then $l(\tau) < l(\sigma)$ and so $\varphi(\tau) = 0$ by the choice of σ. Therefore if

$$\Omega = \{\tau \in \mathbb{S}_n : \ \mathrm{Fix}(\tau) = \mathrm{Fix}(\sigma)\},$$

$$\mu(x) = (\sum_{i=0}^{m-1} x^i)^{|\mathrm{Fix}(\sigma)|},$$

then

$$\begin{aligned} d_{\varphi}^{\mathbb{S}_n}(B(x)) &= \sum_{\tau\in\Omega} \varphi(\tau) \prod_{i=1}^{n} b_{i\tau(i)}(x) \\ &= \mu(x)\Big(\sum_{\tau\in\Omega} \varphi(\tau) \prod_{\substack{i=1 \\ i\notin \mathrm{Fix}(\sigma)}}^{n} b_{i\tau(i)}(x)\Big) \\ &= \mu(x)\Big(\varphi(\sigma) \prod_{\substack{i=1 \\ i\notin \mathrm{Fix}(\sigma)}}^{n} b_{i\sigma(i)}(x) + \sum_{\tau\in\Omega-\{\sigma\}} \varphi(\tau) \prod_{\substack{i=1 \\ i\notin \mathrm{Fix}(\sigma)}}^{n} b_{i\tau(i)}(x)\Big). \end{aligned}$$

On the one hand, by (2) one knows for any $i \notin \mathrm{Fix}(\sigma)$ that $b_{i\sigma(i)}$ is a polynomial in x which is divisible by x but not by x^2 and so

$$\varphi(\sigma) \prod_{\substack{i=1 \\ i\notin \mathrm{Fix}(\sigma)}}^{n} b_{i\sigma(i)}(x) = \varphi(\sigma) x^{l(\sigma)} p(x),$$

where $p(x)$ is a polynomial in x so that $p(0) = 1$.

On the other hand, if $\tau \in \Omega - \{\sigma\}$ is arbitrary, then there exists some $1 \leq t \leq n$ such that $\tau(t) \neq \sigma(t)$. Hence $t \notin \mathrm{Fix}(\tau)$ and so $b_{t\tau(t)}(x)$ is a non-diagonal entry of $B(x)$. Thus one can see using (2) that $b_{t\tau(t)}(x)$ is a (possibly zero) polynomial in x which is divisible by x^2. Again by (2), $b_{i\tau(i)}(x)$, for any $i \notin \mathrm{Fix}(\tau) \cup \{t\}$, is a (possibly zero) polynomial in x which is divisible by x. Hence

$$\varphi(\tau) \prod_{\substack{i=1 \\ i\notin \mathrm{Fix}(\sigma)}}^{n} b_{i\tau(i)}(x) = \varphi(\tau) x^{l(\sigma)+1} q(x),$$

where $q(x)$ is a polynomial in x. Thus

$$\sum_{\tau\in\Omega-\{\sigma\}} \varphi(\tau) \prod_{\substack{i=1 \\ i\notin \mathrm{Fix}(\sigma)}}^{n} b_{i\tau(i)}(x) = x^{l(\sigma)+1} r(x),$$

where $r(x)$ is a polynomial in x.

Therefore

$$0 = d_{\varphi}^{\mathbb{S}_n}(B(x)) = \mu(x)\Big(\varphi(\sigma) x^{l(\sigma)} p(x) + x^{l(\sigma)+1} r(x)\Big),$$

implying that $\varphi(\sigma)$, the coefficient of $x^{l(\sigma)}$, must be zero, which is a contradiction. This completes the proof of the claim. □

To state our next results, we bring the following remark which will be used frequently.

Remark 2.5. (i) *Let Γ be the set of representatives for the equivalence classes of $\sim$ on $\mathbb{S}_n$. Then for any function $\chi : \mathbb{S}_n \to \mathbb{C}$ and for any symmetric matrix $A = (a_{ij}) \in M_n(\mathbb{C})$, one has*

$$d_{\chi}^{\mathbb{S}_n}(A) = \sum_{\sigma\in\Gamma} \sum_{\tau\in[\sigma]} \chi(\tau) \prod_{i=1}^{n} a_{i\tau(i)} = \sum_{\sigma\in\Gamma} \Big(\sum_{\tau\in[\sigma]} \chi(\tau)\Big) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

(ii) *Let $\chi : \mathbb{S}_n \to \mathbb{C}$ be a function. Then, by Theorem* 2.6 *of [3], the following are equivalent:*
(1) $\sum_{\tau \in [\sigma]} \chi(\tau) = |[\sigma]|\chi(1)\varepsilon(\sigma)$ *for all* $\sigma \in \mathbb{S}_n$;
(2) $d_\chi^{\mathbb{S}_n}(A) = \chi(1)\det(A)$ *for all symmetric matrices* $A \in M_n(\mathbb{C})$.

We are now going to prove a theorem for symmetric matrices which is similar to Theorem 2.4.

Theorem 2.6. *Let $\chi : \mathbb{S}_n \to \mathbb{C}$ be a function. Then $d_\chi^{\mathbb{S}_n}(I_n) = d_\chi^{\mathbb{S}_n}(A)d_\chi^{\mathbb{S}_n}(A^{-1})$ for all nonsingular symmetric matrices $A \in M_n(\mathbb{C})$ if and only if $\chi(1) \in \{0,1\}$ and*

$$\sum_{\tau \in [\sigma]} \chi(\tau) = \chi(1)|[\sigma]|\varepsilon(\sigma),$$

for any $\sigma \in \mathbb{S}_n$.

Proof. One part is trivial by part (ii) of Remark 2.5. For the other part, we mimic the proof of Theorem 2.4. By hypothesis

$$\chi(1) = d_\chi^{\mathbb{S}_n}(I_n) = d_\chi^{\mathbb{S}_n}(I_n)d_\chi^{\mathbb{S}_n}(I_n) = \chi(1)^2,$$

so either $\chi(1) = 0$ or $\chi(1) = 1$. If $\chi(1) = 1$, then by hypothesis $d_\chi^{\mathbb{S}_n}(A) \neq 0$ for all nonsingular symmetric matrices $A \in M_n(\mathbb{C})$ and so the result follows by applying Theorem 2.6 of [3].

Now if $\chi(1) = 0$, then

$$0 = d_\chi^{\mathbb{S}_n}(I_n) = d_\chi^{\mathbb{S}_n}(A)d_\chi^{\mathbb{S}_n}(A^{-1}),$$

for all nonsingular symmetric matrices $A \in M_n(\mathbb{C})$. We claim that

$$\sum_{\tau \in [\sigma]} \chi(\tau) = 0,$$

for any $\sigma \in \mathbb{S}_n$. By way of contradiction choose $\sigma \in \mathbb{S}_n$ so that $l(\sigma)$ is minimal and $c(\sigma)$ is maximal and

$$\sum_{\tau \in [\sigma]} \chi(\tau) \neq 0.$$

Thus $l(\sigma) \geq 2$ and if $\tau \in \mathbb{S}_n$ and either $l(\tau) < l(\sigma)$ or $l(\tau) = l(\sigma)$ and $c(\tau) > c(\sigma)$, then

$$\sum_{\alpha \in [\tau]} \chi(\alpha) = 0.$$

Let $m \geq 2$ be the order of σ. For any $1 < x \in \mathbb{R}$, the matrix

$$A(x) = (I_n - xA_\sigma)(I_n - xA_{\sigma^{-1}}) = (x^2+1)I_n - x(A_\sigma + A_{\sigma^{-1}})$$

is symmetric and nonsingular because

$$\lambda I_n = A(x)B(x),$$

where

$$B(x) = (\sum_{i=0}^{m-1} x^i A_{\sigma^{-i}})(\sum_{i=0}^{m-1} x^i A_{\sigma^i}), \quad \lambda = (1-x^m)^2,$$

and so by hypothesis

$$d_\chi^{\mathbb{S}_n}(A(x))d_\chi^{\mathbb{S}_n}(B(x)) = \lambda^n d_\chi^{\mathbb{S}_n}(A(x))d_\chi^{\mathbb{S}_n}(\lambda^{-1}B(x)) = 0.$$

We show that $d_\chi^{\mathbb{S}_n}(A(x)) \neq 0$ and hence $d_\chi^{\mathbb{S}_n}(B(x)) = 0$. Let $A(x) = (a_{ij}(x))$ and let $\tau \in \mathbb{S}_n$ be such that $\prod_{i=1}^n a_{i\tau(i)}(x) \neq 0$. Hence $\tau(i) \in \{i, \sigma(i), \sigma^{-1}(i)\}$ for any $1 \leq i \leq n$ and it then follows by Lemma 2.2 and the choice of σ that if $\tau \notin [\sigma]$ then

$$\sum_{\alpha\in[\tau]} \chi(\alpha) = 0.$$

Note that if (rs) is a transposition in the decomposition of σ into disjoint cycles, then

$$a_{rj}(x) = (x^2+1)\delta_{rj} - 2x\delta_{sj}, \qquad a_{sj}(x) = (x^2+1)\delta_{sj} - 2x\delta_{rj}$$

and so $a_{r\sigma(r)}(x) = a_{s\sigma(s)}(x) = -2x$.

Therefore, if k is the number of transpositions in the decomposition of σ into disjoint cycles, then by Remark 2.5

$$\begin{aligned} d_\chi^{\mathbb{S}_n}(A(x)) &= (\sum_{\tau\in[\sigma]} \chi(\tau)) \prod_{i=1}^n a_{i\sigma(i)}(x) \\ &= (\sum_{\tau\in[\sigma]} \chi(\tau))(1-x)^{2|\mathrm{Fix}(\sigma)|}(-2x)^{2k}(-x)^{l(\sigma)-2k} \\ &\neq 0, \end{aligned}$$

as desired.
Now if $B(x) = (b_{ij}(x))$ and $k \in \mathrm{Fix}(\sigma)$, then for any $1 \leq l \leq n$

$$b_{kl}(x) = b_{lk}(x) = \delta_{kl}(\sum_{i=0}^{m-1} x^i)^2. \qquad (1)$$

To get some information about nonzero non-diagonal entries of $B(x)$ we have

$$\begin{aligned} B(x) &= \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} x^{i+j} A_{\sigma^{j-i}} \\ &= (\sum_{i=0}^{m-1} x^{2i}) I_n + \sum_{i=1}^{m-1}\sum_{j=0}^{i-1} x^{i+j}(A_{\sigma^{j-i}} + A_{\sigma^{i-j}}) \\ &= (\sum_{i=0}^{m-1} x^{2i}) I_n + \sum_{i=1}^{m-1}\sum_{k=1}^{i} x^{2i-k}(A_{\sigma^{-k}} + A_{\sigma^{k}}) \\ &= (\sum_{i=0}^{m-1} x^{2i}) I_n + \sum_{k=1}^{m-1}\sum_{i=k}^{m-1} x^{2i-k}(A_{\sigma^{-k}} + A_{\sigma^{k}}) \\ &= (\sum_{i=0}^{m-1} x^{2i}) I_n + \sum_{k=1}^{m-1} x^k(\sum_{i=0}^{m-k-1} x^{2i})(A_{\sigma^{-k}} + A_{\sigma^{k}}) \\ &= (\sum_{i=0}^{m-1} x^{2i}) I_n + \sum_{k=1}^{m-1} x^k p_k(x)(A_{\sigma^{-k}} + A_{\sigma^{k}}), \end{aligned}$$

where $p_k(x) = \sum_{i=0}^{m-k-1} x^{2i}$ is a polynomial in x with nonnegative coefficients and $p_k(0) = 1$.

Note that each nonzero non-diagonal entry $b_{ij}(x)$ of $B(x)$ is the sum of the elements of the set

$$\{xp_1(x), x^2p_2(x), \ldots, x^{m-1}p_{m-1}(x)\}$$

with a coefficient, that is,

$$b_{ij}(x) = \sum_{k=1}^{m-1} c_k x^k p_k(x), \qquad (2)$$

where

$$c_k = \begin{cases} 0 & \text{if } j \notin \{\sigma^k(i), \sigma^{-k}(i)\} \\ 1 & \text{if } j \in \{\sigma^k(i), \sigma^{-k}(i)\},\ \sigma^k(i) \neq \sigma^{-k}(i) \\ 2 & \text{if } j \in \{\sigma^k(i), \sigma^{-k}(i)\},\ \sigma^k(i) = \sigma^{-k}(i) \end{cases}$$

Suppose now that $\tau \in \mathbb{S}_n$ is chosen so that $\prod_{i=1}^n b_{i\tau(i)}(x) \neq 0$. Then $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\tau)$ by (1). If $\mathrm{Fix}(\sigma) \subset \mathrm{Fix}(\tau)$, then $l(\tau) < l(\sigma)$ and so by the choice of σ

$$\sum_{\alpha\in[\tau]} \chi(\alpha) = 0.$$

Hence $\mathrm{Fix}(\tau) = \mathrm{Fix}(\sigma)$ and so $l(\tau) = l(\sigma)$. Now if $c(\tau) > c(\sigma)$, then again by the choice of σ

$$\sum_{\alpha\in[\tau]} \chi(\alpha) = 0.$$

Therefore if

$$\Gamma_0 = \{\tau \in \Gamma :\ \mathrm{Fix}(\tau) = \mathrm{Fix}(\sigma),\ c(\tau) \leq c(\sigma)\},$$

$$\mu(x) = (\sum_{i=0}^{m-1} x^i)^{2|\mathrm{Fix}(\sigma)|},$$

then by Remark 2.5

$$\begin{aligned} d_\chi^{\mathbb{S}_n}(B(x)) &= \sum_{\tau\in\Gamma_0} \Big(\sum_{\alpha\in[\tau]} \chi(\alpha)\Big) \prod_{i=1}^n b_{i\tau(i)}(x) \\ &= \mu(x) \sum_{\tau\in\Gamma_0} \Big(\sum_{\alpha\in[\tau]} \chi(\alpha)\Big) \prod_{\substack{i=1 \\ i\notin \mathrm{Fix}(\sigma)}}^n b_{i\tau(i)}(x) \\ &= \mu(x)\Big((\sum_{\alpha\in[\sigma]} \chi(\alpha)) \prod_{\substack{i=1 \\ i\notin \mathrm{Fix}(\sigma)}}^n b_{i\sigma(i)}(x) + \sum_{\tau\in\Gamma_0-\{\sigma\}} (\sum_{\alpha\in[\tau]} \chi(\alpha)) \prod_{\substack{i=1 \\ i\notin \mathrm{Fix}(\sigma)}}^n b_{i\tau(i)}(x)\Big). \end{aligned}$$

On the one hand, by (2) one knows for any $i \notin \mathrm{Fix}(\sigma)$ that $b_{i\sigma(i)}(x)$ is a polynomial in x which is divisible by x but not by x^2 and so

$$(\sum_{\alpha\in[\sigma]} \chi(\alpha)) \prod_{\substack{i=1 \\ i\notin \mathrm{Fix}(\sigma)}}^n b_{i\sigma(i)}(x) = (\sum_{\alpha\in[\sigma]} \chi(\alpha)) x^{l(\sigma)} p(x),$$

where $p(x)$ is a polynomial in x so that $p(0) \neq 0$.

On the other hand, if $\tau \in \Gamma_0 - \{\sigma\}$ is arbitrary, then $\tau \notin [\sigma]$ and $c(\tau) \leq c(\sigma)$, and so by Lemma 2.1 there exists some $1 \leq t \leq n$ such that $\tau(t) \notin \{\sigma(t), \sigma^{-1}(t)\}$. Hence $t \notin \mathrm{Fix}(\tau)$ and so $b_{t\tau(t)}(x)$ is a non-diagonal entry of $B(x)$. Thus one can see using (2) that $b_{t\tau(t)}(x)$ is a (possibly zero) polynomial in x which is divisible by x^2. Again by (2), $b_{i\tau(i)}(x)$, for any $i \notin \mathrm{Fix}(\tau) \cup \{t\}$, is a (possibly zero) polynomial in x which is divisible by x. Hence

$$(\sum_{\alpha\in[\tau]} \chi(\alpha)) \prod_{\substack{i=1 \\ i\notin \mathrm{Fix}(\sigma)}}^n b_{i\tau(i)}(x) = (\sum_{\alpha\in[\tau]} \chi(\alpha)) x^{l(\sigma)+1} q(x),$$

where $q(x)$ is a polynomial in x. Thus

$$\sum_{\tau\in\Gamma_0-\{\sigma\}}(\sum_{\alpha\in[\tau]}\chi(\alpha))\prod_{\substack{i=1\\ i\notin \mathrm{Fix}(\sigma)}}^{n} b_{i\tau(i)}(x)=x^{l(\sigma)+1}r(x),$$

where $r(x)$ is a polynomial in x.

Therefore

$$0=d_{\varphi}^{\mathbb{S}_n}(B(x))=\mu(x)\Big((\sum_{\alpha\in[\sigma]}\chi(\alpha))x^{l(\sigma)}p(x)+x^{l(\sigma)+1}r(x)\Big),$$

implying that $\sum_{\alpha\in[\sigma]}\chi(\alpha)$, the coefficient of $x^{l(\sigma)}$, must be zero, which is a contradiction. This completes the proof of the claim. $\square$

For a subset Ω of $\{1,2,\ldots,n\}$, we say that the matrix $A=(a_{ij})\in M_n(\mathbb{C})$ is an Ω-block matrix if $a_{ij}=a_{ji}=0$ for any $i\in\Omega$ and $j\notin\Omega$. Obviously if $B=(b_{ij})\in M_n(\mathbb{C})$ is another matrix and $AB=(c_{ij})$, then

$$c_{ij}=\begin{cases}\sum_{k\in\Omega}a_{ik}b_{kj} & \text{if } i\in\Omega\\ \sum_{k\notin\Omega}a_{ik}b_{kj} & \text{if } i\notin\Omega\end{cases}$$

If B is an Ω-block matrix too, then so is AB. In particular, A^2, AA^t, AA^*, and $A\bar{A}$ are Ω-block matrices.

We now prove our next theorem.

Theorem 2.7. *Let $G\leqslant\mathbb{S}_n$ and $\chi:G\to\mathbb{C}$ be a function. Then $d_\chi^G=\chi(1)\det$ if and only if $d_\chi^G(A^2)=d_\chi^G(A)^2$ for all singular matrices $A\in M_n(\mathbb{C})$.*

Proof. First it is claimed that if $\varphi:\mathbb{S}_n\to\mathbb{C}$ is a function such that $\varphi(1)=0$ and

$$d_{\varphi}^{\mathbb{S}_n}(A^2)=d_{\varphi}^{\mathbb{S}_n}(A)^2$$

for all singular matrices $A\in M_n(\mathbb{C})$, then $\varphi=0$.

By way of contradiction choose $\sigma\in\mathbb{S}_n$ so that $l(\sigma)$ is minimal and $c(\sigma)$ is maximal and $\varphi(\sigma)\neq 0$. Thus $l(\sigma)\geq 2$ and if $\tau\in\mathbb{S}_n$ and either $l(\tau)<l(\sigma)$ or $l(\tau)=l(\sigma)$ and $c(\tau)>c(\sigma)$, then $\varphi(\tau)=0$.

In the sequel, let $\sigma=\sigma_1\ldots\sigma_s$ be the decomposition of σ into disjoint cycles.

Step 1: The cycles σ_i are even permutations.

Without loss of generality, we may assume that $\sigma_1=(a_1a_2\ldots a_m)$ is an odd permutation with the set of moving points Ω. So m is even and define the matrix $A=(a_{ij})\in M_n(\mathbb{C})$ as follows:

$$a_{ij}=\begin{cases}-\delta_{ij}-\delta_{\sigma(i)j} & \text{if } i=a_1\\ \delta_{ij}+\delta_{\sigma(i)j} & \text{if } i\in\Omega-\{a_1\}\\ \delta_{\sigma(i)j} & \text{if } i\notin\Omega\end{cases}$$

Note that A is an Ω-block matrix because $\sigma(\Omega)=\Omega$. Suppose that $\tau\in\mathbb{S}_n$ so that $\prod_{i=1}^n a_{i\tau(i)}\neq 0$. Hence $\tau(i)\in\{i,\sigma(i)\}=\{i,\sigma_1(i)\}$ if $i\in\Omega$ and $\tau(i)=\sigma(i)$ if $i\notin\Omega$. One obtains using Lemma 2.3 for $k=0$ that $\tau\in\{\sigma,\sigma_2\ldots\sigma_s\}$. Therefore

$$\det(A)=-\varepsilon(\sigma)-\varepsilon(\sigma_2\ldots\sigma_s)=0,$$

which means that A is singular, and also

$$d_{\varphi}^{\mathbb{S}_n}(A) = -\varphi(\sigma) - \varphi(\sigma_2 \ldots \sigma_s).$$

Since $l(\sigma_2 \ldots \sigma_s) < l(\sigma)$, hence $\varphi(\sigma_2 \ldots \sigma_s) = 0$ by the choice of σ and so $d_{\varphi}^{\mathbb{S}_n}(A) = -\varphi(\sigma)$.

Let us now compute the Ω-block matrix A^2. If $i \in \Omega$, then

$$\begin{aligned}
\sum_{k=1}^{n} a_{ik}a_{kj} &= \sum_{k\in\Omega} a_{ik}a_{kj} \\
&= \mp \sum_{k\in\Omega} (\delta_{ik} + \delta_{\sigma(i)k})a_{kj} \\
&= \mp\Big((\delta_{ia_1} + \delta_{\sigma(i)a_1})(-\delta_{a_1j} - \delta_{\sigma(a_1)j}) + \sum_{k\in\Omega-\{a_1\}} (\delta_{ik} + \delta_{\sigma(i)k})(\delta_{kj} + \delta_{\sigma(k)j})\Big) \\
&= \mp(-\delta_{ia_1}\delta_{a_1j} - \delta_{\sigma(i)a_1}\delta_{a_1j} - \delta_{ia_1}\delta_{\sigma(a_1)j} - \delta_{\sigma(i)a_1}\delta_{\sigma(a_1)j}) \\
&\quad \mp \sum_{k\in\Omega-\{a_1\}} \delta_{ik}\delta_{kj} + \delta_{\sigma(i)k}\delta_{kj} + \delta_{ik}\delta_{\sigma(k)j} + \delta_{\sigma(i)k}\delta_{\sigma(k)j}.
\end{aligned}$$

It can be easily verified that if $i \in \{a_1, a_m\}$, then

$$\sum_{k=1}^{n} a_{ik}a_{kj} = \delta_{ij} - \delta_{\sigma^2(i)j},$$

and if $i \in \Omega - \{a_1, a_m\}$, then

$$\sum_{k=1}^{n} a_{ik}a_{kj} = \delta_{ij} + 2\delta_{\sigma(i)j} + \delta_{\sigma^2(i)j}.$$

Now if $i \notin \Omega$, then

$$\begin{aligned}
\sum_{k=1}^{n} a_{ik}a_{kj} &= \sum_{k\notin\Omega} a_{ik}a_{kj} \\
&= \sum_{k\notin\Omega} \delta_{\sigma(i)k}\delta_{\sigma(k)j} \\
&= \delta_{\sigma^2(i)j}.
\end{aligned}$$

Therefore if $A^2 = (b_{ij})$, then

$$b_{ij} = \begin{cases} \delta_{ij} - \delta_{\sigma^2(i)j} & \text{if } i \in \{a_1, a_m\} \\ \delta_{ij} + 2\delta_{\sigma(i)j} + \delta_{\sigma^2(i)j} & \text{if } i \in \Omega - \{a_1, a_m\} \\ \delta_{\sigma^2(i)j} & \text{if } i \notin \Omega \end{cases}$$

We show that $d_{\varphi}^{\mathbb{S}_n}(A^2) = 0$. If not, then there exists some $\tau \in \mathbb{S}_n$ such that $\varphi(\tau) \neq 0$ and $\prod_{i=1}^{n} b_{i\tau(i)} \neq 0$. Hence $\tau(i) = \sigma^2(i)$ if $i \notin \Omega$, $\tau(i) \in \{i, \sigma^2(i)\}$ if $i \in \{a_1, a_m\}$, and $\tau(i) \in \{i, \sigma(i), \sigma^2(i)\}$ if $i \in \Omega - \{a_1, a_m\}$. In particular, $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\tau)$. If $\mathrm{Fix}(\sigma) \subset \mathrm{Fix}(\tau)$, then $l(\tau) < l(\sigma)$ and so $\varphi(\tau) = 0$ by the choice of σ, a contradiction. Thus $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\tau)$ and so $\tau(i) \in \{\sigma_1(i), \sigma_1^2(i)\}$ if $i \in \Omega$. Now by using Lemma 2.3 for $k = 1$ one has $\tau \in \{\sigma^2, \sigma_1\sigma_2^2 \ldots \sigma_s^2\}$. But $\tau(a_1) = \sigma^2(a_1) \neq \sigma(a_1)$ and so $\tau = \sigma^2$. Since m is even, hence one has $c(\tau) = c(\sigma^2) > c(\sigma)$. It then follows from $l(\tau) = l(\sigma)$ and the choice of σ that $\varphi(\tau) = 0$, again a contradiction.

Now by hypothesis

$$\varphi(\sigma)^2 = d_\varphi^{\mathbb{S}_n}(A)^2 = d_\varphi^{\mathbb{S}_n}(A^2) = 0,$$

a final contradiction.

From now on, let $\theta = \theta_1 \dots \theta_s \in \mathbb{S}_n$ be a permutation with the same cycle structure as σ, where $\theta_1 = (a_1 a_2 \dots a_m)$ is a cycle with the set of moving points Ω. Notice that all the cycles θ_i are even permutations by Step 1.

Step 2: $\varphi(\theta)^2 = \varphi(\theta^2)$.

We define the Ω-block matrix $A = (a_{ij}) \in M_n(\mathbb{C})$ as follows:

$$a_{ij} = \begin{cases} -\delta_{ij} + \delta_{\theta(i)j} & \text{if } i = a_1 \\ \delta_{ij} + \delta_{\theta(i)j} & \text{if } i \in \Omega - \{a_1\} \\ \delta_{\theta(i)j} & \text{if } i \notin \Omega \end{cases}$$

One can obtain similar to the proof of Step 1 that

$$\det(A) = \varepsilon(\theta) - \varepsilon(\theta_2 \dots \theta_s) = 0,$$

meaning that A is singular, and also

$$d_\varphi^{\mathbb{S}_n}(A) = \varphi(\theta) - \varphi(\theta_2 \dots \theta_s).$$

Since $l(\theta_2 \dots \theta_s) < l(\theta) = l(\sigma)$, hence $\varphi(\theta_2 \dots \theta_s) = 0$ by the choice of σ and so $d_\varphi^{\mathbb{S}_n}(A) = \varphi(\theta)$.

Also similar computations as Step 1 show that if $A^2 = (b_{ij})$, then

$$b_{ij} = \begin{cases} \delta_{ij} + \delta_{\theta^2(i)j} & \text{if } i \in \{a_1, a_m\} \\ \delta_{ij} + 2\delta_{\theta(i)j} + \delta_{\theta^2(i)j} & \text{if } i \in \Omega - \{a_1, a_m\} \\ \delta_{\theta^2(i)j} & \text{if } i \notin \Omega \end{cases}$$

Now if $\tau \in \mathbb{S}_n$ is so that $\prod_{i=1}^n b_{i\tau(i)} \neq 0$, then one has in a similar manner as Step 1 that either $\varphi(\tau) = 0$ or $\tau = \theta^2$. Therefore $d_\varphi^{\mathbb{S}_n}(A^2) = \varphi(\theta^2)$ and using hypothesis the proof is completed.

Step 3: $\varphi(\theta_1 \theta_2^2 \dots \theta_s^2) = 0$.

Recall that θ_1 is an even permutation and so m is odd. Now define the Ω-block matrix $A = (a_{ij}) \in M_n(\mathbb{C})$ as follows:

$$a_{ij} = \begin{cases} \delta_{ij} - \delta_{\theta(i)j} & \text{if } i \in \Omega \\ \delta_{\theta(i)j} & \text{if } i \notin \Omega \end{cases}$$

If $\tau \in \mathbb{S}_n$ is so that $\prod_{i=1}^n a_{i\tau(i)} \neq 0$, then using Lemma 2.3 for $k = 0$ one obtains that $\tau \in \{\theta, \theta_2 \dots \theta_s\}$ and so

$$\det(A) = (-1)^m \varepsilon(\theta) + \varepsilon(\theta_2 \dots \theta_s) = 0,$$

which means that A is singular, and also

$$d_\varphi^{\mathbb{S}_n}(A) = (-1)^m \varphi(\theta) + \varphi(\theta_2 \dots \theta_s).$$

Since $l(\theta_2 \dots \theta_s) < l(\theta) = l(\sigma)$, hence $\varphi(\theta_2 \dots \theta_s) = 0$ by the choice of σ and so $d_\varphi^{\mathbb{S}_n}(A) = -\varphi(\theta)$.

One has by similar computations as before that if $A^2 = (b_{ij})$, then

$$b_{ij} = \begin{cases} \delta_{ij} - 2\delta_{\theta(i)j} + \delta_{\theta^2(i)j} & \text{if } i \in \Omega \\ \delta_{\theta^2(i)j} & \text{if } i \notin \Omega \end{cases}$$

and if $\tau \in \mathbb{S}_n$ is so that $\prod_{i=1}^n b_{i\tau(i)} \neq 0$, then either $\varphi(\tau) = 0$ or $\tau \in \{\theta^2, \theta_1\theta_2^2 \dots \theta_s^2\}$. Now by hypothesis

$$\varphi(\theta)^2 = d_\varphi^{\mathbb{S}_n}(A)^2 = d_\varphi^{\mathbb{S}_n}(A^2) = \varphi(\theta^2) + (-2)^m\varphi(\theta_1\theta_2^2 \dots \theta_s^2),$$

and the proof is completed by Step 2.

To complete the proof of the claim, let r be the least common multiple of the odd numbers $l(\sigma_2), \dots, l(\sigma_s)$. Hence by Euler's Theorem there is a natural number k such that r divides $2^k - 1$. Hence $2^k = 1 + rt$ for some natural number t. Now the permutation $\sigma_1\sigma_2^{2^{k-1}} \dots \sigma_s^{2^{k-1}}$ clearly has the same cycle structure as σ and so by applying Step 3 to this permutation we obtain

$$\begin{aligned} 0 &= \varphi(\sigma_1(\sigma_2^{2^{k-1}})^2 \dots (\sigma_s^{2^{k-1}})^2) \\ &= \varphi(\sigma_1\sigma_2^{2^k} \dots \sigma_s^{2^k}) \\ &= \varphi(\sigma_1\sigma_2^{1+rt} \dots \sigma_s^{1+rt}) \\ &= \varphi(\sigma_1\sigma_2 \dots \sigma_s) \\ &= \varphi(\sigma), \end{aligned}$$

a contradiction.

To complete the proof, suppose that $\varphi = \hat{\chi} - \chi(1)\varepsilon$ and so $\varphi(1) = 0$. Thus for all singular matrices $A \in M_n(\mathbb{C})$

$$\begin{aligned} d_\varphi^{\mathbb{S}_n}(A^2) &= d_{\hat{\chi}}^{\mathbb{S}_n}(A^2) - \chi(1)\det(A^2) \\ &= d_{\hat{\chi}}^{\mathbb{S}_n}(A^2) \\ &= d_\chi^G(A^2) \\ &= d_\chi^G(A)^2 \\ &= (d_{\hat{\chi}}^{\mathbb{S}_n}(A) - \chi(1)\det(A))^2 \\ &= d_\varphi^{\mathbb{S}_n}(A)^2, \end{aligned}$$

which implies that $\varphi = 0$ by the claim and the proof of the theorem is completed. □

As a consequence we obtain:

Corollary 2.8. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a nonzero function. Then $d_\chi^G = \det$ if and only if $d_\chi^G(A^2) = d_\chi^G(A)^2$ for all nonsingular matrices $A \in M_n(\mathbb{C})$.*

Proof. For any singular matrix $A \in M_n(\mathbb{C})$ there exists some $\epsilon > 0$ such that for all $0 < x < \epsilon$ the matrix $xI_n + A$ is nonsingular and so by hypothesis

$$d_\chi^G((xI_n + A)^2) = d_\chi^G(xI_n + A)^2.$$

But both sides of the above equality are polynomials in x and therefore their constant coefficients are equal, that is,

$$d_\chi^G(A^2) = d_\chi^G(A)^2$$

for all singular matrices $A \in M_n(\mathbb{C})$. It now follows by Theorem 2.7 that $\hat{\chi} = \chi(1)\varepsilon$. By hypothesis

$$\chi(1) = d_\chi^G(I_n^2) = d_\chi^G(I_n)^2 = \chi(1)^2,$$

so either $\chi(1) = 0$ or $\chi(1) = 1$. If $\chi(1) = 0$, then $\hat{\chi} = 0$, a contradiction. Thus $\chi(1) = 1$ and so $d_\chi^G = \det$. This completes the proof. □

To prove our next theorem, we need a useful group theoretical lemma which has been used frequently in [3]. Its easy proof is omitted.

Lemma 2.9. (i) *Let $\sigma = (a_1a_2 \dots a_m) \in \mathbb{S}_n$, where $m \geq 2$. Then $\sigma = \alpha\beta$, where $\alpha, \beta \in \mathbb{T}_n$ are defined as follows:*

$$\alpha = (a_1a_m)(a_2a_{m-1}) \cdots (a_{l-1}a_{l+2})(a_la_{l+1}),$$

$$\beta = (a_ma_2)(a_{m-1}a_3) \cdots (a_{l+3}a_{l-1})(a_{l+2}a_l),$$

if $m = 2l$ is even, and

$$\alpha = (a_1a_m)(a_2a_{m-1}) \cdots (a_{l-1}a_{l+3})(a_la_{l+2}),$$

$$\beta = (a_ma_2)(a_{m-1}a_3) \cdots (a_{l+3}a_l)(a_{l+2}a_{l+1}),$$

if $m = 2l + 1$ is odd;
(ii) *For each $\sigma \in \mathbb{S}_n$ there exist $\alpha, \beta \in \mathbb{T}_n$ such that $\sigma = \alpha\beta$ and* $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\alpha) \cap \mathrm{Fix}(\beta)$.

Now we can state the following theorem for symmetric matrices which is similar to Theorem 2.7. As it will be seen, its proof is entirely different than that of Theorem 2.7.

Theorem 2.10. *Let $\chi : \mathbb{S}_n \to \mathbb{C}$ be a function. Then $d_\chi^{\mathbb{S}_n}(A^2) = d_\chi^{\mathbb{S}_n}(A)^2$ for all singular symmetric matrices $A \in M_n(\mathbb{C})$ if and only if*

$$\sum_{\tau \in [\sigma]} \chi(\tau) = \chi(1)|[\sigma]|\varepsilon(\sigma),$$

for any $\sigma \in \mathbb{S}_n$.

Proof. One part is trivial by part (ii) of Remark 2.5. For the other part, first we claim that if $\varphi : \mathbb{S}_n \to \mathbb{C}$ is a function such that $\varphi(1) = 0$ and

$$d_\varphi^{\mathbb{S}_n}(A^2) = d_\varphi^{\mathbb{S}_n}(A)^2$$

for all singular matrices $A \in M_n(\mathbb{C})$, then

$$\sum_{\tau \in [\sigma]} \varphi(\tau) = 0,$$

for any $\sigma \in \mathbb{S}_n$.

First we prove by induction on $c(\tau)$ that $\varphi(\tau) = 0$ for any $\tau \in \mathbb{T}_n - \{1\}$. Let $1 \neq \tau \in \mathbb{T}_n$, $c(\tau) = s$, and $\tau = \tau_1 \dots \tau_s$ be the decomposition of τ into disjoint transpositions with $\tau_1 = (a_1a_2)$. Then the matrices

$$A = A_\tau + E_{a_1a_1} + E_{a_2a_2}, \quad B = A_\tau + E_{a_1a_1} + 4E_{a_2a_2} + E_{a_1a_2} + E_{a_2a_1}$$

are singular and symmetric and

$$A^2 = I_n + E_{a_1a_1} + E_{a_2a_2} + 2E_{a_1a_2} + 2E_{a_2a_1},$$

$$B^2 = I_n + 4E_{a_1a_1} + 19E_{a_2a_2} + 10E_{a_1a_2} + 10E_{a_2a_1}.$$

Hence by hypothesis

$$4(\varphi(1) + \varphi(\tau_1)) = d_\varphi^{\mathbb{S}_n}(A^2) = d_\varphi^{\mathbb{S}_n}(A)^2 = (\varphi(\tau) + \varphi(\tau_2 \dots \tau_s))^2,$$

$$100(\varphi(1) + \varphi(\tau_1)) = d_\varphi^{\mathbb{S}_n}(B^2) = d_\varphi^{\mathbb{S}_n}(B)^2 = 16(\varphi(\tau) + \varphi(\tau_2 \dots \tau_s))^2.$$

If $s = 1$, then one has

$$4\varphi(\tau) = 4(\varphi(1) + \varphi(\tau_1)) = (\varphi(\tau) + \varphi(1))^2 = \varphi(\tau)^2,$$

$$100\varphi(\tau) = 100(\varphi(1) + \varphi(\tau_1)) = 16(\varphi(\tau) + \varphi(1))^2 = 16\varphi(\tau)^2,$$

which imply that $\varphi(\tau) = 0$. Now assume that $s > 1$ and so $\tau_1, \tau_2 \dots \tau_s \in \mathbb{T}_n - \{1\}$ with $c(\tau_1) = 1$ and $c(\tau_2 \dots \tau_s) = s - 1$ and hence by induction $\varphi(\tau_1) = \varphi(\tau_2 \dots \tau_s) = 0$. Thus

$$0 = 4(\varphi(1) + \varphi(\tau_1)) = (\varphi(\tau) + \varphi(\tau_2 \dots \tau_s))^2 = \varphi(\tau)^2,$$

which means that $\varphi(\tau) = 0$. Therefore the claim is true for the elements of $\mathbb{T}_n$.

To complete the proof, by way of contradiction choose $\sigma \in \mathbb{S}_n$ so that $l(\sigma)$ is minimal and $c(\sigma)$ is maximal and

$$\sum_{\tau \in [\sigma]} \varphi(\tau) \neq 0.$$

Thus $\sigma \notin \mathbb{T}_n$ and if $\tau \in \mathbb{S}_n$ and either $l(\tau) < l(\sigma)$ or $l(\tau) = l(\sigma)$ and $c(\tau) > c(\sigma)$, then

$$\sum_{\alpha \in [\tau]} \varphi(\alpha) = 0.$$

We see by Lemma 2.9 that there exist $\alpha, \beta \in \mathbb{T}_n$ such that $\sigma = \alpha\beta$ and $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\alpha) \cap \mathrm{Fix}(\beta)$. More precisely, if $\sigma = \sigma_1 \dots \sigma_s$ is the decomposition of σ into disjoint cycles, then there exist disjoint permutations $\alpha_1, \dots, \alpha_s \in \mathbb{T}_n$ and disjoint permutations $\beta_1, \dots, \beta_s \in \mathbb{T}_n$ so that

$$\alpha = \alpha_1 \dots \alpha_s, \quad \beta = \beta_1 \dots \beta_s, \quad \sigma = \alpha\beta,$$

where if $\sigma_j = (a_1a_2 \dots a_m)$, then

$$\alpha_j = (a_1a_m)(a_2a_{m-1}) \cdots (a_{l-1}a_{l+2})(a_la_{l+1}),$$

$$\beta_j = (a_ma_2)(a_{m-1}a_3) \cdots (a_{l+3}a_{l-1})(a_{l+2}a_l),$$

if $m = 2l$ is even, and

$$\alpha_j = (a_1a_m)(a_2a_{m-1}) \cdots (a_{l-1}a_{l+3})(a_la_{l+2}),$$

$$\beta_j = (a_ma_2)(a_{m-1}a_3) \cdots (a_{l+3}a_l)(a_{l+2}a_{l+1}),$$

if $m = 2l + 1$ is odd.

Now we define the matrix $A = (a_{ij}) \in M_n(\mathbb{C})$ as follows:

$$a_{ij} = \begin{cases} \delta_{\alpha(i)j} - \delta_{\beta(i)j} & \text{if } i \notin \Omega \\ \delta_{ij} & \text{if } i \in \Omega \end{cases}$$

where $\Omega = \mathrm{Fix}(\sigma)$. Obviously, A is an Ω-block matrix because $\alpha(\Omega) = \beta(\Omega) = \Omega$. Since $\alpha, \beta \in \mathbb{T}_n$, hence

$$\delta_{\alpha(i)j} - \delta_{\beta(i)j} = \delta_{i\alpha^{-1}(j)} - \delta_{i\beta^{-1}(j)} = \delta_{i\alpha(j)} - \delta_{i\beta(j)},$$

which means that A is symmetric. Now consider the vector $X = (x_1, \ldots, x_n)^t$ with $x_i = 1$ if $i \notin \Omega$ and $x_i = 0$ otherwise. Since $\sigma \neq 1$, hence $X \neq \mathbf{0}$ and one can easily see that $AX = \mathbf{0}$, meaning that A is singular.

First we show that $d_\varphi^{\mathbb{S}_n}(A) = 0$. Assuming by way of contradiction that $d_\varphi^{\mathbb{S}_n}(A) \neq 0$, there is some $\theta \in \mathbb{S}_n$ by Remark 2.5 such that

$$\sum_{\tau \in [\theta]} \varphi(\tau) \neq 0, \qquad \prod_{i=1}^{n} a_{i\theta(i)} \neq 0.$$

Hence $\theta(i) \in \{i, \alpha(i), \beta(i)\}$ for any $1 \leq i \leq n$. It then follows that if Ω_j is the set of moving points of σ_j, then $\theta(\Omega_j) \subseteq \Omega_j$, $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\theta)$, and so $l(\theta) \leq l(\sigma)$. But we deduce by the choice of σ that $l(\theta) = l(\sigma)$ and so $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\theta)$. Now if θ_j is the restriction of θ to Ω_j, then $\theta_j \in \mathbb{S}_{\Omega_j}$ and $\theta = \theta_1 \ldots \theta_s$ which implies that $c(\theta) \geq c(\sigma)$. Again by the choice of σ we have $c(\theta) = c(\sigma)$. We show that $\theta = \alpha$. To this end, we know that $\theta_j(i) \in \{\alpha_j(i), \beta_j(i)\}$, for any $i \in \Omega_j = \{a_1, a_2, \ldots, a_m\}$, where $1 \leq j \leq s$. Notice that θ_j fixes no element of Ω_j and β_j fixes a_1 in both cases for m, so $\theta_j(a_1) = \alpha_j(a_1) = a_m$. It follows from $\beta_j(a_2) = a_m = \theta_j(a_1) \neq \theta_j(a_2)$ that $\theta_j(a_2) = \alpha_j(a_2) = a_{m-1}$. Continuing in this way, we obtain that $\theta_j = \alpha_j$ and so $\theta = \alpha$. It then follows from $\alpha \in \mathbb{T}_n$ that $[\theta] = [\alpha] = \{\alpha\}$ and so

$$0 \neq \sum_{\tau \in [\theta]} \varphi(\tau) = \varphi(\alpha) = 0,$$

a contradiction.

Let us now compute the Ω-block matrix A^2. If $i \notin \Omega$, then

$$\begin{aligned}
\sum_{k=1}^{n} a_{ik}a_{kj} &= \sum_{k \notin \Omega} (\delta_{\alpha(i)k} - \delta_{\beta(i)k})(\delta_{\alpha(k)j} - \delta_{\beta(k)j}) \\
&= \sum_{k \notin \Omega} \delta_{\alpha(i)k}\delta_{\alpha(k)j} + \delta_{\beta(i)k}\delta_{\beta(k)j} - \delta_{\alpha(i)k}\delta_{\beta(k)j} - \delta_{\beta(i)k}\delta_{\alpha(k)j} \\
&= \delta_{\alpha^2(i)j} + \delta_{\beta^2(i)j} - \delta_{\beta(\alpha(i))j} - \delta_{\alpha(\beta(i))j} \\
&= 2\delta_{ij} - \delta_{(\alpha\beta)(i)j} - \delta_{(\beta\alpha)(i)j} \\
&= 2\delta_{ij} - \delta_{\sigma(i)j} - \delta_{\sigma^{-1}(i)j}.
\end{aligned}$$

If $i \in \Omega$, then

$$\sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k \in \Omega} \delta_{ik}\delta_{kj} = \delta_{ij}$$

Therefore $A^2 = (b_{ij}) \in M_n(\mathbb{C})$, where

$$b_{ij} = \begin{cases} 2\delta_{ij} - \delta_{\sigma(i)j} - \delta_{\sigma^{-1}(i)j} & \text{if } i \notin \Omega \\ \delta_{ij} & \text{if } i \in \Omega \end{cases}$$

If $\tau \in \mathbb{S}_n$ is so that $\prod_{i=1}^{n} b_{i\tau(i)} \neq 0$, then $\tau(i) \in \{i, \sigma(i), \sigma^{-1}(i)\}$ for any $1 \leq i \leq n$ and so we can apply Lemma 2.2. Also note that if (rs) is a transposition in the decomposition of σ into disjoint cycles, then

$$b_{rj} = 2\delta_{rj} - 2\delta_{sj}, \qquad b_{sj} = 2\delta_{sj} - 2\delta_{rj}$$

and so $b_{r\sigma(r)} = b_{s\sigma(s)} = -2$.

Therefore, if k is the number of transpositions in the decomposition of σ into disjoint cycles, then, using hypothesis, Remark 2.5, and Lemma 2.2, one obtains by the choice of σ that

$$\begin{aligned}
0 &= d_{\varphi}^{\mathbb{S}_n}(A)^2 \\
&= d_{\varphi}^{\mathbb{S}_n}(A^2) \\
&= \sum_{\gamma\in\Gamma}\Big(\sum_{\tau\in[\gamma]}\varphi(\tau)\Big)\prod_{i=1}^{n} b_{i\gamma(i)} \\
&= (\sum_{\tau\in[\sigma]}\varphi(\tau))\prod_{i=1}^{n} b_{i\sigma(i)} \\
&= (-1)^{l(\sigma)-2k}(-2)^{2k}\sum_{\tau\in[\sigma]}\varphi(\tau), \\
&\neq 0,
\end{aligned}$$

which is a contradiction. This completes the proof of the claim.

To complete the proof, suppose that $\varphi = \chi - \chi(1)\varepsilon$ and so $\varphi(1) = 0$. Thus by hypothesis for all singular matrices $A \in M_n(\mathbb{C})$

$$\begin{aligned}
d_{\varphi}^{\mathbb{S}_n}(A^2) &= d_{\chi}^{\mathbb{S}_n}(A^2) - \chi(1)\det(A^2) \\
&= d_{\chi}^{\mathbb{S}_n}(A^2) \\
&= d_{\chi}^{\mathbb{S}_n}(A)^2 \\
&= (d_{\chi}^{\mathbb{S}_n}(A) - \chi(1)\det(A))^2 \\
&= d_{\varphi}^{\mathbb{S}_n}(A)^2,
\end{aligned}$$

which implies by the claim that

$$0 = \sum_{\tau\in[\sigma]}\varphi(\tau) = \sum_{\tau\in[\sigma]}\chi(\tau) - \chi(1)|[\sigma]|\varepsilon(\sigma),$$

and the proof of the theorem is completed. □

As a consequence we obtain:

Corollary 2.11. *Let $\chi : \mathbb{S}_n \to \mathbb{C}$ be a class function. Then $d_{\chi}^{\mathbb{S}_n}(A^2) = d_{\chi}^{\mathbb{S}_n}(A)^2$ for all singular symmetric matrices $A \in M_n(\mathbb{C})$ if and only if $d_{\chi}^{\mathbb{S}_n} = \chi(1)\det$.*

As another consequence we obtain:

Corollary 2.12. *Let $\chi : \mathbb{S}_n \to \mathbb{C}$ be a function. Then $d_{\chi}^{\mathbb{S}_n}(A^2) = d_{\chi}^{\mathbb{S}_n}(A)^2$ for all nonsingular symmetric matrices $A \in M_n(\mathbb{C})$ if and only if $\chi(1) \in \{0, 1\}$ and*

$$\sum_{\tau\in[\sigma]}\chi(\tau) = \chi(1)|[\sigma]|\varepsilon(\sigma),$$

for any $\sigma \in \mathbb{S}_n$.

Proof. For any singular symmetric matrix $A \in M_n(\mathbb{C})$ there exists some $\epsilon > 0$ such that for all $0 < x < \epsilon$ the matrix $xI_n + A$ is a nonsingular symmetric matrix and so by hypothesis

$$d_{\chi}^{\mathbb{S}_n}((xI_n + A)^2) = d_{\chi}^{\mathbb{S}_n}(xI_n + A)^2.$$

But both sides of the above equality are polynomials in x and therefore their constant coefficients are equal, that is,

$$d_\chi^{\mathbb{S}_n}(A^2) = d_\chi^{\mathbb{S}_n}(A)^2$$

for all singular symmetric matrices $A \in M_n(\mathbb{C})$. It now follows by Theorem 2.10 that

$$\sum_{\tau\in[\sigma]} \chi(\tau) = \chi(1)|[\sigma]|\varepsilon(\sigma),$$

for any $\sigma \in \mathbb{S}_n$. Also by hypothesis

$$\chi(1) = d_\chi^{\mathbb{S}_n}(I_n^2) = d_\chi^{\mathbb{S}_n}(I_n)^2 = \chi(1)^2,$$

so either $\chi(1) = 0$ or $\chi(1) = 1$. This completes the proof. □

The following is an immediate consequence of Corollary 2.12.

Corollary 2.13. *Let $\chi : \mathbb{S}_n \to \mathbb{C}$ be a nonzero class function. Then $d_\chi^{\mathbb{S}_n}(A^2) = d_\chi^{\mathbb{S}_n}(A)^2$ for all nonsingular symmetric matrices $A \in M_n(\mathbb{C})$ if and only if $d_\chi^{\mathbb{S}_n} = \det$.*

The next is our final theorem.

Theorem 2.14. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a function. Then the following are equivalent:*
(i) $d_\chi^G = \chi(1)\det$;
(ii) $d_\chi^G(AA^t) = d_\chi^G(A)d_\chi^G(A^t)$ *for all singular matrices $A \in M_n(\mathbb{C})$;*
(iii) $d_\chi^G(AA^*) = d_\chi^G(A)d_\chi^G(A^*)$ *for all singular matrices $A \in M_n(\mathbb{C})$;*
(iv) $d_\chi^G(A\bar{A}) = d_\chi^G(A)d_\chi^G(\bar{A})$ *for all singular matrices $A \in M_n(\mathbb{C})$.*

Proof. (i) ⇒ (ii), (i) ⇒ (iii), and (i) ⇒ (iv) are obvious.

(ii) ⇒ (i): First we claim that if $\varphi : \mathbb{S}_n \to \mathbb{C}$ is a function such that $\varphi(1) = 0$ and

$$d_\varphi^{\mathbb{S}_n}(AA^t) = d_\varphi^{\mathbb{S}_n}(A)d_\varphi^{\mathbb{S}_n}(A^t)$$

for all singular matrices $A \in M_n(\mathbb{C})$, then $\varphi = 0$.

The first lines of the proof of Theorem 2.10 show that if $\tau \in \mathbb{T}_n$, then $\varphi(\tau) = 0$. By way of contradiction choose $\sigma \in \mathbb{S}_n$ so that $l(\sigma)$ is minimal and $c(\sigma)$ is maximal and $\varphi(\sigma) \neq 0$. Thus $\sigma \notin \mathbb{T}_n$ and if $\tau \in \mathbb{S}_n$ and either $l(\tau) < l(\sigma)$ or $l(\tau) = l(\sigma)$ and $c(\tau) > c(\sigma)$, then $\varphi(\tau) = 0$.

Now let $\sigma = \sigma_1 \dots \sigma_s$ be the decomposition of σ into disjoint cycles, where one of the cycles σ_i is of length at least 3 because $\sigma \notin \mathbb{T}_n$.

Step 1: $\varphi(\theta)\varphi(\theta^{-1}) = 0$ if $\theta \in \mathbb{S}_n$ has the same cycle structure as σ.

Suppose that $\theta = \theta_1 \dots \theta_s$ is the decomposition of θ into disjoint cycles, where $\theta_1 = (a_1a_2 \dots a_m)$ is an m-cycle with $m \geq 3$. If $\alpha = (a_1a_2)$ and $\tau = \alpha\theta$, then θ is even if and only if τ is odd, and so $\varepsilon(\theta) = -\varepsilon(\tau)$. Moreover, $l(\tau) = l(\theta) - 1 < l(\sigma)$ and so $\varphi(\tau) = \varphi(\tau^{-1}) = 0$ by the choice of σ. Now the matrix $A = A_\theta + E_{a_1a_3} + E_{a_2a_2}$ is singular, because

$$\det(A) = \varepsilon(\theta) + \varepsilon(\tau) = 0.$$

Also $A^t = A_{\theta^{-1}} + E_{a_3a_1} + E_{a_2a_2}$ and it can be easily seen that

$$AA^t = I_n + E_{a_1a_1} + E_{a_2a_2} + 2E_{a_1a_2} + 2E_{a_2a_1}.$$

Since φ is zero on $\mathbb{T}_n$, one has by hypothesis

$$\begin{aligned} 0 &= 4\varphi(1) + 4\varphi(\alpha) \\ &= d_\varphi^{\mathbb{S}_n}(AA^t) \\ &= d_\varphi^{\mathbb{S}_n}(A) d_\varphi^{\mathbb{S}_n}(A^t) \\ &= (\varphi(\theta) + \varphi(\tau))(\varphi(\theta^{-1}) + \varphi(\tau^{-1})) \\ &= \varphi(\theta)\varphi(\theta^{-1}), \end{aligned}$$

which completes the proof.

Step 2: $\sum_{\tau\in[\theta]} \varphi(\tau) = 0$ if $\theta \in \mathbb{S}_n$ has the same cycle structure as σ.

Define the Ω-block matrix $A = (a_{ij}) \in M_n(\mathbb{C})$ as follows:

$$a_{ij} = \begin{cases} \delta_{ij} - \delta_{\theta(i)j} & \text{if } i \notin \Omega \\ \delta_{ij} & \text{if } i \in \Omega \end{cases}$$

where $\Omega = \mathrm{Fix}(\theta)$. Consider the vector $X = (x_1, \ldots, x_n)^t$ with $x_i = 1$ if $i \notin \Omega$ and $x_i = 0$ otherwise. Since $\theta \notin \mathbb{T}_n$, hence $X \neq \mathbf{0}$ and one can see that $AX = \mathbf{0}$, which means that A is singular. Now if $\prod_{i=1}^n a_{i\tau(i)} \neq 0$ for some $\tau \in \mathbb{S}_n$, then $\tau(i) \in \{i, \theta(i)\}$ for any $1 \leq i \leq n$ and so $\mathrm{Fix}(\theta) \subseteq \mathrm{Fix}(\tau)$. If $\mathrm{Fix}(\theta) \subset \mathrm{Fix}(\tau)$, then $l(\tau) < l(\theta) = l(\sigma)$ and so by the choice of σ one has $\varphi(\tau) = 0$. Thus if $\mathrm{Fix}(\theta) = \mathrm{Fix}(\tau)$, then $\tau = \theta$. Therefore

$$d_\varphi^{\mathbb{S}_n}(A) = (-1)^{l(\theta)}\varphi(\theta).$$

If $A^t = (b_{ij})$, then obviously

$$b_{ij} = \begin{cases} \delta_{ij} - \delta_{\theta^{-1}(i)j} & \text{if } i \notin \Omega \\ \delta_{ij} & \text{if } i \in \Omega \end{cases}$$

and similar as above

$$d_\varphi^{\mathbb{S}_n}(A^t) = (-1)^{l(\theta)}\varphi(\theta^{-1}).$$

Now if $AA^t = (c_{ij})$, then one can easily see that

$$c_{ij} = \begin{cases} 2\delta_{ij} - \delta_{\theta(i)j} - \delta_{\theta^{-1}(i)j} & \text{if } i \notin \Omega \\ \delta_{ij} & \text{if } i \in \Omega \end{cases}$$

Choose $\tau \in \mathbb{S}_n$ such that $\prod_{i=1}^n c_{i\tau(i)} \neq 0$. Then $\tau(i) \in \{i, \theta(i), \theta^{-1}(i)\}$ for any $1 \leq i \leq n$. Since θ and σ have the same cycle structures, we see by Lemma 2.2 that if $\tau \notin [\theta]$, then $\varphi(\tau) = 0$ by the choice of σ.

Therefore using Step 1, hypothesis, and Remark 2.5 one obtains

$$\begin{aligned} 0 &= (-1)^{2l(\theta)}\varphi(\theta)\varphi(\theta^{-1}) \\ &= d_\varphi^{\mathbb{S}_n}(A) d_\varphi^{\mathbb{S}_n}(A^t) \\ &= d_\varphi^{\mathbb{S}_n}(AA^t) \\ &= \sum_{\tau\in[\theta]} \varphi(\tau) \prod_{i=1}^n c_{i\theta(i)} \\ &= (-1)^{l(\theta)-2k}(-2)^{2k} \sum_{\tau\in[\theta]} \varphi(\tau), \end{aligned}$$

where k is the number of transpositions in the decomposition of θ into disjoint cycles. This completes the proof.

Step 3: If $\theta = \theta_1\theta_2 \dots \theta_s \in \mathbb{S}_n$ has the same cycle structure as σ such that $\varphi(\theta) \neq 0$, then $\varphi(\theta_1^{-1}\theta_2 \dots \theta_s) \neq 0$.

There is nothing to prove if θ_1 is a transposition. So let θ_1 be a cycle with the set of moving points Ω and $m = |\Omega| \geq 3$. Define the Ω-block matrix $A(x) = (a_{ij}(x))$ as follows:

$$a_{ij}(x) = \begin{cases} \delta_{\theta(i)j} - \delta_{\theta^{-1}(i)j} & \text{if } i \in \Omega \\ x\delta_{\theta(i)j} - \delta_{\theta^{-1}(i)j} & \text{if } i \notin \Omega \end{cases}$$

where $x \in \mathbb{R}$ is arbitrary. Consider the vector $X = (x_1, \dots, x_n)^t$ with $x_i = 1$ if $i \in \Omega$ and $x_i = 0$ otherwise. Then $AX = \mathbf{0}$, which means that $A(x)$ is singular. We compute the Ω-block symmetric matrix $A(x)A(x)^t = (b_{ij}(x))$. If $i \notin \Omega$, then

$$\begin{aligned} b_{ij}(x) &= \sum_{k\notin\Omega} a_{ik}(x)a_{jk}(x) \\ &= \sum_{k\notin\Omega} (x\delta_{\theta(i)k} - \delta_{\theta^{-1}(i)k})(x\delta_{\theta(j)k} - \delta_{\theta^{-1}(j)k}) \\ &= \sum_{k\notin\Omega} x^2\delta_{\theta(i)k}\delta_{\theta(j)k} - x\delta_{\theta(i)k}\delta_{\theta^{-1}(j)k} - x\delta_{\theta^{-1}(i)k}\delta_{\theta(j)k} + \delta_{\theta^{-1}(i)k}\delta_{\theta^{-1}(j)k} \\ &= x^2\delta_{\theta(i)\theta(j)} - x\delta_{\theta(i)\theta^{-1}(j)} - x\delta_{\theta^{-1}(i)\theta(j)} + \delta_{\theta^{-1}(i)\theta^{-1}(j)} \\ &= x^2\delta_{ij} - x\delta_{\theta^2(i)j} - x\delta_{\theta^{-2}(i)j} + \delta_{ij} \\ &= (x^2+1)\delta_{ij} - x(\delta_{\theta^2(i)j} + \delta_{\theta^{-2}(i)j}). \end{aligned}$$

Similarly, if $i \in \Omega$, then

$$b_{ij}(x) = 2\delta_{ij} - (\delta_{\theta^2(i)j} + \delta_{\theta^{-2}(i)j}).$$

Hence

$$b_{ij}(x) = \begin{cases} 2\delta_{ij} - (\delta_{\theta^2(i)j} + \delta_{\theta^{-2}(i)j}) & \text{if } i \in \Omega \\ (x^2+1)\delta_{ij} - x(\delta_{\theta^2(i)j} + \delta_{\theta^{-2}(i)j}) & \text{if } i \notin \Omega \end{cases}$$

First we show that $d_\varphi^{\mathbb{S}_n}(A(x)A(x)^t) = 0$. If not, then there exists some $\tau \in \mathbb{S}_n$ such that $\prod_{i=1}^n b_{i\tau(i)}(x) \neq 0$ and $\varphi(\tau) \neq 0$. Thus $\tau(i) \in \{i, \theta^2(i), \theta^{-2}(i)\}$ for any $1 \leq i \leq n$ and so by Lemma 2.2 three distinct cases can happen:

(1) $l(\tau) < l(\theta^2)$;
(2) $l(\tau) = l(\theta^2)$ and $c(\tau) > c(\theta^2)$;
(3) $\tau \in [\theta^2]$.

We know that $l(\theta^2) \leq l(\theta)$, $l(\theta) = l(\sigma)$, and $c(\theta) = c(\sigma)$. Also recall that if $\tau \in [\theta^2]$, then $l(\tau) = l(\theta^2)$ and $c(\tau) = c(\theta^2)$. Now if $l(\theta^2) < l(\theta)$, then in all three cases we have $\varphi(\tau) = 0$ by the choice of σ, a contradiction. Hence we must have $l(\theta^2) = l(\theta)$ which implies that there is no transposition in the decomposition of θ into disjoint cycles and so $c(\theta^2) \geq c(\theta)$. Since $\varphi(\tau) \neq 0$, we see by the choice of σ that cases (1) and (2) cannot occur at all and case (3) cannot occur if $c(\theta^2) > c(\theta)$. It then follows that $l(\theta^2) = l(\theta)$, $c(\theta^2) = c(\theta)$, and $\tau \in [\theta^2]$. Two former conditions imply that θ^2 and θ (and hence θ^2 and σ) have the same cycle structures and so by Remark 2.5 and Step 2 one has

$$d_\varphi^{\mathbb{S}_n}(A(x)A(x)^t) = \Big(\sum_{\alpha\in[\theta^2]} \varphi(\alpha)\Big) \prod_{i=1}^n b_{i\theta^2(i)}(x) = 0,$$

which is a contradiction. Hence it was shown that $d_\varphi^{\mathbb{S}_n}(A(x)A(x)^t) = 0$ and so by hypothesis

$$d_\varphi^{\mathbb{S}_n}(A(x))d_\varphi^{\mathbb{S}_n}(A(x)^t) = 0.$$

It is clear that $\prod_{i=1}^{n} a_{i\tau(i)}(x)$ is a polynomial in x of degree at most $n-m$ for any $\tau \in \mathbb{S}_n$ and so both $d_{\varphi}^{\mathbb{S}_n}(A(x))$ and $d_{\varphi}^{\mathbb{S}_n}(A(x)^t)$ are polynomials in x and so at least one of them must be zero.

Assume first that $d_{\varphi}^{\mathbb{S}_n}(A(x))$ is the zero polynomial. If $\tau \in \mathbb{S}_n$ is such that $\prod_{i=1}^{n} a_{i\tau(i)}(x) \neq 0$ and $\varphi(\tau) \neq 0$, then $\tau(i) \in \{\theta(i), \theta^{-1}(i)\}$ for any $1 \leq i \leq n$ and so by Lemma 2.2 and the choice σ we have $\tau \in [\theta]$. Therefore, if $\prod_{i=1}^{n} a_{i\tau(i)}(x)$ is a polynomial in x of degree $n-m$, then $\tau \in \{\theta, \theta_1^{-1}\theta_2 \dots \theta_s\}$. But the coefficient of x^{n-m} in $d_{\varphi}^{\mathbb{S}_n}(A(x))$ is zero, which implies that

$$\varphi(\theta) + (-1)^m \varphi(\theta_1^{-1}\theta_2 \dots \theta_s) = 0,$$

implying that $\varphi(\theta_1^{-1}\theta_2 \dots \theta_s) \neq 0$, as desired.

Assume now that $d_{\varphi}^{\mathbb{S}_n}(A(x)^t)$ is the zero polynomial, where $A(x)^t = (c_{ij}(x))$ is as follows:

$$c_{ij}(x) = a_{ji}(x) = \begin{cases} \delta_{\theta^{-1}(i)j} - \delta_{\theta(i)j} & \text{if } i \in \Omega \\ x\delta_{\theta^{-1}(i)j} - \delta_{\theta(i)j} & \text{if } i \notin \Omega \end{cases}$$

Hence the constant coefficient in $d_{\varphi}^{\mathbb{S}_n}(A(x)^t)$ is zero, that is, $d_{\varphi}^{\mathbb{S}_n}(A(0)^t) = 0$. One can show similar to the first case that if $\tau \in \mathbb{S}_n$ is such that $\prod_{i=1}^{n} c_{i\tau(i)}(0) \neq 0$ and $\varphi(\tau) \neq 0$, then $\tau \in \{\theta, \theta_1^{-1}\theta_2 \dots \theta_s\}$. Therefore

$$0 = d_{\varphi}^{\mathbb{S}_n}(A(0)^t) = (-1)^n \varphi(\theta) + (-1)^{n-m} \varphi(\theta_1^{-1}\theta_2 \dots \theta_s),$$

again implying that $\varphi(\theta_1^{-1}\theta_2 \dots \theta_s) \neq 0$, and the proof of Step 3 is completed.

To get a final contradiction, since $\varphi(\sigma) \neq 0$, by Step 3 we have $\varphi(\sigma_1^{-1}\sigma_2 \dots \sigma_s) \neq 0$. Again since $\sigma_1^{-1}\sigma_2 \dots \sigma_s$ and σ have the same cycle structures, by Step 3 we conclude that $\varphi(\sigma_1^{-1}\sigma_2^{-1}\sigma_3 \dots \sigma_s) \neq 0$. Continuing in this way, we see that

$$\varphi(\sigma^{-1}) = \varphi(\sigma_1^{-1}\sigma_2^{-1} \dots \sigma_s^{-1}) \neq 0,$$

which contradicts Step 1, and the proof of the claim is completed.

To complete the proof of the theorem, suppose that $\varphi = \hat{\chi} - \chi(1)\varepsilon$ and so $\varphi(1) = 0$. Thus by hypothesis for all singular matrices $A \in M_n(\mathbb{C})$

$$\begin{aligned} d_{\varphi}^{\mathbb{S}_n}(AA^t) &= d_{\hat{\chi}}^{\mathbb{S}_n}(AA^t) - \chi(1)\det(AA^t) \\ &= d_{\chi}^{G}(AA^t) \\ &= d_{\chi}^{G}(A)d_{\chi}^{G}(A^t) \\ &= (d_{\hat{\chi}}^{\mathbb{S}_n}(A) - \chi(1)\det(A))(d_{\hat{\chi}}^{\mathbb{S}_n}(A^t) - \chi(1)\det(A^t)) \\ &= d_{\varphi}^{\mathbb{S}_n}(A)d_{\varphi}^{\mathbb{S}_n}(A^t), \end{aligned}$$

which implies that $\varphi = 0$ by the claim and so $d_{\chi}^{G} = \chi(1)\det$, as required.

(iii) ⇒ (i): The proof of (ii) ⇒ (i) will work here because all matrices used there have real entries.

(iv) ⇒ (i): The proof of Theorem 2.7 will work here because all matrices used there have real entries. □

As a final consequence we have:

Corollary 2.15. *Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a nonzero function. Then the following are equivalent:*
(i) $d_{\chi}^{G} = \det$;
(ii) $d_{\chi}^{G}(AA^t) = d_{\chi}^{G}(A)d_{\chi}^{G}(A^t)$ *for all nonsingular matrices* $A \in M_n(\mathbb{C})$;
(iii) $d_{\chi}^{G}(AA^*) = d_{\chi}^{G}(A)d_{\chi}^{G}(A^*)$ *for all nonsingular matrices* $A \in M_n(\mathbb{C})$;
(iv) $d_{\chi}^{G}(A\bar{A}) = d_{\chi}^{G}(A)d_{\chi}^{G}(\bar{A})$ *for all nonsingular matrices* $A \in M_n(\mathbb{C})$.

Proof. (ii) ⇒ (i): For any singular matrix $A \in M_n(\mathbb{C})$ there exists some $\epsilon > 0$ such that for all $0 < x < \epsilon$ the matrix $xI_n + A$ is nonsingular and so by hypothesis

$$d_\chi^G((xI_n + A)(xI_n + A)^t) = d_\chi^G(xI_n + A)d_\chi^G((xI_n + A)^t).$$

But both sides of the above equality are polynomials in x and therefore their constant coefficients are equal, that is,

$$d_\chi^G(AA^t) = d_\chi^G(A)d_\chi^G(A^t),$$

for all singular matrices $A \in M_n(\mathbb{C})$. It now follows by Theorem 2.14 that $\hat{\chi} = \chi(1)\varepsilon$. By hypothesis

$$\chi(1) = d_\chi^G(I_n) = d_\chi^G(I_n)d_\chi^G(I_n) = \chi(1)^2,$$

so either $\chi(1) = 0$ or $\chi(1) = 1$. If $\chi(1) = 0$, then $\hat{\chi} = 0$, a contradiction. Thus $\chi(1) = 1$ and so $d_\chi^G = \det$, as desired.

The proofs of (iii) ⇒ (i) and (iv) ⇒ (i) are similar to that of (ii) ⇒ (i). This completes the proof. □

We close this paper with two interesting research problems.

Let $G \leqslant \mathbb{S}_n$ and $\chi : G \to \mathbb{C}$ be a function.

Problem 1. Let φ be a ring automorphism or a ring anti-automorphism of $M_n(\mathbb{C})$. Is it true that $d_\chi^G = \chi(1)\det$ if $d_\chi^G(A\varphi(A)) = d_\chi^G(A)d_\chi^G(\varphi(A))$ for all singular matrices $A \in M_n(\mathbb{C})$?

By Theorems 2.7 and 2.14, this problem is true for the ring automorphisms $\varphi_1(A) = A$ and $\varphi_2(A) = \bar{A}$ and for the ring anti-automorphisms $\varphi_3(A) = A^t$ and $\varphi_4(A) = A^*$.

Problem 2. Let φ be a group automorphism or a group anti-automorphism of $GL_n(\mathbb{C})$ and χ be a nonzero function. Is it true that $d_\chi^G = \det$ if $d_\chi^G(A\varphi(A)) = d_\chi^G(A)d_\chi^G(\varphi(A))$ for all nonsingular matrices $A \in M_n(\mathbb{C})$?

By Theorem 2.4 and Corollaries 2.8 and 2.15, this problem is true for the group automorphisms $\varphi_1(A) = A$ and $\varphi_2(A) = \bar{A}$ and for the group anti-automorphisms $\varphi_3(A) = A^{-1}$, $\varphi_4(A) = A^t$, and $\varphi_5(A) = A^*$.

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