Filomat 37:30 (2023), 10199–10207
https://doi.org/10.2298/FIL2330199D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Ideal convergence of multiset sequences

Nihal Demir[a], **Hafize Gümüş**[b,*]

[a]*Necmettin Erbakan University, Institute of Sciences Konya, Turkey*
[b]*Necmettin Erbakan University Faculty of Eregli Education Department of Math. Education, Eregli, Konya, Turkey*

Abstract. Although elements are written only once in classical set theory, it is possible to find many examples in daily life where an element is repeated more than once in a set and each repetition is indispensable. This situation is an indication of how important multisets are. On the other hand, ideal convergence is a type of convergence that generalizes many convergence types, and it is quite interesting how this convergence type can be defined for multiset sequences and what properties it will provide. For this purpose, in this paper we introduce the ideal convergence of multiset sequences and we investigate some basic algebraic and topological properties of a multiset sequences.

1. Introduction and Background

1.1. History of multiset sequences

In classical set theory, the elements of a set are written only once. Unlike this, multiset is a collection of objects in which elements are allowed to repeat. In fact, it is possible to see multisets in many areas of our lives. For example:

Telephone numbers: 0 535 7597...

Computer codes: 1110001101101...

Water molecule: H_2O

Coincident roots of equations: $(x-3)^2 = 0$.

In each example, there are same numbers and same molecules that play different roles. If these numbers are used once rather than multiple times, it is clear that there will be problems. Weyl explained this situation as there can be more than one white ball, more than one red ball, and more than one green ball in the same sack and he tried to apply his notion of multiset (a set with an equivalence relation) to a variety of problems in physics, chemistry, and genetics [37]. Hence, multisets are very interesting in mathematics, physics, philosophy, logic, linguistics, computer science, etc. Manna and Waldinger develop an elementary theory of bags using a primitive binary insertion symbol $\odot$. If an atom u has multiplicity $n \geq 0$ in bag x, then u has multiplicity $n+1$ in bag $u \odot x$. Their theory BAG admits only finite collections of atoms (no hierarchy of bags) and is developed to the point of a simple algebra of bags [21]. Actually, the development of multiset theory is, in fact, one small part of the remarkable proliferation of non-classical set theories such as Zadeh's fuzzy set theory, Vopënka's alternative (or semi-) set theory, Church's set theory with a universal set, intuitionistic

2020 *Mathematics Subject Classification*. 40G15, 40A35.
Keywords. Statistical Convergence; Ideal Convergence; Multisets; Multisequences.
Received: 15 April 2023; Revised: 24 June 2023; Accepted: 26 June 2023
Communicated by Eberhard Malkowsky
* Corresponding author: Hafize Gümüş
Email addresses: nihaldemircise@gmail.com (Nihal Demir), gumushafize2@gmail.com (Hafize Gümüş)

and constructive set theory and Pawlak's rough set theory. It is also possible to say that in social sciences, multisets can be used to model social structures, etc. [35].

It is observed from the survey of literature, multisets have been studied under the names of bags, occurrence set, sample and weighted set in the past years. Since the 1970s, Bender [2], Lake [19], Hickman [14], Meyer [23], and Monro [24] investigated some important properties of multisets. In 1981, Knuth [17] studied computer programing and multisets. On the other hand, Blizard studied on multisets in his doctoral thesis [5], [4], [3]. In multisets, the order of the elements is not important. So, $\{1,3,5,3,4,1,1\}$ and $\{1,1,1,3,3,4,5\}$ sets are same. On the other hand, it is very important how many times the elements are repeated in the set. We denote $\{1,3,5,3,4,1,1\}$ multiset by $\{1,3,4,5\}_{3,2,1,1}$ or $\{1|3,\ 3|2,\ 4|1,\ 5|1\}$ and it means 1 appearing 3 times, 3 appearing 2 times, 4 appearing 2 times and 5 appearing 2 times. The cardinality of a multiset is the sum of the multiplicities of its elements.

Studies on multisets continued in the 2000s and the studies titled "Mathematics of multisets" published by Syropoulos in 2001 [35], "An overview of the application of multiset" published by Singh in 2007 [33] , "Soft multisets" published by Majumdar in 2012 [20], "On multisets and multigroups" published by Nazmul in 2013 [24] and "Multigroup actions on multisets" published by İbrahim in 2017 [15] took its place among the important studies.

Despite all these studies several mathematicians have voiced their belief that the lack of adequate terminology and notation for this common concept has been a definite handicap to the development of mathematics, logic and philosophy. After these studies on multisets, the multiset sequences and their properties have started to be the subject of research. Usual convergence of multiset sequences was studied by Pachilangode and John in 2021 [29] and statistical convergence on $\mathbb{R}$ was studied by Debnath and Debnath in 2021 [8].

Definition 1.1. [29] *Let $\mathbb{R}$ be the set of real numbers. A sequence in which all the terms are multiset is known as a multiset sequence. For any sequence $x = (x_i) \in \mathbb{R}$, a multiset sequence is defined by*

$$mx = \{x_i|c_i : x_i \in \mathbb{R},\ c_i \in \mathbb{N}_0\}.$$

Example 1.2. [29] *Let $N_n = \{1|1, 2|2, \cdots, n|n\}$. Then $\{N_n\}$ is an multiset sequence and n^{th} terms has $\frac{n(n+1)}{2}$ elements.*

Example 1.3. [29] *The prime factorises n completely, and let F_n be the mset of these factors, including 1. Then, $F_1 = \{1\}$, $F_2 = \{1,2\}$, $F_3 = \{1,3\}$, $F_4 = \{1,2,2\}$ and $F_{36} = \{1,2,2,3,3\}$. In this case $\{F_n\}$ is an mset sequence.*

1.2. Statistical convergence of multiset sequences

Statistical convergence was formally introduced by Fast [10] and Steinhaus [34] , independently. After the 1950s, studies on the concept of statistical convergence made rapid progress and many studies were conducted on this subject [11] , [32], [36] . Since it was defined, many researchers have studies on this subject [26] , [28] .

Definition 1.4. [10] *A number sequence (x_i) is statistically convergent to L provided that for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} |i \le n : |x_i - L| \ge \varepsilon| = 0.$$

In this case we write $st - \lim x_i = L$ and usually the set of statistically convergent sequences is denoted by S.

Debnath and Debnath studied statistical convergence of multiset sequences on $\mathbb{R}$ [8] and definitions of statistical convergence of multiset sequences are basis of our study.

Definition 1.5. [8] *Let $\mathbb{N}_0$ is the set of non-negative integers. The set*

$$m\mathbb{R} = \{mx = x_i|c_i : x_i \in \mathbb{R} \text{ and } c_i \in \mathbb{N}_0\}$$

is called multiset of real numbers.

Definition 1.6. [8] *Let $x = (x_i)$ be a real sequence and $c = (c_i)$ be a sequence of $\mathbb{N}_0$. A multiset sequence $mx = (x_i|c_i)$ of $m\mathbb{R}$ is statistically convergent to $l|c$ of $m\mathbb{R}$ if given for any $\varepsilon > 0$,*

$$\lim_{n\to\infty}\frac{1}{n}\left|i \le n : \sqrt{(x_i - l)^2 + (c_i - c)^2} \ge \varepsilon\right| = 0.$$

Example 1.7. [8] *Consider a multisequence $mx = (x_i|c_i)$, given by*

$$x_i = \begin{cases} i, & i = n^2; n = 1, 2, 3, ... \\ 1, & \text{otherwise.} \end{cases} \quad \text{and } c_i = \begin{cases} i, & i = n^3; n = 1, 2, 3, ... \\ 5, & \text{otherwise.} \end{cases} .$$

Then for any $\varepsilon > 0$,

$$\begin{aligned} & \lim_{n\to\infty}\tfrac{1}{n}\left|i \le n : \sqrt{(x_i - 1)^2 + (c_i - 5)^2} \ge \varepsilon\right| \\ = \; & \text{Density of perfect squared or perfect cubic positive integers or both} \\ \le \; & \lim_{n\to\infty}\tfrac{1}{n}\left(n^{\frac{1}{2}} + n^{\frac{1}{3}} - n^{\frac{1}{6}}\right) = 0. \end{aligned}$$

Therefore, the multisequence mx statistically converges to $1|5$.

1.3. $\mathcal{I}$−convergence

$\mathcal{I}$−convergence has emerged as a generalized form of many types of convergences. This means that, if we choose different ideals we will have different convergences. Kostyrko et al. [18] introduced this concept in a metric space. Due to its generalization property, it has been and continues to be the center of many researches [1], [12], [13], [16], [27], [31]. We will explain this situation with two examples later. Before defining $\mathcal{I}$−convergence, the definitions of ideal and filter will be needed.

Definition 1.8. *A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an ideal if the following properties are provided:*

i) $\emptyset \in \mathcal{I}$
ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$
iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ implies $B \in \mathcal{I}$.
We say that $\mathcal{I}$ is non-trivial if $\mathbb{N} \notin \mathcal{I}$ and $\mathcal{I}$ is admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 1.9. *A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter if the following properties are provided:*

i) $\emptyset \notin \mathcal{F}$,
ii) If $A, B \in \mathcal{F}$ then we have $A \cap B \in \mathcal{F}$,
iii) For each $A \in \mathcal{F}$ and each $A \subseteq B$ we have $B \in \mathcal{F}$.

Proposition 1.10. *If $\mathcal{I}$ is an ideal in $\mathbb{N}$ then the collection,*

$$\mathcal{F}(\mathcal{I}) = \{A \subset \mathbb{N} : \mathbb{N}\backslash A \in \mathcal{I}\}$$

forms in a filter in $\mathbb{N}$ which is called the filter associated with $\mathcal{I}$.

Definition 1.11. [18] *A sequence of reals $x = (x_i)$ is $\mathcal{I}$−convergent to $L \in \mathbb{R}$ if and only if the set*

$$A_\varepsilon = \{i \in \mathbb{N} : |x_i - L| \ge \varepsilon\} \in \mathcal{I}$$

for each $\varepsilon > 0$. In this case, we say that L is the $\mathcal{I}$−limit of the sequence x.

Example 1.12. *Define the set of all finite subsets of $\mathbb{N}$ by $\mathcal{I}_f$. Then, $\mathcal{I}_f$ is an ideal and $\mathcal{I}_f$ −convergence coincides with the usual convergence.*

Example 1.13. *Define the set $\mathcal{I}_d$ by $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$ where $d(A) = \lim_n \frac{|A_n|}{n}$. Then, $\mathcal{I}_d$ is an ideal and $\mathcal{I}_d$−convergence gives the statistical convergence.*

2. Main Results

After specifying our purpose, let's start by giving the definition of ideal convergence for multiset sequences on $\mathbb{R}$. The aim of the study is to generalize the concepts of convergence and statistical convergence which were previously defined for multiset sequences with the help of ideals. As it is known, multisets are sets whose elements can repeat a finite number of times. Due to repetitive elements of the multisets, it is necessery to define a new metric in order to work on multisets. Let (X, d) be a metric space and M be a multiset in a metric space (X, d). The d metric is not very functional on M because of the repetitive elements of M. Hence, if a new d_M metric is defined on M, then (M, d_M) is a metric space. In this study, it is defined as

$$d_M(mx, my) = d_M(x_i|c_i, y_i|t_i) = \sqrt{(x_i - y_i)^2 + (c_i - t_i)^2} \quad \text{where} \quad d_M : M \times M \to \mathbb{R}$$

for each $i \in \mathbb{N}$. It is easily seen that d_M satisfies the metric conditions with Minkowsky inequality. With the help of all this information, the ideal convergence of multiset sequences on $\mathbb{R}$ is defined.

Definition 2.1. *A multiset sequence $mx = (x_i|c_i)$ of $m\mathbb{R}$ is convergent to $l|c$ if given the set*

$$\lim_{i\to\infty} d_M(x_i|c_i, l|c) = \lim_{i\to\infty} \sqrt{(x_i - l)^2 + (c_i - c)^2} = 0.$$

In this case, $x_i \to l$ and $c_i \to c$ i.e. for each $\varepsilon > 0$, $|x_i - l| < \varepsilon$ and $|c_i - c| < \varepsilon$.

Definition 2.2. *A multiset sequence $mx = (x_i|c_i)$ of $m\mathbb{R}$ is $\mathcal{I}-$convergent to $l|c$ if given the set*

$$\{i \in \mathbb{N} : d_M(x_i|c_i, l|c) \geq \varepsilon\} = \left\{i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\} \in \mathcal{I}$$

for each $\varepsilon > 0$. In this case, we say that $l|c$ is the $\mathcal{I}-$ limit of the sequence x and it is denoted by $\mathcal{I}-\lim mx = l|c$.

Theorem 2.3. *Every constant $(l|c)$ multiset sequence is $\mathcal{I}-$convergent to $l|c$.*

Proof. Let $mx = (l|c)$ be a constant multiset sequence. Then,

$$\left\{i \in \mathbb{N} : \sqrt{(l - l)^2 + (c - c)^2} \geq \varepsilon\right\} = \emptyset \in \mathcal{I}$$

and we have the proof. □

Theorem 2.4. *The convergent multiset sequences are also $\mathcal{I}-$convergent.*

Proof. Let $mx = (x_i|c_i)$ be a convergent multiset sequence which converges to $l|c$ of $m\mathbb{R}$. Then, from the definition, (x_i) converges to l and (c_i) converges to c. We know that, every convergent sequence is also $\mathcal{I}-$convergent to the same limit. Hence, for each $\varepsilon > 0$ we have

$$A = \{i \in \mathbb{N} : |x_i - l| \geq \varepsilon\} \in \mathcal{I}$$

and

$$B = \{i \in \mathbb{N} : |c_i - c| \geq \varepsilon\} \in \mathcal{I}.$$

On the other hand,

$$\begin{aligned} \sqrt{(x_i - l)^2 + (c_i - c)^2} &\leq \sqrt{(x_i - l)^2} + \sqrt{(c_i - c)^2} \\ &= |x_i - l| + |c_i - c|. \end{aligned}$$

For each $i \in A \cap B$ we can write,

$$\left\{ i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \quad \subseteq \quad \{ i \in \mathbb{N} : |x_i - l| \geq \varepsilon \} \cup \{ i \in \mathbb{N} : |c_i - c| \geq \varepsilon \} .$$

According to the first property of the ideal, the right side belongs to the ideal. Again, in accordance with the second property of the ideal,

$$\left\{ i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \in \mathcal{I}$$

and this completes the proof. □

Considering that $\mathcal{I}-$convergence is a type of convergence that generalizes usual convergence, it is clear that the inverse of this theorem may not always be true. We chose to leave this situation as an open problem.

Theorem 2.5. *The $\mathcal{I}-$limit of a multiset sequence is unique.*

Proof. Let a multiset sequence $mx = (x_i|c_i)$ has two limits such as $(l_1|c_1)$ and $(l_2|c_2)$. Then for each $\varepsilon > 0$,

$$\left\{ i \in \mathbb{N} : \sqrt{(x_i - l_1)^2 + (c_i - c_1)^2} \geq \tfrac{\varepsilon}{2} \right\} \in \mathcal{I}$$

and

$$\left\{ i \in \mathbb{N} : \sqrt{(x_i - l_2)^2 + (c_i - c_2)^2} \geq \frac{\varepsilon}{2} \right\} \in \mathcal{I}.$$

This means that,

$$A_1(\varepsilon) = \left\{ i \in \mathbb{N} : \sqrt{(x_i - l_1)^2 + (c_i - c_1)^2} < \frac{\varepsilon}{2} \right\} \in \mathcal{F}(\mathcal{I})$$

and

$$A_2(\varepsilon) = \left\{ i \in \mathbb{N} : \sqrt{(x_i - l_2)^2 + (c_i - c_2)^2} < \frac{\varepsilon}{2} \right\} \in \mathcal{F}(\mathcal{I})$$

where $F(\mathcal{I})$ is the filter produced by $\mathcal{I}$.

For each $i \in A_1(\varepsilon) \cap A_2(\varepsilon)$, from Minkowsky inequality we have,

$$\begin{aligned} \sqrt{(l_1 - l_2)^2 + (c_1 - c_2)^2} &= \sqrt{[(l_1 - x_i) + (x_i - l_2)]^2 + [(c_1 - c_i) + (c_i - c_2)]^2} \\ &\leq \sqrt{(x_i - l_1)^2 + (c_i - c_1)^2} + \sqrt{(x_2 - l_2)^2 + (c_i - c_2)^2} \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore, $\sqrt{(l_1 - l_2)^2 + (c_1 - c_2)^2} < \varepsilon$ for infinitely many $i \in \mathbb{N}$ which implies $(l_1 - l_2)^2 + (c_1 - c_2)^2 = 0$. So, $l_1 = l_2$ and $c_1 = c_2$ which completes the proof. □

Definition 2.6. *A multiset sequence $mx = (x_i|c_i)$ is said to be $\mathcal{I}-$bounded if there exists a non-negative real number K such that*

$$\left\{ i \in \mathbb{N} : \sqrt{x_i^2 + (c_i - 1)^2} > K \right\} \in \mathcal{I}.$$

Example 2.7. *Consider a multisequence $mx = (x_i|c_i)$, given by*

$$x_i = \begin{cases} i, & i = n^2; n = 1,2,3,... \\ 1, & otherwise. \end{cases} \quad and\ c_i = \begin{cases} i, & i = n^3; n = 1,2,3,... \\ 5, & otherwise. \end{cases} .$$

Then,

$$\sqrt{x_i^2 + (c_i-1)^2} \leq \sqrt{x_i^2} + \sqrt{(c_i-1)^2} = |x_i| + |c_i - 1|$$

$$\left\{ i \in \mathbb{N} : \sqrt{x_i^2} + \sqrt{(c_i-1)^2} > K \right\} \subseteq \{i \in \mathbb{N} : |x_i| > K\} \cup \{i \in \mathbb{N} : |c_i - 1| > K\}.$$

The right side belongs to the ideal so $mx = (x_i|c_i)$ is bounded.

Theorem 2.8. *An $\mathcal{I}-$convergent multiset sequence is $\mathcal{I}-$bounded.*

Proof. Let $mx = (x_i|c_i)$ is ideal convergent to $l|c$. Then for any $\varepsilon > 0$,

$$\left\{ i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \in \mathcal{I}$$

which is equivalent to

$$\begin{aligned} \left\{ i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \in \mathcal{I} &\implies \{i \in \mathbb{N} : |x_i - l| < \varepsilon,\ |c_i - c| < \varepsilon\} \in \mathcal{F}(\mathcal{I}) \\ &\implies \{i \in \mathbb{N} : l - \varepsilon < x_i < l + \varepsilon,\ c - \varepsilon < c_i < c + \varepsilon\} \in \mathcal{F}(\mathcal{I}) \\ &\implies \left\{ i \in \mathbb{N} : x_i < K',\ c_i < K' \right\} \in \mathcal{F}(\mathcal{I}), \\ &\qquad \text{where } K' = \max(l + \varepsilon, c + \varepsilon) \\ &\implies \left\{ i \in \mathbb{N} : x_i^2 + c_i^2 < 2K'^2 \right\} \in \mathcal{F}(\mathcal{I}) \\ &\implies \left\{ i \in \mathbb{N} : x_i^2 + (c_i - 1)^2 < 2K'^2 \right\} \in \mathcal{F}(\mathcal{I}) \\ &\implies \left\{ i \in \mathbb{N} : \sqrt{x_i^2 + (c_i - 1)^2} < K \right\} \in \mathcal{F}(\mathcal{I}) \\ &\qquad \text{where } K^2 = 2K'^2. \\ &\implies \left\{ i \in \mathbb{N} : \sqrt{x_i^2 + (c_i - 1)^2} \geq K \right\} \in \mathcal{I} \end{aligned}$$

Then, for any number N greater than K,

$$\left\{ i \in \mathbb{N} : \sqrt{x_i^2 + (c_i - 1)^2} > N \right\} \in \mathcal{I}.$$

Hence, mx is ideal bounded. □

Now let's give the definition of $\mathcal{I}^*-$convergence, which is an important definition, and examine its relationship with $\mathcal{I}-$convergence.

Definition 2.9. *A multisequence $mx = (x_i|c_i)$ is said to be $\mathcal{I}^*-$convergent to $l|c$ if and only if there exists a set $M = \{m_1 < m_2 < ... < m_i < ...\} \in \mathcal{F}(\mathcal{I})$ such that $(x_{m_i}|c_{m_i})$ submultiset sequence converges to $l|c$.*

Theorem 2.10. *If a multisequence mx is $\mathcal{I}^*-$convergent to $l|c$ then it is $\mathcal{I}-$convergent to $l|c$.*

Proof. Assume that mx is a $\mathcal{I}^*-$convergent sequence to $l|c$. Then, there exists a set $H \in \mathcal{I}$ such that $H = \mathbb{N} \setminus M$ and $\lim\limits_{i\to\infty} d_M(x_{m_i}|x_{m_i}, l|c) = 0$. Let $\varepsilon > 0$. From the convergence of the submultiset sequence, there exists $i_0 \in \mathbb{N}$ such that

$$d_M(x_{m_i}|c_{m_i}, l|c) = \sqrt{(x_i - l)^2 + (c_i - c)^2} < \varepsilon$$

for each $i > i_0$. When the H set and the definition of convergence are considered together we can write,

$$\left\{ i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \subseteq H \cup \left\{ m_1 < m_2 < ... < m_{i_0} \right\}.$$

Since the right side belongs to the ideal we have the proof. □

We will state in the next theorem that the inverse of Theorem 2.11 will be satisfied when the ideal $\mathcal{I}$ has the property (AP). If $\mathcal{I}$ has the property (AP), then $\mathcal{I}-$convergence and $\mathcal{I}^*-$convergence are equivalent. What would be the situation when the (AP) property is not provided we chose to leave it as an open problem. In this case, let's define the (AP) property first.

Definition 2.11. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ is an admissible ideal. $\mathcal{I}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j$ is finite set for $j \in \mathbb{N}$ and $B = \cup_{j=1}^{\infty} B_j \in \mathcal{I}$. In this definition, $A_j \Delta B_j = \left(A_j \setminus B_j\right) \cup \left(B_j \setminus A_j\right)$.*

Theorem 2.12. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ is an admissible ideal. If $\mathcal{I}$ has property (AP), for the multiset sequence $mx = (x_i|c_i)$, $\mathcal{I} - \lim_{i\to\infty} mx = l|c$ implies $\mathcal{I}^* - \lim_{i\to\infty} mx = l|c$.*

Proof. Assume that $\mathcal{I}$ is an ideal that property that has (AP) and $\mathcal{I} - \lim_{i\to\infty} mx = l|c$. Then, for $\varepsilon > 0$

$$A_\varepsilon = \left\{ i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon \right\} \in \mathcal{I}.$$

Now, let's create mutually disjoint and countable sets with the help of the elements of the multiset sequence. Put

$$A_1 = \left\{ i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq 1 \right\}$$

$$A_2 = \left\{ i \in \mathbb{N} : \tfrac{1}{2} \leq \sqrt{(x_i - l)^2 + (c_i - c)^2} < 1 \right\}$$

$$\vdots$$

$$A_i = \left\{ i \in \mathbb{N} : \tfrac{1}{i} \leq \sqrt{(x_i - l)^2 + (c_i - c)^2} < \tfrac{1}{i-1} \right\}$$

for $i \in \mathbb{N}$. By property (AP), there exists a sequences of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j$ is finite set for $j \in \mathbb{N}$ and $B = \cup_{j=1}^{\infty} B_i \in \mathcal{I}$. Here, since both A_j, B_j are infinite sets and $A_j \Delta B_j$ is a finite set, this means that there are infinite elements at the $A_j \cap B_j$. We can also say that, for a given $j > i_0$, the elements of A_j and B_j are same.

We know that $B = \cup_{j=1}^{\infty} B_j \in \mathcal{I}$. Choose $M = \{m_1 < m_2 < ...\} = \mathbb{N} \backslash B \in \mathcal{F}(\mathcal{I})$ and let's see that $\lim_{i\to\infty} d_M(x_{m_i}|c_{m_i}, l|c) = 0$. Let $\eta > 0$ and for $k \in \mathbb{N}$, $\frac{1}{k+1} < \eta$. It is obvious,

$$\left\{ i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \eta \right\} \subset \cup_{j=1}^{k+1} A_j.$$

Since $A_j \Delta B_j$ finite for $j = 1, 2, ..., k+1$ there exists $i_0 \in \mathbb{N}$ such that

$$\left(\cup_{j=1}^{k+1} B_j \right) \cap \{ i \in \mathbb{N} : i > i_0 \} = \left(\cup_{j=1}^{k+1} A_j \right) \cap \{ i \in \mathbb{N} : i > i_0 \}. \tag{1}$$

If $i > i_0$ and $i \notin B$ then, $i \notin \left(\cup_{j=1}^{k+1} B_j \right)$ and by (2.1), $i \notin \left(\cup_{j=1}^{k+1} A_j \right)$. But then,

$$d_M(x_{m_i}|c_{m_i}, l|c) = \sqrt{(x_{m_i} - l)^2 + (c_{m_i} - c)^2} < \frac{1}{k+1} < \eta$$

for $\varepsilon > 0$ and $i > i_0$, we have the proof. □

3. Conclusions

Multisets have a very important place in many areas of science and even in our daily lives. Therefore, the properties of multiset sequences consisting of these sets are very interesting. On the other hand, how the concept of $\mathcal{I}-$convergence, which generalizes many convergence types, can be defined for multiset sequences is the main purpose of this study. We think that, this study will hold an important place for other studies in this subject.

4. Acknowledgements

The authors are grateful to the referees and the editor for their corrections and suggestions, which have greatly improved the readability of the paper.

References

[1] N. Akın, E. Dündar, F. Nuray, *Wijsman $\mathcal{I}-$invariant convergenceof sequences of sets*, Bulletin of Mathematical Analysis and Applications **11(1)** (2019), 1–9.
[2] E. A. Bender, *Partitions of Multisets*, Discrete Mathematics (1974), 301–311.
[3] W. D. Blizard, *The Development of Multiset Theory*, Modern Logic 1 (1991), 319–352.
[4] W. D. Blizard, *Real-Valued Multisets and Fuzzy Sets*, Fuzzy Sets and Systems**33)** (1989), 77–97.
[5] W. D. Blizard, *Multiset Theory*, Notre Dame Journal of Formal Logic **30** (1989), 36–66.
[6] J. S. Connor, *The statistical and strong $p-$Cesaro convergence of sequences*, Analysis**10** (1990), 373–385.
[7] P. Das, E. Savaş, S. Ghosal, *On generalized of certain summability methods using ideals*, Appl. Math. Letters **36** (2011), 1509–1514.
[8] S. Debnath, A. Debnath, *Statistical Convergence of Multisequences on R*, Applied Sciences **23** (2021), 17–28.
[9] N. Demir, H. Gümüş, *A study on $\mathcal{I}-$lacunary statistical convergence of multiset sequences*, Sigma Jour. of Engineering and Natural Sciences, in press.
[10] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.
[11] J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.
[12] H. Gümüş, *A new approach to the concept of $A^{\mathcal{I}}-$statistical convergence with the number of α*, Commun. Fac. Sci.Univ. Ank. Series A1 **67(1))** (2018), 37–45.
[13] J. Connor, H. Gümüş, *Summability matrices that preserve asymptotic equivalence for ideal convergence*, Sarajevo Journal of Mathematics **12(1)** (2016), 107–124.
[14] J. L. Hickman, *A note on the concept of multiset*, Bulletin of the Australian Mathematical Society **22** (1980), 211–217.
[15] A. M. Ibrahim, P. A. Ejegwa, *Multigroup actions on multisets*, Annals Fuzzy Math. Inform. **14(5)** (2017), 515–526.
[16] Ö. Kişi, *On $\mathcal{I}-$lacunary arithmetic statistical convergence*, Journal of applied mathematics &informatics **40** (2022), 327–339.
[17] D. Knuth, *The Art of Computer Programming*, (Second edition), Addison-Wesley, Reading, Massachusetts, 1981.
[18] P. Kostyrko, T. Šalát, W. Wileynski, *$\mathcal{I}-$Convergence*, Real Analysis Exchange **26(2)** (2000), 669–680.
[19] J. Lake, *Sets, fuzzy sets, multisets and functions*, Journal of the London Mathematical Society **2(12)** (1976), 323–326.
[20] P. Majumdar, *Soft multisets*, J. Math. Comput. Sci. **2(6)** (2012), 1700–1711.

[21] Z. Manna, R. Waldinger, *The Logical Basis of Computer Programming*, Addison-Wesley, Reading, Massachusetts, 1985.
[22] R. K. Meyer, M. A. McRobbie, *Multisets and relevant implication I and II*, Australasian Journal of Philosophy **60(2)** (1982), 107–139.
[23] H. I. Miller, *A measure theoretical subsequence characterization of statistical convergence*, Trans. Amer. Math. Soc. **347** (1995), 1811–1819.
[24] G. P. Monro, *The concept of multiset*, Zeitschrift fur mathematische Logik und Grunlagen der Mathematik **33(8)** (1987), 171–178.
[25] S. K. Nazmul, P. Majumdar, S. K. Samanta, *On multisets and multigroups*, Annals of Fuzzy Math. Inform. **6(3)** (2013), 643–656.
[26] F. Nuray, R. Rhodes, *Statistical convergence of sequences of sets*, Fasciculi Mathematici **49** (2012), 87–99.
[27] F. Nuray, H. Gök (Gümüş), U. Ulusu, $\mathcal{I}_\sigma-$*convergence*, Mathematical communications **16** (2011), 531–538.
[28] F. Nuray, *Statistical convergence in partial metric spaces*, Korean Journal of Math. **30(1)** (2022), 155–160.
[29] S. Pachilangode, S. C. John, *Convergence of Multiset Sequences*, Journal of New Theory **34** (2021), 20–27.
[30] E. Savaş,M. Gürdal, $\mathcal{I}-$*statistical convergence in probabilistic normed spaces*, U.P.B. Sci. Bull., Series A **77(4)** (2015), 195–204.
[31] E. Savaş, H. Gümüş, *A generalization on* $\mathcal{I}-$*asymptotically lacunary statistical equivalent sequences*, Journal of Inequalities and Applications **270** (2013), 2-9.
[32] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (2021), 361–375.
[33] D. Singh, A. M. Ibrahim, T. Yohanna, J. N. Singh, *TAn Overview of The Application of Multiset*, Novi Sad Journal of Mathematics **37** (2007), 73–92.
[34] H. Steinhaus, *Sur la convergence ordiniaire et la convergence asymptotique*, Colloquium Mathematicum **2** (1951), 73–84.
[35] A. Syropoulos, *Mathematics of Multisets*, Lecture Notes in Computer Science**2235** (2001), 347–358.
[36] B. C. Tripathy, *On statistically convergent and statistically bounded sequences*, Bull. Calcutta Math. Soc. **20** (1997), 31–33.
[37] H. Weyl, *Philosophy of Mathematics and Natural Science*, Revised and Augmen ted English Edition, Atheneum, New York, 1963.