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# Existence of solutions for infinite system of nonlinear $q$-fractional boundary value problem in Banach spaces

**S. A. Mohiuddine[a,b,*], Anupam Das[c], Abdullah Alotaibi[b]**

[a] *Department of General Required Courses, Mathematics, The Applied College, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[b] *Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[c] *Department of Mathematics, Cotton University, Panbazar, Guwahati 781001, Assam, India*

**Abstract.** Studying various fixed point theorems on Banach spaces such as the Darbo's fixed point theorem, has recently proved to be quite effective while doing research on existence problems. We here use a contraction operator to demonstrate a modified Darbo-type fixed point result with the intention to study the existence of solutions of infinite system of nonlinear $q$-fractional boundary value problem in the Banach spaces. Towards the end, reasonable example is presented to validate our findings.

## 1. Introduction and preliminaries

Kuratowski [23] was a person who first explain the idea of measure of noncompactness (shortly, MNC). Darbo [13] presented the world with a fixed point theorem making use of the concepts of MNC, later popularly known by his name as the "Darbo fixed point theorem". Since then, it has become a crucial tool for the researchers while studying existence or solvability of nonlinear functional equations forming new Banach spaces and employing "Darbo fixed point theorem" on them. Recently, Banaś and Lecko [9], Mohiuddine et al. [26], Mursaleen et al. [28, 29] made efforts on studying certain special kinds of differential equations (infinite system) to instate the existence of solutions in certain Banach spaces. Some works are related to the fractional derivatives are done in the following articles [11, 12, 14, 24, 31, 32, 34, 35]. For studying real world problems in science and engineering such as the theory of neural nets, kinetic theory of gas, radiation, mechanics, neutron transportation etc, integral and differential equations are of great help. Most recently, using infinite system integral and differential equations along with MNCs on specified Banach spaces, many researches have shown existence of solutions to various real world problems [6, 15–17, 19, 20, 25, 27, 30, 33].

Let $\mathfrak{E}$ be a Banach space and $\mathbb{B}(\theta, \hat{r}) = \{x \in \mathfrak{E} : \|x - \theta\| \leq \hat{r}\}$ be a closed ball in $\mathfrak{E}$. If $\mathcal{Y}\left(\neq \phi\right) \subseteq \mathfrak{E}$, $\bar{\mathcal{Y}}$ and $\mathrm{Conv}\mathcal{Y}$ denote the closure and the convex closure of the set $\mathcal{Y}$. Further, the family of nonempty bounded and relatively compact spaces are expressed with the symbols $\mathfrak{M}_{\mathfrak{E}}$ and $\mathfrak{N}_{\mathfrak{E}}$ respectively.

The definition of MNC in [7] (see also [10]) as follows:

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* Corresponding author: S. A. Mohiuddine
*Email addresses:* mohiuddine@gmail.com (S. A. Mohiuddine), math.anupam@gmail.com (Anupam Das), mathker11@hotmail.com (Abdullah Alotaibi)

**Definition 1.1.** *A mapping $\mathcal{G} : \mathfrak{M}_{\mathfrak{E}} \to \mathbb{R}_+ (= [0, \infty))$ is called MNC in $\mathfrak{E}$ if*

*(i)* $\mathcal{J} \in \mathfrak{M}_{\mathfrak{E}}$, *implies* $\mathcal{G}(\mathcal{J}) = 0$ *gives* $\mathcal{J}$ *be relatively compact.*
*(ii)* *ker* $\mathcal{G} = \{\mathcal{J} \in \mathfrak{M}_{\mathfrak{E}} : \mathcal{G}(\mathcal{J}) = 0\} \neq \phi$. *Also ker* $\mathcal{G} \subset \mathfrak{N}_{\mathfrak{E}}$.
*(iii)* $\mathcal{J} \subseteq \mathcal{J}_1 \implies \mathcal{G}(\mathcal{J}) \leq \mathcal{G}(\mathcal{J}_1)$.
*(iv)* $\mathcal{G}\left(\bar{\mathcal{J}}\right) = \mathcal{G}(\mathcal{J})$.
*(v)* $\mathcal{G}(Conv\mathcal{J}) = \mathcal{G}(\mathcal{J})$.
*(vi)* $\mathcal{G}(\varsigma\mathcal{J} + (1-\varsigma)\mathcal{J}_1) \leq \varsigma\mathcal{G}(\mathcal{J}) + (1-\varsigma)\mathcal{G}(\mathcal{J}_1), \varsigma \in [0,1]$.
*(vii)* *for a nested sequence of sets* $\mathcal{S}_k \in \mathfrak{M}_{\mathfrak{E}}$, *where* $\mathcal{S}_{k+1} \subset \mathcal{S}_k$ *for* $k \in \mathbb{N}$ *and* $\mathcal{S}_k = \bar{\mathcal{S}}_k$ *such that* $\lim_{k\to\infty} \mathcal{G}(\mathcal{S}_k) = 0$ *then* $\bigcap_{k=1}^{\infty} \mathcal{S}_k \neq \phi$.

Now we recall a Banach space with it's respective norm, namely

$$c_0 = \left\{\theta \in w : \lim_{k\to\infty} \theta_k = 0,\ \| \theta \| = \sup_k |\theta_k|\right\}.$$

In [7] the MNC $\chi$ in $(c_0, \|\|)$ is as follows

$$\chi(B) = \lim_{n\to\infty}\left[\sup_{y\in B}\left(\max_{k\geq n}|y_k|\right)\right]. \tag{1}$$

$C(I, c_0)$ represents the set of all continuous functions from $I = [0, 1]$ to $c_0$ and $C(I, c_0)$ is a Banach space with

$$\|\theta\|_{C(I,c_0)} = \sup_{s\in I} \|\theta(s)\|_{c_0},$$

where $x(s) = (x_n(s))_{n=1}^{\infty} \in C(I, c_0)$. For any nonempty $E \subseteq C(I, c_0)$ and $E(s) = \{x(s) : x(s) \in E\}$ for $s \in I$and its MNC is as follows

$$\chi_{C(I,c_0)}(E) = \sup_{s\in I} \chi_{c_0}(E(s)).$$

Recall the following in [13] as follows:

**Theorem 1.2.** *For a nonempty, closed, bounded and convex (NCBC) subset $\mathcal{J}$ of Banach space, assume a continuous function $\mathfrak{S} : \mathcal{J} \to \mathcal{J}$ and for some $\kappa \in [0, 1)$ we have*

$$\mathcal{G}(\mathfrak{S}\Lambda) \leq \kappa\mathcal{G}(\Lambda),\ \Lambda \subseteq \mathcal{J}.$$

*Then $\mathfrak{S}$ has a fixed point.*

## 2. Darbo-type results

We now start to recall certain class of functions quite essential for proving our generalized form of Dorbo-fixed theorem. The following has already been discussed in [4].

Let $\mathfrak{F}(\mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$, denotes the collection of functions $\mathfrak{f} : \mathbb{R}_+ \to \mathbb{R}_+$. Suppose $\hat{\Theta}$ denotes the set of operators

$$\mathfrak{O}(\cdot;\cdot) : \mathfrak{F}(\mathbb{R}_+) \to \mathfrak{F}(\mathbb{R}_+),\ \mathfrak{f} \to \mathfrak{O}(\mathfrak{f}; .)$$

such that

(1) $\mathfrak{O}(\mathfrak{f}; \tau) \leq \mathfrak{O}(\mathfrak{f}; \varsigma)$ for $\tau \leq \varsigma$,
(2) $\mathfrak{O}(\mathfrak{f}; \max\{\tau, \varsigma\}) = \max\{\mathfrak{O}(\mathfrak{f}; \tau), \mathfrak{O}(\mathfrak{f}; \varsigma)\}$ for some $\mathfrak{f} \in \mathfrak{F}(\mathbb{R}_+)$,

(3) $\mathfrak{O}(\mathfrak{f};\tau) > 0$ for $\tau > 0$ and $\mathfrak{O}(\mathfrak{f};0) = 0$,

(4) $\lim_{n\to\infty} \mathfrak{O}(\mathfrak{f};\tau_n) = \mathfrak{O}\left(\mathfrak{f};\lim_{n\to\infty}\tau_n\right)$

hold. An example of such function is $\mathfrak{O}(\mathfrak{f};\tau) = \tau$.

Also, let us define the following class of functions:

$\Theta'$ is the collection of all functions $v : \mathbb{R} \to \mathbb{R}$ so that $\sum_{n=1}^{\infty} v(s_n) = \infty$ for all $\{s_n\} \subseteq \mathbb{R}$ (e.g. $v(s) = \tau \geq 0$).

$\Omega'$ is the collection of all functions $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

(1) $\mathcal{F}$ is strictly increasing,

(2) $\lim_{n\to\infty} \lambda_n = 0$ if and only if $\lim_{n\to\infty} \mathcal{F}(\lambda_n) = -\infty$ for each sequence $\{\lambda_n\}$ in $\mathbb{R}_+ = [0,\infty)$ (e.g. $\mathcal{F}(s) = \ln s$ for all $s > 0$).

**Theorem 2.1.** *For a NCBC subset $D$ of a Banach space $\mathfrak{E}$ with an arbitrary MNC $\mathfrak{G}$, let $\mathfrak{T} : D \to D$ be a continuous function satisfying the inequality*

$$\mathfrak{O}(\mathfrak{f};\mathcal{F}(\mathfrak{G}(\mathfrak{T}X))) \leq \mathfrak{O}[\mathfrak{f};\mathcal{F}(\mathfrak{G}(X))] - v(\mathcal{F}(\mathfrak{G}(X))) \tag{2}$$

*for any $X(\neq \phi) \subseteq D$ where $\mathcal{F} \in \Omega'$, $v \in \Theta'$, $\mathfrak{O} \in \hat{\Theta}$ and $\mathfrak{f} \in \mathfrak{F}(\mathbb{R}_+)$. Then*

$$\mathfrak{T} \text{ has at least one fixed point in } D. \tag{3}$$

*Proof.* We form a sequence of sets $\{D_n\}_{n=1}^{\infty}$ satisfying $D_1 = D$ and $D_{n+1} = Conv(\mathfrak{T}D_n)$ for $n \in \mathbb{N}$. Clearly we have, $\mathfrak{T}D_1 = \mathfrak{T}D \subseteq D = D_1$, $D_2 = Conv(\mathfrak{T}D_1) \subseteq D = D_1$. Proceeding this way we get the following nested sequence of sets,

$$D_1 \supseteq D_2 \supseteq D_3 \supseteq \ldots \supseteq D_n \supseteq D_{n+1} \supseteq \ldots.$$

Let $\exists\, n_0 \in \mathbb{N}$ with $\mathfrak{G}(D_{n_0}) = 0$, then $D_{n_0}$ is compact. And using Schauder theorem [2], we conclude $\mathfrak{T}$ has a fixed point in $D \subseteq \mathfrak{E}$.

Otherwise, $\mathfrak{G}(D_n) > 0$ for $n \in \mathbb{N}$. The sequence $\{\mathfrak{G}(D_n)\}_{n=1}^{\infty}$ is nonnegative, decreasing and bounded below. Hence, $\{\mathfrak{G}(D_n)\}_{n=1}^{\infty}$ is convergent.

If $\mathfrak{G}(D_{n+1}) > 0$ then $\mathcal{F}(\mathfrak{G}(D_{n+1})) \geq 0$ which gives $\mathfrak{O}(\mathfrak{f};\mathcal{F}(\mathfrak{G}(D_{n+1}))) \geq 0$.

Again, inequality (2) gives

$$\begin{aligned}
\mathfrak{O}(\mathfrak{f};\mathcal{F}(\mathfrak{G}(D_{n+1}))) &= \mathfrak{O}(\mathfrak{f};\mathcal{F}(\mathfrak{G}(Conv\mathfrak{T}D_n))) \\
&= \mathfrak{O}(\mathfrak{f};\mathcal{F}(\mathfrak{G}(\mathfrak{T}D_n))) \\
&\leq \mathfrak{O}[\mathfrak{f};\mathcal{F}(\mathfrak{G}(D_n))] - v(\mathcal{F}(\mathfrak{G}(D_n))) \\
&\leq \mathfrak{O}[\mathfrak{f};\mathcal{F}(\mathfrak{G}(D_{n-1}))] - v(\mathcal{F}(\mathfrak{G}(D_{n-1}))) - v(\mathcal{F}(\mathfrak{G}(D_n))) \\
&\ \ \ldots \\
&\leq \mathfrak{O}[\mathfrak{f};\mathcal{F}(\mathfrak{G}(D_1))] - \sum_{k=1}^{n} v(\mathcal{F}(\mathfrak{G}(D_k))).
\end{aligned}$$

Since $\sum_{k=1}^{n} v(\mathcal{F}(\mathfrak{G}(D_k))) \to \infty$ as $n \to \infty$ which gives

$$\mathfrak{O}(\mathfrak{f};\mathcal{F}(\mathfrak{G}(D_{n+1}))) \to -\infty$$

as $n \to \infty$ which is a contradiction. Hence we can not take $\mathfrak{G}(D_{n+1}) > 0$ for all $n$. Therefore $\mathfrak{G}(D_{n+1}) = 0$ for all $n$ i.e.

$$\lim_{n\to\infty} \mathfrak{G}(D_{n+1}) = 0,$$

that is,

$$\lim_{n\to\infty} \mathfrak{G}(D_n) = 0.$$

Since $D_n \supseteq D_{n+1}$ and from our definition at 1.1, we get

$$D_\infty = \bigcap_{j=1}^{\infty} D_n \subseteq D$$

must be a convex, nonempty, closed subset of $D$. As, $D_\infty$ is $\mathfrak{T}$ invariant, from Schauder theorem [2] we can say that Eq. (3) holds. □

**Theorem 2.2.** *For a NCBC set $D$ of a Banach space $\mathfrak{E}$ with an arbitrary MNC $\mathfrak{G}$, if $\mathfrak{T} : D \to D$ be a continuous function having the inequality*

$$\mathcal{F}(\mathfrak{G}(\mathfrak{T}X)) \leq \mathcal{F}(\mathfrak{G}(X)) - v(\mathcal{F}(\mathfrak{G}(X))) \tag{4}$$

*for any $X(\neq \phi) \subseteq D$ where $\mathcal{F} \in \Omega'$, $v \in \Theta'$. Then*

$$\mathfrak{T} \text{ must have a fixed point in } D. \tag{5}$$

*Proof.* Considering the function $\mathfrak{O}(\mathfrak{f}; \tau) = \tau$ in Theorem 2.1 we obtain the result. □

**Theorem 2.3.** *For a NCBC set $D$ of $\mathfrak{E}$ with an arbitrary MNC $\mathfrak{G}$, let $\mathfrak{T} : D \to D$ be a continuous function having the inequality*

$$\tau + \mathcal{F}(\mathfrak{G}(\mathfrak{T}X)) \leq \mathcal{F}(\mathfrak{G}(X)) \tag{6}$$

*for any $X(\neq \phi) \subseteq D$ where $\mathcal{F} \in \Omega'$. Then*

$$\text{there exists a fixed point for } \mathfrak{T} \text{ in } D. \tag{7}$$

*Proof.* Taking the function $v(s) = \tau \geq 0$ in Theorem 2.2, we get the required. □

**Remark 2.4.** *Taking $\tau = \ln \frac{1}{k}$, $0 < k < 1$ and $\mathcal{F}(s) = \ln s$ in Theorem 2.3 we obtain*

$$\ln \frac{1}{k} + \ln(\mathfrak{G}(\mathfrak{T}X)) \leq \ln(\mathfrak{G}(X)),$$

*that is,*

$$\mathfrak{G}(\mathfrak{T}X) \leq k\mathfrak{G}(X), \ 0 < k < 1.$$

## 3. An infinite system of $q$-fractional differential equations with boundary conditions

We recall some $q$-calculus concepts. For deeper understanding, the reader might read these literature [1, 3, 5, 18, 22].

Let $q \in (0, 1)$. and define

$$[\xi]_q = \frac{1 - q^\xi}{1 - q}, \ \xi \in \mathbb{R}.$$

The $q$-analogue of $(\eta - \zeta)^\nu$ with $\nu = 0, 1, 2, \ldots$ is

$$(\eta - \zeta)^0 = 1, \ (\eta - \zeta)^\nu = \prod_{k=0}^{\nu-1} (\eta - \zeta q^k), \ \nu \in \mathbb{N}, \ \eta, \zeta \in \mathbb{R}.$$

If $\xi \in \mathbb{R}$, then

$$(\eta - \zeta)^{(\xi)} = \eta^\xi \prod_{\nu=0}^{\infty} \frac{\eta - \zeta q^\nu}{\eta - \zeta q^{\xi+\nu}}.$$

If $\zeta = 0$ then $\eta^{(\nu)} = \eta^{\nu}$. The $q$-gamma function is as follows

$$\Gamma_q(\varpi) = \frac{(1-q)^{(\varpi-1)}}{(1-q)^{\varpi-1}},\ \varpi \in \mathbb{R}\setminus\{0,-1,-2,\ldots\}$$

and $\Gamma_q(\varpi+1) = [\varpi]_q\,\Gamma_q(\varpi)$.

The $q$-derivative of a function $\mathfrak{f}$ is as follows

$$(D_q\mathfrak{f})(\varpi) = \frac{\mathfrak{f}(\varpi) - \mathfrak{f}(q\varpi)}{(1-q)\varpi}$$

and the $q$-integral of $\mathfrak{f}$ on $[0,b]$ is as follows

$$(I_q\mathfrak{f})(\varpi) = \int_0^{\varpi} \mathfrak{f}(\iota)d_q\iota = \varpi(1-q)\sum_{\nu=0}^{\infty}\mathfrak{f}(\varpi q^{\nu})q^{\nu},\ \varpi \in [0,b].$$

**Lemma 3.1.** *[21] If $\mathfrak{f}:[0,1]\to\mathbb{R}$ is a continuous function, then*

$$\left|\int_0^{\varpi}\mathfrak{f}(\iota)d_q\iota\right| \le \int_0^{\varpi}|\mathfrak{f}(\iota)|\,d_q\iota,\ \varpi\in[0,1].$$

**Lemma 3.2.** *[21] If $\varepsilon > 0$ and $0 \le \zeta \le \rho \le \tau$ implies*

$$(\tau-\rho)^{(\varepsilon)} \le (\tau-\zeta)^{(\varepsilon)}.$$

**Definition 3.3.** *For $\alpha \ge 0$ and $\mathfrak{f}:[0,1]\to\mathbb{R}$ , the fractional $q$-integral of Riemann-Liouville type is defined as*

$$(I_q^0\mathfrak{f})(\varpi) = \mathfrak{f}(\varpi)$$

*and*

$$(I_q^{\alpha}\mathfrak{f})(\varpi) = \frac{1}{\Gamma_q(\alpha)}\int_0^{\varpi}(\varpi - q\varsigma)^{(\alpha-1)}\mathfrak{f}(\varsigma)d_q\varsigma,\ \alpha > 0.$$

**Definition 3.4.** *The fractional $q$-derivative of Riemann–Liouville type of order $\alpha \ge 0$ is defined by*

$$(D_q^{\lambda}\mathfrak{f})(\varpi) = \left(D_q^{[\lambda]}I_q^{[\lambda]-\lambda}\mathfrak{f}\right)(\varpi),\ \lambda > 0,\ \varpi\in[0,1]$$

*where $[\lambda]$ is the smallest integer greater than or equal to $\lambda$. Evidently, $(D_q^{\lambda}\mathfrak{f})(\varpi) = (D_q\mathfrak{f})(\varpi)$ when $\lambda = 1$.*

For a continuous mapping $\mathfrak{f} : [0,1]\times\mathbb{R} \to \mathbb{R}_+$, consider the nonlinear $q$-fractional boundary value problem (BVP) [18] as

$$(D_q^{\alpha}\mathfrak{y})(\varpi) = -\mathfrak{f}(\varpi,\mathfrak{y}(\varpi)),\ 0<\varpi<1,\ \mathfrak{y}(0)=\mathfrak{y}(1)=0 \tag{8}$$

where , $1 < \alpha \le 2,\ 0 < q < 1$.

$\mathfrak{y}$ is a solution of the BVP (8) iff $\mathfrak{y}$ satisfies the integral equation:

$$\mathfrak{y}(\varpi) = \int_0^1 \mathfrak{H}(\varpi, q\varrho)\mathfrak{f}(\varrho,\mathfrak{y}(\varrho))d_q\varrho$$

where $\mathfrak{H}$ is as follows

$$\mathfrak{H}(\varpi,\varrho) = \frac{1}{\Gamma_q(\alpha)}\left\{(\varpi(1-\varrho))^{(\alpha-1)} - (\varpi-\varrho)^{(\alpha-1)}\right\},\ 0\le\varrho\le\varpi\le1$$

and

$$\mathfrak{H}(\varpi,\varrho) = \frac{1}{\Gamma_q(\alpha)}(\varpi(1-\varrho))^{(\alpha-1)},\ 0 \le \varpi \le \varrho \le 1.$$

Here, we are investigating the solvability of following infinite system:

$$(D_q^\alpha \mathfrak{y}_n)(\varpi) = -\mathfrak{f}_n(\varpi, \mathfrak{y}(\varpi)),\ 0 < \varpi < 1,\ \mathfrak{y}(0) = \mathfrak{y}(1) = 0, \tag{9}$$

where $1 < \alpha \le 2,\ 0 < q < 1$ and $\mathfrak{f}_n : [0,1] \times E$ ($E :=$ sequence space) $\to \mathbb{R}_+$ are continuous functions for all $n \in \mathbb{N}$. Also $\mathfrak{y}(\varpi) = (\mathfrak{y}_n(\varpi))_{n=1}^\infty \in E$ where $E$ is a sequence space.

$\mathfrak{y}_n(\varpi)$ is a solution of problem (9) if and only if $\mathfrak{y}_n(\varpi)$ satisfies the following,

$$\mathfrak{y}_n(\varpi) = \int_0^1 \mathfrak{H}(\varpi, q\varrho)\mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))d_q\varrho. \tag{10}$$

We required the following assumptions to demonstrate our result.

(1) $\forall n \in \mathbb{N}$, $\mathfrak{f}_n : I \times C(I, c_0) \to \mathbb{R}_+ = [0,\infty)$ are continuous where $I = [0,1]$. The operator $\mathfrak{f}$ is defined from $I \times C(I, c_0)$ to $C(I, c_0)$ as

$$(\varpi, \mathfrak{y}(\varpi)) \to (\mathfrak{f}\mathfrak{y})(\varpi) = (\mathfrak{f}_n(\varpi, \mathfrak{y}(\varpi)))_{n=1}^\infty$$

where $((\mathfrak{f}\mathfrak{y})(\varpi))_{\varpi\in I}$ is equicontinuous at every point of $C(I, c_0)$.

(2) For every $\mathfrak{y}(\varpi) \in C(I, c_0),\ n \in \mathbb{N},\ \varpi \in I$, gives

$$\mathfrak{f}_n(\varpi, \mathfrak{y}(\varpi)) \le A_n(\varpi) + B_n(\varpi)\,|\mathfrak{y}_n(\varpi)|$$

where both $A_n(\varpi),\ B_n(\varpi)$ are nonnegative real continuous mappings on $I$ satisfying $\{A_n(\varpi)\}_{n=1}^\infty$ converges uniformly to zero on $I$ and $\{B_n(\varpi)\}_{n=1}^\infty$ is equibounded on $I$.

Let us assume

$$B(\varpi) = \sup_{n\in\mathbb{N}} B_n(\varpi),\ \mathfrak{B} = \sup_{\varpi\in I} B(\varpi) < \Gamma_q(\alpha),\ \mathfrak{A} = \sup_{n\in\mathbb{N},\, \varpi\in I} A_n(\varpi).$$

**Theorem 3.5.** *If assumptions (1)-(2) hold, system (9) has a solution in $C(I, c_0)$.*

*Proof.* We have $(\varpi(1-\varrho))^{(\alpha-1)} \le (\varpi)^{(\alpha-1)} = \varpi^{\alpha-1} \le 1$.

Therefore

$$(\varpi(1-\varrho))^{(\alpha-1)} - (\varpi-\varrho)^{(\alpha-1)} \le \varpi^{\alpha-1} - (\varpi-\varrho)^{(\alpha-1)} \le \varpi^{\alpha-1} \le 1$$

and

$$(\varpi(1-\varrho))^{(\alpha-1)} \le \varpi^{\alpha-1} \le 1.$$

Thus we have

$$\left|\mathfrak{H}(\varpi,\varrho)\right| \le \frac{1}{\Gamma_q(\alpha)}.$$

For arbitrary fixed $\varpi \in I$, using (2) and (9)

$$\begin{aligned}
\|\mathfrak{y}(\varpi)\|_{c_0} &= \sup_{n\geq 1}\left|\int_0^1 \mathfrak{H}(\varpi, q\varrho)\mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))d_q\varrho\right| \\
&\leq \sup_{n\geq 1}\int_0^1 \left|\mathfrak{H}(\varpi, q\varrho)\right| \mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))d_q\varrho \\
&\leq \frac{1}{\Gamma_q(\alpha)}\sup_{n\geq 1}\int_0^1 \mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))d_q\varrho \\
&\leq \frac{1}{\Gamma_q(\alpha)}\sup_{n\geq 1}\int_0^1 \left\{A_n(\varrho) + B_n(\varrho)\left|\mathfrak{y}_n(\varrho)\right|\right\} d_q\varrho \\
&\leq \frac{1}{\Gamma_q(\alpha)}\sup_{n\geq 1}\int_0^1 \left\{\mathfrak{A} + \mathfrak{B} \parallel \mathfrak{y} \parallel_{C(I,c_0)}\right\} d_q\varrho \\
&= \frac{\mathfrak{A} + \mathfrak{B} \parallel \mathfrak{y} \parallel_{C(I,c_0)}}{\Gamma_q(\alpha)}
\end{aligned}$$

which gives

$$(\Gamma_q(\alpha) - \mathfrak{B}) \parallel \mathfrak{y} \parallel_{C(I,c_0)} \leq \mathfrak{A}$$

i.e.

$$\|y\|_{C(I,c_0)} \leq \frac{\mathfrak{A}}{\Gamma_q(\alpha) - \mathfrak{B}} = d \ (say).$$

Suppose,

$$B = \left\{\mathfrak{y} \in C\,(I, c_0) : \|\mathfrak{y}\|_{C(I,c_0)} \leq d\right\}$$

which is a NCBC subset of $C\,(I, c_0)$.

For fixed $\varpi \in I$, we define the operator from $C(I, c_0)$ to $C(I, c_0)$ as follows:

$$(T\mathfrak{y})(\varpi) = \{(T_n\mathfrak{y})(\varpi)\}_{n=1}^{\infty} = \left\{\int_0^1 \mathfrak{H}(\varpi, q\varrho)\mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))d_q\varrho\right\}_{n=1}^{\infty}$$

where $\mathfrak{y}(\varpi) = \{\mathfrak{y}_n(\varpi)\}_{n=1}^{\infty} \in C\,(I, c_0)$ and $\mathfrak{y}_n(\varpi) \in C(I, \mathbb{R})$.

As $\{\mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))\}_{n=1}^{\infty} \in C(I, c_0)$ we have

$$\lim_{n\to\infty}(T_n\mathfrak{y})(\varpi) = \lim_{n\to\infty}\int_0^1 \mathfrak{H}(\varpi, q\varrho)\mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))d_q\varrho = \int_0^1 \mathfrak{H}(\varpi, q\varrho)\lim_{n\to\infty}\mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))d_q\varrho = 0.$$

Hence $(T\mathfrak{y})(\varpi) \in C\,(I, c_0)$.

Also $(T_n\mathfrak{y})(\varpi)$satisfies the boundary conditions

$$(T_n\mathfrak{y})(0) = \int_0^1 \mathfrak{H}(0, q\varrho)\mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))d_q\varrho = \int_0^1 0.\mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))d_q\varrho = 0$$

and

$$(T_n\mathfrak{y})(1) = \int_0^1 \mathfrak{H}(1, q\varrho)\mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))d_q\varrho = \int_0^1 0.\mathfrak{f}_n(\varrho, \mathfrak{y}(\varrho))d_q\varrho = 0.$$

Again, $T$ maps $B$ to $B$ itself.

Taking the assumption (1) in our consideration we take $\mathfrak{z}(\varpi) = \{\mathfrak{z}_n(\varpi)\}_{n=1}^{\infty} \in B$ and $\exists\, \epsilon > 0$ for each $\delta > 0$ satisfying

$$\| (f\mathfrak{y})(\varpi) - (f\mathfrak{z})(\varpi) \|_{c_0} \leq \epsilon\Gamma_q(\alpha) \text{ for } \mathfrak{y}(\varpi),\ \mathfrak{z}(\varpi) \in B$$

whenever $\| \mathfrak{y}(\varpi) - \mathfrak{z}(\varpi) \|_{c_0} \leq \delta$ for all $\varpi \in I$.

For fixed $\varpi \in I$,

$$\| (T\mathfrak{y})(\varpi) - (T\mathfrak{z})(\varpi) \|_{c_0} \leq \sup_{n\geq 1} \int_0^1 \left|\mathfrak{H}(\varpi, q\varrho)\right| \left|\mathfrak{f}_n(\varrho, y(\varrho)) - \mathfrak{f}_n(\varrho, y(\varrho))\right| d_q\varrho < \epsilon$$

thus $T$ is continuous in $B$ as $\varpi$ is arbitrarily fixed.

Now,

$$\begin{aligned}
\chi_{c_0}(TB) &= \lim_{n\to\infty} \sup_{\mathfrak{y}\in B} \sup_{k\geq n} \left| \int_0^1 \mathfrak{H}(\varpi, q\varrho)\mathfrak{f}_k(\varrho, \mathfrak{y}y(\varrho)) d_q\varrho \right| \\
&\leq \frac{1}{\Gamma_q(\alpha)} \lim_{n\to\infty} \sup_{\mathfrak{y}\in B} \sup_{k\geq n} \int_0^1 \left\{A_k(\varrho) + B_k(\varrho) \left|\mathfrak{y}_k(\varrho)\right|\right\} d_q\varrho \\
&\leq \frac{\mathfrak{B}}{\Gamma_q(\alpha)} \lim_{n\to\infty} \sup_{\mathfrak{y}\in B} \sup_{k\geq n} \int_0^1 \left|\mathfrak{y}_k(\varrho)\right| d_q\varrho \\
&\leq \frac{\mathfrak{B}}{\Gamma_q(\alpha)} \chi_{c_0}(B),
\end{aligned}$$

which gives

$$\sup_{x\in I} \chi_{c_0}(TB) \leq \sup_{x\in I} \frac{\mathfrak{B}}{\Gamma_q(\alpha)} \chi_{c_0}(B), \tag{11}$$

implies

$$\chi_{C(I,c_0)}(TB) \leq \frac{\mathfrak{B}}{\Gamma_q(\alpha)} \chi_{C(I,c_0)}(B). \tag{12}$$

Now using assumption (2) and Remark 2.4 we can say $\exists$ a fixed point for $T$ in $B \subseteq C(I, c_0)$, i.e., the system we considered has a solution in $C(I, c_0)$. □

**Example 3.6.** *To validate the last Theorem 3.5, we suppose an infinite system as:*

$$D_{0.5}^{1.5}\mathfrak{y}_n(\varpi) = -\frac{\varpi}{n^2} - \Gamma_{0.5}(1.5) \left( \sum_{k=n}^{\infty} \frac{1}{6k^2} \right) \mathfrak{y}_n(\varpi),\ \mathfrak{y}_n(0) = \mathfrak{y}_n(1) = 0, \tag{13}$$

*for* $\varpi \in [0, 1] = I$ *and* $n \in \mathbb{N}$.

*And we write,*

$$\mathfrak{f}_n(\varpi, \mathfrak{y}(\varpi)) = \frac{\varpi}{n^2} + \Gamma_{0.5}(1.5) \left( \sum_{k=n}^{\infty} \frac{1}{6k^2} \right) \mathfrak{y}_n(\varpi)$$

*and*

$$q = 0.5,\ \alpha = 1.5.$$

*Here*

$$|\mathfrak{f}_n(\varpi, \mathfrak{y}(\varpi))| \leq \frac{\varpi}{n^2} + \Gamma_{0.5}(1.5) \left( \sum_{k=n}^{\infty} \frac{1}{6k^2} \right) |\mathfrak{y}_n(\varpi)|$$

*which implies*

$$|\mathfrak{f}_n(\varpi, \mathfrak{y}(\varpi))| \leq \frac{1}{n^2} + \frac{\Gamma_{0.5}(1.5)\,\pi^2}{36}\,|\mathfrak{y}_n(\varpi)|.$$

*We get* $A_n(\varpi) = \frac{1}{n^2},\ B_n(\varpi) = \frac{\Gamma_{0.5}(1.5)\pi^2}{36}$. *Clearly the sequence* $\left\{\frac{1}{n^2}\right\}$ *is uniformly convergent to zero on* $I = [0,1]$ *and* $\left\{\frac{\Gamma_{0.5}(1.5)\pi^2}{36}\right\}$ *is equibounded on* $I$. *Also,*

$$\mathfrak{A} = 1,\ \mathfrak{B} = \frac{\Gamma_{0.5}(1.5)\,\pi^2}{36}$$

*and* $\frac{\mathfrak{B}}{\Gamma_{0.5}(1.5)} < 1$ *i.e.* $\mathfrak{B} < \Gamma_{0.5}(1.5)$.

*If* $\mathfrak{y}(\varpi) \in C(I, c_0)$ *then for all* $\varpi \in I$, $\lim_{n\to\infty} \mathfrak{f}_n(\varpi, \mathfrak{y}(\varpi)) = 0$ *i.e.* $\{\mathfrak{f}_n(\varpi, \mathfrak{y}(\varpi))\}_{n=1}^{\infty} \in C(I, c_0)$.

*Let* $\mathfrak{z}(x) = \{\mathfrak{z}_n(\varpi)\}_{n=1}^{\infty} \in C(I, c_0)$ *and* $\epsilon > 0$.

*For arbitrary fixed* $\varpi \in I$ *we have*

$$\| (\mathfrak{f}\mathfrak{y})(\varpi) - (\mathfrak{f}\mathfrak{z})(\varpi) \|_{c_0} \leq \sup_{n\geq 1} |\mathfrak{f}_n(\varpi, \mathfrak{y}(\varpi)) - \mathfrak{f}_n(\varpi, \mathfrak{z}(\varpi))| \leq \frac{\pi^2\Gamma_{0.5}(1.5)}{36} \| \mathfrak{y}(\varpi) - \mathfrak{z}(\varpi) \|_{c_0}.$$

*Thus*

$$\| (\mathfrak{f}\mathfrak{y})(\varpi) - (\mathfrak{f}\mathfrak{z})(\varpi) \|_{C(I,c_0)} < \epsilon$$

*whenever*

$$\| \mathfrak{y}(\varpi) - \mathfrak{z}(\varpi) \|_{C(I,c_0)} < \frac{36\epsilon}{\pi^2\Gamma_{0.5}(1.5)}$$

*which proves the equicontinuity of* $((\mathfrak{f}\mathfrak{y})(\varpi))_{x\in I}$ *on* $C(I, c_0)$. *Since all the required conditions are satisfied, using Theorem 3.5 we can conclude that the considered system (13) has a solution in* $C(I, c_0)$.

## 4. Concluding remarks

Here in this paper, we draw connections among three disciplines of mathematics namely the concept of MNC, infinite system of nonlinear $q$-fractional equations with boundary conditions and Banach space theory. We begin our discussion by proving a Darbo-type fixed theorem, later to use it on a sequence space with a suitable MNC predefined on it. Henceforth, we prove the existence of solution to our nonlinear $q$-fractional boundary value problem (infinite system). We conclude our discussion with an example that validate our findings.

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