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Deformed intermadiate and complete lifts of $1-$forms to the bundle of $2-$jets

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Abstract. Using an algebraic approach to the lift problems, we introduce deformed lifts of $1-$forms to the bundle of $2-$jets and investigate some properties of these lifts.

1. Introduction

1.1. Problems of lifts in the tangent bundles of $2-$jets has been studied by Yano and Ishihara [1],[2] (see also [3],[4]). The purpose of this paper is to study the deformed lift of $1-$forms which is a generalization already known lifts and appear in the context of algebraic approach to problems of lifts.

Let $\Pi = \left\{ \left(\underset{\alpha}{J_j^i} \right) \right\}, \alpha = 1, ..., m; i, j = 1, ..., n$ be a $\Pi-$structure on a smooth manifold M_n [8]. If there exists a frame $\{\partial_i\}, i = 1, ..., n$ such that $\partial_i \underset{\alpha}{J_j^k} = 0$, then the $\Pi-$structure is said to be integrable. Let $\mathfrak{A}_m$ be an associative, commutative and Frobenius algebra with the unit element $e_1 = 1$. An algebraic structure on M_n is an integrable $\Pi-$structure such that $\underset{\alpha}{J_j^m} \underset{\beta}{J_m^i} = C_{\alpha\beta}^{\gamma} \underset{\gamma}{J_j^i}$, i.e. if there exists an isomorphism $\mathfrak{A}_m \leftrightarrow \Pi$, where $C_{\alpha\beta}^{\gamma}$ are structure constants of $\mathfrak{A}_m$. An algebraic structure is said to be an $r-$regular $\Pi-$structure if the matrices $\left(\underset{\alpha}{J_j^i} \right)$ of order $n \times n$, $\alpha = 1, ..., m$ simultaneously reduce to the form

$$\left(\underset{\alpha}{J_j^i} \right) = \begin{pmatrix} C_\alpha & 0 & \cdots & 0 \\ 0 & C_\alpha & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & C_\alpha \end{pmatrix}, \quad \alpha = 1, ..., m; \quad i, j = 1, ..., n \tag{1}$$

with respect to the adapted frame $\{\partial_i\}$, where $C_\alpha = \left(C_{\alpha\beta}^{\gamma} \right)$ is the regular representation of $\mathfrak{A}_m$ and r is a number of $C_\alpha-$blocks. We note that the $r-$regular $\Pi-$structure is integrable if a structure-preserving connection with free-torsion exists on M_n [5].

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From (1) we easily see that $n = rm$ and the structure tensors $\underset{\sigma}{J}$ have the components $\underset{\sigma}{J}{}^i_j = \underset{\sigma}{J}{}^{u\alpha}_{v\beta} = \delta^u_v C^\alpha_{\sigma\beta}, u, v = 1, ..., r$, where δ^u_v is the Kronecker delta and $u\alpha = (u-1)m + \alpha, v\beta = (v-1)m + \beta$.

An $\mathfrak{A}-$holomorphic manifold [6] $X_r(\mathfrak{A})$ over algebra $\mathfrak{A}_m$ of dimension r is a Hausdorff space with a fixed complete atlas compatible with a group of $\mathfrak{A}-$holomorphic transformations of space $\mathfrak{A}^r_m$, where $\mathfrak{A}^r_m = \mathfrak{A}_m \times \cdots \times \mathfrak{A}_m$ is the space of $r-$tuples of algebraic numbers $\left(z^1, z^2, \ldots, z^r\right)$ with $z^u = x^{u\alpha} e_\alpha \in \mathfrak{A}_m$, $x^{u\alpha} = x^i \in R, i = 1, \ldots, n; u = 1, \ldots, r; \alpha = 1, ..., m$.

Let now $\Pi = \left\{\underset{\sigma}{J}\right\}$ be an integrable $r-$regular structure on M_{rm}. The transformation $z^{u'} = z^{u'}(z^u)$ of local coordinates on $X_r(\mathfrak{A})$ is $\mathfrak{A}-$holomorphic if and only if the transformation $x^{i'} = x^{i'}(x^i)$ of local coordinates on M_{rm} is a structure-preserving transformation (an admissible transformation), i.e. [6]

$$\underset{\alpha}{J}A = A\underset{\alpha}{J},\ A = \left(\frac{\partial x^j}{\partial x^{j'}}\right),\ \underset{\alpha}{J} = \left(\underset{\alpha}{J}{}^i_j\right).$$

Thus the real smooth manifold M_{rm} with an integrable $r-$regular $\Pi-$structure and with a structure-preserving transformations of local coordinates is a real modeling of an $\mathfrak{A}-$holomorphic manifold $X_r(\mathfrak{A})$ over algebra $\mathfrak{A}_m$.

Let now $\Pi = \left\{\underset{\sigma}{J}\right\}$ be the integrable regular $\Pi-$structure on manifold M_{rm} and let $\omega = \omega_i(x^1, ..., x^{rm})dx^i = \omega_{u\alpha}(x^1, ..., x^{rm})dx^{u\alpha}$ be an 1−form on M_{rm}. An $\mathfrak{A}$ - algebraic 1−form $\overset{*}{\omega} = (\overset{*}{\omega}_u) = (\overset{*}{\omega}_{u\alpha} e^\alpha)$, $u = 1, ..., r$, $e^\alpha = \varphi^{\alpha\beta} e_\beta$ (where $\varphi^{\alpha\beta}$ are contravariant coordinates of Frobenius metric) on $\mathfrak{A}-$holomorphic manifold $X_r(\mathfrak{A})$ corresponding to an 1−form $\omega = (\omega_i) = (\omega_{u\alpha}), i = 1, ..., rm$ on M_{rm} is not $\mathfrak{A}-$holomorphic, in general. To investigate a holomorphic algebraic 1−form $\overset{*}{\omega}$, we consider the Tachibana $\Phi_{\underset{\sigma}{J}}-$operators M_{rm} associated with the Π −structure and applied to ω [7]:

$$(\Phi_{\underset{\alpha}{J}}\omega)(X, Y) = (L_{\underset{\sigma}{J}X}\omega - L_X(\omega \circ \underset{\sigma}{J}))(Y),$$

where $\Phi_{\underset{\alpha}{J}}\omega$ is a tensor field of type (0, 2), L_X is the Lie derivations with respect to X. In terms of the coordinate systems, we have

$$(\Phi_{\underset{\alpha}{J}}\omega)_{ji} = \underset{\sigma}{J}{}^h_j \partial_h \omega_i - \underset{\sigma}{J}{}^m_i \partial_j \omega_m - \omega_m(\partial_j \underset{\sigma}{J}{}^m_i - \partial_i \underset{\sigma}{J}{}^m_j).$$

Theorem 1.1. ([8]) *An algebraic* 1*−form* $\overset{*}{\omega}$ *on* $\mathfrak{A}$*−holomorphic manifold* $X_r(\mathfrak{A})$ *corresponding to an* 1*−form* ω *on* M_{rm} *is an* $\mathfrak{A}$*−holomorphic tensor field if and only if*

$$\underset{\sigma}{J}{}^h_j \partial_h \omega_i - \underset{\sigma}{J}{}^m_i \partial_j \omega_m - \omega_m(\partial_j \underset{\sigma}{J}{}^m_i - \partial_i \underset{\sigma}{J}{}^m_j) = 0,\ \sigma = 1, ..., m.$$

1.2. Let $R(\varepsilon^2)$ be an algebra of order 3 with a canonical basis $\{e_1, e_2, e_3\} = \{1, \varepsilon, \varepsilon^2\}$, $\varepsilon^3 = 0$. From $e_\alpha e_\beta = C^\gamma_{\alpha\beta} e_\gamma$ follows that the (3×3) −matrices $C_\sigma = \left(C^\gamma_{\sigma\beta}\right), \sigma = 1, 2, 3$ of regular representation of $R(\varepsilon^2)$ have the following forms

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $z = x^1 + \varepsilon x^2 + \varepsilon^2 x^3$. Then the generalized Cauchy-Riemann conditions [8]

$$C^\alpha_{\sigma\beta} \frac{\partial y^\beta}{\partial x^\gamma} = \frac{\partial y^\alpha}{\partial x^\beta} C^\beta_{\sigma\gamma}$$

for $R(\varepsilon^2)$-holomorphicity of function

$$w = w(z) = y^1(x^1, x^2, x^3) + \varepsilon y^2(x^1, x^2, x^3) + \varepsilon^2 y^3(x^1, x^2, x^3),$$

reduces to the following equations:

$$\begin{aligned}
\text{(i)}\ \frac{\partial y^1}{\partial x^2} &= \frac{\partial y^1}{\partial x^3} = \frac{\partial y^2}{\partial x^3} = 0,\\
\text{(ii)}\ \frac{\partial y^2}{\partial x^2} &= \frac{\partial y^1}{\partial x^1} = \frac{\partial y^3}{\partial x^3},\\
\text{(iii)}\ \frac{\partial y^3}{\partial x^2} &= \frac{\partial y^2}{\partial x^1}.
\end{aligned}$$

From (i), (ii), (iii) we have

$$\begin{aligned}
y^1 &= y^1(x^1),\\
y^2 &= y^2(x^1, x^2),\\
y^2(x^1, x^2) &= x^2 \frac{dy^1}{dx^1} + G(x^1),\\
y^3(x^1, x^2, x^3) &= x^3 \frac{dy^1}{dx^1} + \frac{1}{2}(x^2)^2 \frac{d^2 y^1}{(dx^1)^2} + x^2 \frac{dG}{dx^1} + H(x^1),
\end{aligned}$$

where $G = G(x^1)$ and $H = H(x^1)$ are arbitrary functions. Thus the $R(\varepsilon^2)$ -holomorphic function $w = w(z)$ has the following expression

$$w(z) = y^1(x^1) + \varepsilon(x^2 \frac{dy^1}{dx^1} + G(x^1)) + \varepsilon^2(x^3 \frac{dy^1}{dx^1} + \frac{1}{2}(x^2)^2 \frac{d^2 y^1}{(dx^1)^2} + x^2 \frac{dG}{dx^1} + H(x^1)).$$

Similarly, if $w(z^1, ..., z^n) = y^1(x^1, ..., x^n) + \varepsilon y^2(x^1, ..., x^n) + \varepsilon^2 y^3(x^1, ..., x^n)$, where $z^i = x^i + \varepsilon x^{n+i} + \varepsilon^2 x^{2n+i}$, $i = 1, ..., n$, is a multi-variable $R(\varepsilon^2)$ -holomorphic function, then the function $w = w(z^1, ..., z^n)$ has the following specific form:

$$\begin{aligned}
w\left(z^1, ..., z^n\right) &= y^1\left(x^1, ..., x^n\right) + \varepsilon\left(x^{n+i}\partial_i y^1 + G\left(x^1, ..., x^n\right)\right) \\
&\quad + \varepsilon^2\left(x^{2n+i}\frac{\partial y^1}{\partial x^i} + \frac{1}{2}x^{n+i}x^{n+j}\frac{\partial^2 y^1}{\partial x^i \partial x^j} + x^{n+i}\frac{\partial G}{\partial x^i} + H\left(x^1, ..., x^n\right)\right).
\end{aligned} \tag{2}$$

From here if $G(x^1, ..., x^n) = H(x^1, ..., x^n) = 0$ and $y^1(x^1, ..., x^n) = f(x^1, ..., x^n)$, then the function

$$w(z^1, ..., z^n) = f(x^1, ..., x^n) + \varepsilon x^{n+i}\partial_i f + \varepsilon^2(x^{2n+i}\frac{\partial f}{\partial x^i} + \frac{1}{2}x^{n+i}x^{n+j}\frac{\partial^2 f}{\partial x^i \partial x^j}) \tag{3}$$

is said to be natural extension of the real C^∞- functions $f = f(x^1, ..., x^n)$ to $\mathbb{R}(\varepsilon^2)$.

1.3. Let now $T^2(M_r)$ be the bundle of 2-jets, i.e. the tangent bundle of order 2 over C^∞−manifold M_r, $dim T^2\,(M_r) = 3r$ and let

$$(x^i, x^{\bar{i}}, x^{\bar{\bar{i}}}) = (x^i, x^{r+i}, x^{2r+i}), x^i = x^i(t), x^{\bar{i}} = \frac{dx^i}{dt}, x^{\bar{\bar{i}}} = \frac{1}{2}\frac{d^2 x^i}{dt^2}, t \in \mathbb{R}, i = 1, ..., r$$

be an induced local coordinates in $T^2(M_r)$. It is clear that there exists an affinor field (a tensor field of type $(1, 1)$) γ in $T^2(M_r)$ which has components of the form

$$\gamma = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \tag{4}$$

with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}, \partial_{\bar{\bar{i}}}\} = \{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}, \frac{\partial}{\partial x^{\bar{\bar{i}}}}\}, i = 1, ..., r$, where I denotes the $r \times r$ identity matrix. From here, we have

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{pmatrix}, \gamma^3 = 0, \tag{5}$$

i.e. $T^2(V_r)$ has a natural integrable structure $\Pi = \left\{I, \gamma, \gamma^2\right\}, I = id_{T^2(M_r)}$, which is an isomorphic representation of the algebra $R\left(\varepsilon^2\right)$, $\varepsilon^3 = 0$. Using $\gamma\partial_i = \partial_{\bar{i}}, \gamma^2\partial_i = \gamma\partial_{\bar{i}} = \partial_{\bar{\bar{i}}}$, we have $\{\partial_i, \partial_{\bar{i}}, \partial_{\bar{\bar{i}}}\} = \{\partial_i, \gamma\partial_i, \gamma^2\partial_i\}$. Also, using a frame

$$\{\partial_1, \gamma\partial_1, \gamma^2\partial_1, \partial_2, \gamma\partial_2, \gamma^2\partial_2, ..., \partial_r, \gamma\partial_r, \gamma^2\partial_r\} = \{\partial_1, \partial_{\bar{1}}, \partial_{\bar{\bar{1}}}, \partial_2, \partial_{\bar{2}}, \partial_{\bar{\bar{2}}}, ..., \partial_r, \partial_{\bar{r}}, \partial_{\bar{\bar{r}}}\}$$

which is obtained from $\{\partial_i, \partial_{\bar{i}}, \partial_{\bar{\bar{i}}}\} = \{\partial_i, \gamma\partial_i, \gamma^2\partial_i\}$ by changing of numbers of frame elements, we see that structure affinors I, γ and γ^2 have the following components

$$I = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & C_1 \end{pmatrix}, \ \gamma = \begin{pmatrix} C_2 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & C_2 \end{pmatrix}, \ \gamma^2 = \begin{pmatrix} C_3 & 0 & \cdots & 0 \\ 0 & C_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & C_3 \end{pmatrix}$$

with respect to the frame $\{\partial_1, \partial_{\bar{1}}, \partial_{\bar{\bar{1}}}, \partial_2, \partial_{\bar{2}}, \partial_{\bar{\bar{2}}}, ..., \partial_r, \partial_{\bar{r}}, \partial_{\bar{\bar{r}}}\}$, where the block matrices $C_\sigma, \sigma = 1, 2, 3$ of order 3 are the regular representation of algebra $\mathbb{R}(\varepsilon^2)$. Thus the bundle $T^2(M_r)$ has a natural integrable structure $\Pi = \left\{I, \gamma, \gamma^2\right\}$, which is an $r-$regular representation of $R(\varepsilon^2)$.

On the other hand, the transformation of induced coordinates $(x^i, x^{\bar{i}}, x^{\bar{\bar{i}}})$ in $T^2(M_r)$ is given by

$$\begin{aligned} x^{i'} &= x^{i'}\left(x^i\right), \\ x^{\bar{i}'} &= \frac{dx^{i'}}{dt} = \frac{\partial x^{i'}}{\partial x^i}\frac{dx^i}{dt} = \frac{\partial x^{i'}}{\partial x^i}x^{\bar{i}}, \\ x^{\bar{\bar{i}}'} &= \frac{1}{2}\frac{d^2x^{i'}}{dt^2} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial x^{i'}}{\partial x^i}\frac{dx^i}{dt}\right) \\ &= \frac{1}{2}\frac{\partial x^{i'}}{\partial x^i}\frac{d^2x^i}{dt^2} + \frac{1}{2}\frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}\frac{dx^i}{dt}\frac{dx^j}{dt} \\ &= \frac{\partial x^{i'}}{\partial x^i}x^{\bar{\bar{i}}} + \frac{1}{2}\frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}x^{\bar{i}}x^{\bar{j}} \end{aligned}$$

and its Jacobian matrix by

$$A = \begin{pmatrix} \frac{\partial x^{i'}}{\partial x^i} & \frac{\partial x^{i'}}{\partial x^{\bar{i}}} & \frac{\partial x^{i'}}{\partial x^{\bar{\bar{i}}}} \\ \frac{\partial x^{\bar{i}'}}{\partial x^i} & \frac{\partial x^{\bar{i}'}}{\partial x^{\bar{i}}} & \frac{\partial x^{\bar{i}'}}{\partial x^{\bar{\bar{i}}}} \\ \frac{\partial x^{\bar{\bar{i}}'}}{\partial x^i} & \frac{\partial x^{\bar{\bar{i}}'}}{\partial x^{\bar{i}}} & \frac{\partial x^{\bar{\bar{i}}'}}{\partial x^{\bar{\bar{i}}}} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^{i'}}{\partial x^i} & 0 & 0 \\ \frac{\partial^2 x^{i'}}{\partial x^i \partial x^s}x^{\bar{s}} & \frac{\partial x^{i'}}{\partial x^i} & 0 \\ \frac{\partial^2 x^{i'}}{\partial x^i \partial x^s}x^{\bar{\bar{s}}} + \frac{\partial^3 x^{i'}}{\partial x^i \partial x^s \partial x^t}x^{\bar{s}}x^{\bar{t}} & \frac{\partial^2 x^{i'}}{\partial x^i \partial x^s}x^{\bar{s}} & \frac{\partial x^{i'}}{\partial x^i} \end{pmatrix}. \tag{6}$$

From (4), (5) and (6) follows that $A^{-1}\gamma A = \gamma, A^{-1}\gamma^2 A = \gamma^2$, i.e. the transformation of local coordinates $(x^i, x^{\bar{i}}, x^{\bar{\bar{i}}})$ in $T^2(M_r)$ is a structure-preserving transformation. Then the transition functions

$$z^{i'}(z^i) = x^{i'} + \varepsilon x^{\bar{i}'} + \varepsilon^2 x^{\bar{\bar{i}}'} = x^{i'}(x^i) + \varepsilon\frac{\partial x^{i'}}{\partial x^i}x^{\bar{i}} + \varepsilon^2\left(\frac{\partial x^{i'}}{\partial x^i}x^{\bar{\bar{i}}} + \frac{1}{2}\frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}x^{\bar{i}}x^{\bar{j}}\right)$$

of charts on $X_r(R(\varepsilon^2))$ are $R(\varepsilon^2)$ -holomorphic functions by virtue of (3), i.e. we have the bundle $T^2(M_r)$ is a real modeling of $R(\varepsilon^2)$ -holomorphic manifold $X_r(R(\varepsilon^2))$.

1.4. Since the bundle $T^2(M_r)$ is a real modeling of $X_r(R(\varepsilon^2))$ and any holomorphic function

$$w(z^1,...,z^r) = f^1(x^1,...,x^r) + \varepsilon f^2(x^1,...,x^r) + \varepsilon^2 f^3(x^1,...,x^r),$$

on $X_r(R(\varepsilon^2))$, where $z^i = x^i + \varepsilon x^{r+i} + \varepsilon^2 x^{2r+i}, i = 1,...,r,$ is expressed by (see (2))

$$\begin{aligned} w\left(z^1,...,z^r\right) &= f\left(x^1,...,x^r\right) + \varepsilon\left(x^{r+i}\partial_i f + g\left(x^1,...,x^r\right)\right) \\ &\quad +\varepsilon^2\left(x^{2r+i}\frac{\partial f}{\partial x^i} + \frac{1}{2}x^{r+i}x^{r+j}\frac{\partial^2 f}{\partial x^i \partial x^j} + x^{r+i}\frac{\partial g}{\partial x^i} + h\left(x^1,...,x^r\right)\right), \\ f &= f^1, \end{aligned}$$

in the bundle of 2−jets we introduce the following three functions:

$$\begin{aligned} &{}^V f = f(x^1,...,x^r), \\ &{}^I f = x^{r+i}\partial_i f + g(x^1,...,x^r), \\ &{}^C f = x^{2r+i}\frac{\partial f}{\partial x^i} + \frac{1}{2}x^{r+i}x^{r+j}\frac{\partial^2 f}{\partial x^i \partial x^j} + x^{r+i}\frac{\partial g}{\partial x^i} + h(x^1,...,x^r), \end{aligned} \tag{7}$$

where f, g and h are any functions on M_r. These functions ${}^V f, {}^I f$ and ${}^C f$ are called respectively the *vertical, intermediate and complete* lifts of f in M_r to $T^2(M_r)$ [1]. If $g = h = 0$, then we have the 0−*th* f^0, 1−*th* f^1 *and* 2−*th* f^2 lifts of f [7], [8], i.e. the lifts ${}^I f$ and ${}^C f$ of f to $T^2(M_r)$ are respectively the *deformed lifts* of 1−th and 2−th lifts of f.

Thus we have

$${}^V f = f^0, {}^I f = f^1 + g^0, {}^C f = f^2 + g^1 + h^0. \tag{8}$$

2. Deformed complete lifts of 1−forms

Let $\widetilde{\omega} = \widetilde{\omega}_I dx^I = \widetilde{\omega}_i dx^i + \widetilde{\omega}_{r+i} dx^{r+i} + \widetilde{\omega}_{2r+i} dx^{2r+i}$ be an 1−form in $T^2(M_r)$, and $\Pi = \left\{I, \gamma, \gamma^2\right\}$, $I = id_{T^2(M_r)}$ be a Π −structure naturally existing in $T^2(M_r)$. We would like to find local expression of $\widetilde{\omega} = (\widetilde{\omega}_I)$ in $T^2(M_r)$ which is corresponding to the $R(\varepsilon^2)$−holomorphic 1−form $\overset{*}{\omega} = (\overset{*}{\omega}_u) = (\overset{*}{\omega}_{u\alpha} e^\alpha), e^\alpha = \varphi^{\alpha\beta} e_\beta, u = 1,...,r; \alpha, \beta = 1,2,3$ in $X_r(R(\varepsilon^2))$.

Using Theorem 1.1, we obtain

$$(\Phi_\gamma \widetilde{\omega})_{JI} = \gamma_J^H \partial_H \widetilde{\omega}_I - \gamma_I^H \partial_J \widetilde{\omega}_H = 0, (\Phi_{\gamma^2} \widetilde{\omega})_{JI} = (\gamma^2)_J^H \partial_H \widetilde{\omega}_I - (\gamma^2)_I^H \partial_J \widetilde{\omega}_H = 0.$$

From here, after straightforward calculations (see Section 1), we find the following covector field

$$\widetilde{\omega} = (\widetilde{\omega}_I) = (x^{2r+h}\partial_h \omega_i + \frac{1}{2}x^{r+h}x^{r+m}\partial^2_{hm}\omega_i + x^{h+i}\partial_h G_i + H_i, x^{r+h}\partial_h \omega_i + G_i, \omega_i), \tag{9}$$

where $G = \left(G_i\left(x^1,...,x^r\right)\right)$, $H = (H_i(x^1,...,x^r))$ any covector fields in M_r. In fact, by means of (13), we easily see that $\widetilde{\omega} = (\widetilde{\omega}_I)$ determine 1−form in $T^2(M_r)$ which are called the *deformed complete lifts* of ω from M_r to $T^2(M_r)$ and denoted by ${}^C\omega = ({}^C\omega_I)$.

From (9), we have

$${}^C\omega = \omega^2 + G^1 + H^0, \tag{10}$$

where

$$\begin{aligned} H^0 &= (H_i, 0, 0), G^1 = (x^{r+h}\partial_h G_i, G_i, 0), \\ \omega^2 &= (x^{2r+h}\partial_h \omega_i + \frac{1}{2}x^{r+h}x^{r+m}\partial^2_{hm}\omega_i, x^{r+h}\partial_h \omega_i, \omega_i) \end{aligned}$$

are respectively the 0−*th (vertical)* , 1−*th and* 2−*th (complete) lifts* of H, G and ω [7]. It is clear that the deformed complete lift ${}^C\omega = ({}^C\omega_I)$ is deformation of 2−th lift of ω.

Thus we have

Theorem 2.1. *Let $\omega = \omega_i dx^i$ be an $1-$form on M_r. The deformed complete lift $^C\omega$ of ω to the bundle of $2-$jets $T^2(M_r)$ have the following expression*

$$^C\omega = \omega^2 + G^1 + H^0,$$

where H^0, G^1 and ω^2 are respectively the $0-th$, $1-th$ and $2-th$ lifts of any $1-$forms H, G and ω.

3. Deformed intermediate lifts of 1−forms

Putting $\omega = G$ in (10), we see that

$$\omega^1 + H^0 = {}^C\omega - \omega^2 = (x^{r+i}\partial_i\omega_h + H_h, \omega_h, 0) \tag{11}$$

determine a new 1−form in $T^2(M_r)$, which are called the *deformed intermediate lift* of 1−form ω from M_r to $T^2(M_r)$ and denoted by $^I\omega = \omega^1 + H^0$. We note that the deformed intermediate lift $^I\omega$ of ω to $T^2(M_r)$ is deformation of 1−th lift of ω.

Thus we have

$$^V\omega = \omega^0, {}^I\omega = \omega^1 + H^0, {}^C\omega = \omega^2 + G^1 + H^0, \tag{12}$$

where

$$^V\omega = (\omega_h, 0, 0). \tag{13}$$

Thus we have

Theorem 3.1. *Let $\omega = \omega_i dx^i$ be an $1-$form on M_r. The deformed intermediate lift $^I\omega$ of ω to the bundle of $2-$jets $T^2(M_r)$ have the following expression*

$$^I\omega = \omega^1 + H^0,$$

where H^0 is the $0-th$ lift of $1-$form H.

From (9), (11) and (13) we have

Theorem 3.2. *Deformed complete lifts satisfies the following matrix formulas*

$$^C\omega\gamma = \omega^1 + G^0, {}^C\omega\gamma^2 = \omega^0 = {}^V\omega,$$

where γ and γ^2 are matrices in the form (4) and (5), respectively.

Let now $\omega = dx^i$, $G = dx^j$, $H = dx^k$, $i, j, k = 1, ..., r$. Then from (11), (12) and (13) we have

Theorem 3.3. *Deformed complete, intermediate and vertical lifts of differentials dx^i has the following linear combination of differentials in $T^2(M_r)$:*

$$^V\left(dx^i\right) = dx^i,\ {}^I(dx^i) = dx^{r+i} + dx^k, {}^C(dx^i) = dx^{2r+i} + dx^{r+j} + dx^k.$$

Let now X be a vector field in M_r. It is well known that the vertical and deformed lifts $^IX, {}^CX$ of X has the following expressions (see [4])

$$^VX = X^0 = \begin{pmatrix} 0 \\ 0 \\ X^h \end{pmatrix}, {}^IX = X^1 + Y^0 = \begin{pmatrix} 0 \\ X^h \\ x^{r+i}\partial_i X^h \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ Y^h \end{pmatrix} = \begin{pmatrix} 0 \\ X^h \\ x^{r+i}\partial_i X^h + Y^h \end{pmatrix},$$

$$
\begin{aligned}
{}^{C}X &= X^2 + Y^1 + Z^0 = \begin{pmatrix} X^h \\ x^{r+i}\partial_i X^h \\ x^{2r+i}\partial_i X^h + \frac{1}{2}x^{r+i}x^{r+j}\partial^2_{ij}X^h \end{pmatrix} + \begin{pmatrix} 0 \\ Y^h \\ x^{r+i}\partial_i Y^h \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ Z^h \end{pmatrix} \\
&= \begin{pmatrix} X^h \\ x^{r+i}\partial_i X^h + Y^h \\ x^{2r+i}\partial_i X^h + \frac{1}{2}x^{r+i}x^{r+j}\partial^2_{ij}X^h + x^{r+i}\partial_i Y^h + Z^h \end{pmatrix}
\end{aligned}
$$

for any vector fields Y, Z in M_r. Using the last formulas and also (7), (8), (11), (12), (13) we have

Theorem 3.4. *Let X, ω and f are respectively any vector field, $1-$form and function in M_r. Then*

$$
{}^{V}(f\omega) = f^0\omega^0,\ {}^{I}(f\omega) = f^1\omega^0 + f^0\omega^1 + G^0,\ {}^{C}(f\omega) = (f^2 + f^0)\omega^0 + f^1\omega^1 + G^1 + H^0,
$$

$$
{}^{V}\omega({}^{V}X) = 0,\ {}^{V}\omega({}^{I}X) = 0,\ {}^{V}\omega({}^{C}X) = (\omega(X))^0,
$$

$$
{}^{I}\omega({}^{V}X) = 0,\ {}^{I}\omega({}^{I}X) = (\omega(X))^0,\ {}^{I}\omega({}^{C}X) = (\omega(X))^1 + (\omega(Y))^0 + (H(X))^0,
$$

$$
{}^{C}\omega({}^{V}X) = (\omega(X))^0,\ {}^{C}\omega({}^{I}X) = (\omega(X))^1 + (\omega(Y))^0 + (G(X))^0,
$$

$$
{}^{C}\omega({}^{C}X) = (\omega(X))^2 + (\omega(Y))^1 + (\omega(Z))^0 + (G(X))^1 + (G(Y))^0 + (H(X))^0.
$$

4. Exterior differentials of deformed complete and intermediate lifts

Let now Ω be a tensor field of type $(0,2)$ in M_r. We define an $1-$form $\gamma_Y\Omega$ by

$$
(\gamma_Y\Omega)X = \Omega(X, Y)
$$

for any vector fields X and Y. If Ω has local components Ω_{ij}, then $\gamma_Y\Omega$ has local components $\Omega_{ij}Y^j$.

It is well known that the deformed intermediate and complete lifts of Ω has respectively components (see [3])

$$
{}^{I}\Omega = \begin{pmatrix} x^{r+s}\partial_s\Omega_{ji} + \pi_{ji} & \Omega_{ji} & 0 \\ \Omega_{ji} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \Omega^1 + \pi^0,
$$

$$
\begin{aligned}
{}^{C}\Omega &= \begin{pmatrix} x^{2r+s}\partial_s\Omega_{ji} + \frac{1}{2}x^{r+t}x^{r+s}\partial^2_{ts}\Omega_{ji} + x^{r+s}\partial_s\Omega_{ji} + \pi_{ji} & x^{r+s}\partial_s\Omega_{ji} + \Omega_{ji} & \Omega_{ji} \\ x^{r+s}\partial_s\Omega_{ji} + \Omega_{ji} & \Omega_{ji} & 0 \\ \Omega_{ji} & 0 & 0 \end{pmatrix} \\
&= \Omega^2 + \Omega^1 + \pi^0,
\end{aligned}
$$

where

$$
{}^{V}\pi = {}^{0}\pi = \begin{pmatrix} \pi_{ji} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

is the vertical lift of any tensor field π of type $(0,2)$. Using the expression of lifts ${}^V\pi, {}^I\Omega, {}^C\Omega$ and (9), (11), (13) we have

$$\begin{aligned}
\gamma_{X^2}{}^V\Omega &= (\Omega_{ij}X^j, 0, 0) = ((\gamma_X\Omega)_i, 0, 0) = {}^V(\gamma_X\Omega),\\
\gamma_{X^2}{}^I\Omega &= ((x^{r+s}\partial_s\Omega_{ij} + \pi_{ij})X^j + \Omega_{ij}(x^{r+s}\partial_s X^j), \Omega_{ij}X^j, 0)\\
&= (x^{r+s}\partial_s(\gamma_X\Omega)_i + (\gamma_X\pi)_i, (\gamma_X\Omega)_i, 0)\\
&= (\gamma_X\Omega)^1 + (\gamma_X\pi)^0 = {}^I(\gamma_X\Omega),\\
\gamma_{X^2}{}^C\Omega &= ((x^{2r+s}\partial_s\Omega_{ij} + \frac{1}{2}x^{r+s}x^{r+t}\partial^2_{st}\Omega_{ij} + x^{r+s}\partial_s\Omega_{ij} + \pi_{ij})X^j\\
&\quad +(x^{r+s}\partial_s\Omega_{ij} + \Omega_{ij})x^{r+t}\partial_t X^j\\
&\quad +\Omega_{ij}(x^{2r+s}\partial_s X^j + \frac{1}{2}x^{r+s}x^{r+t}\partial^2_{st}X^j, (x^{r+s}\partial_s\Omega_{ij} + \Omega_{ij})X^j\\
&\quad +\Omega_{ij}x^{r+t}\partial_t X^j, \Omega_{ij}X^j)\\
&= (x^{2r+s}\partial_s(\gamma_X\Omega)_i + \frac{1}{2}x^{r+s}x^{r+t}\partial^2_{st}(\gamma_X\Omega)_i + x^{r+s}\partial_s(\gamma_X\Omega)_i\\
&\quad +(\gamma_X\pi)_i, x^{r+s}\partial_s(\gamma_X\Omega)_i + (\gamma_X\Omega)_i, (\gamma_X\Omega)_i)\\
&= (\gamma_X\Omega)^2 + (\gamma_X\Omega)^1 + (\gamma_X\pi)^0 = {}^C(\gamma_X\Omega).
\end{aligned}$$

Thus we have

Theorem 4.1. *Let Ω be a tensor field of type $(0,2)$ in M_r. Then*

$$\begin{aligned}
\gamma_{X^2}{}^V\Omega &= (\gamma_X\Omega)^0 = {}^V(\gamma_X\Omega),\\
\gamma_{X^2}{}^I\Omega &= (\gamma_X\Omega)^1 + (\gamma_X\pi)^0 = {}^I(\gamma_X\Omega),\\
\gamma_{X^2}{}^C\Omega &= (\gamma_X\Omega)^2 + (\gamma_X\Omega)^1 + (\gamma_X\pi)^0 = {}^C(\gamma_X\Omega).
\end{aligned}$$

We shall now study the deformed lifts of exterior differentials of 1−forms $\omega = \omega_i dx^i$, $i = 1, ..., r$. Using $[X^2, Y^2] = [X, Y]^2$ and linearity of mappings $X \to X^0$, $X \to X^1$, $X \to X^2$, from Theorem 3.4 and Theorem 4.1 we have

$$\begin{aligned}
2(d^I\omega)(X^2, Y^2) &= X^2({}^I\omega(Y^2)) - Y^2({}^I\omega(X^2)) - {}^I\omega([X^2, Y^2])\\
&= X^2((\omega(Y))^1 + (H(Y))^0) - Y^2((\omega(X))^1\\
&\quad +(H(X))^0) - {}^I\omega([X, Y]^2)\\
&= (X\omega(Y))^1 + (XH(Y))^0 - (Y\omega(X))^1 - (YH(X))^0\\
&\quad -(\omega([X, Y]))^1 - (H([X, Y]))^0\\
&= (X\omega(Y) - Y\omega(X) - \omega([X, Y]))^1\\
&\quad +(XH(Y) - YH(X) - H([X, Y]))^1\\
&= 2((d\omega)(X, Y))^1 + 2((dH)(X, Y))^0\\
&= 2(\gamma_Y(d\omega)(X))^1 + 2(\gamma_Y(dH)(X))^0\\
&= 2(\gamma_Y(d\omega))^1(X^2) + 2(\gamma_Y(dH))^0(X^2)\\
&= 2(\gamma_{Y^2}(d\omega)^1)(X^2) + 2(\gamma_{Y^2}(dH)^0)(X^2)\\
&= 2((d\omega)^1 + (dH)^0)(X^2, Y^2).
\end{aligned}$$

By similar devices, we have

$$2(d^C\omega)(X^2, Y^2) = 2((d\omega)^2 + (dG)^1 + (dH)^0)(X^2, Y^2).$$

Since the any tensor field Ω of type $(0,2)$ in $T^2(M_r)$ is completely determined by its action on lifts X^2, Y^2 (see [2, p.324]), i.e. if $\Omega(X^2, Y^2) = \widetilde{\Omega}(X^2, Y^2)$ for any X, Y, then $\Omega = \widetilde{\Omega}$, we have

Theorem 4.2. *Let ω, G and H be 1-forms in M_r . Then the exterior differentials of deformed intermediate and complete lifts of ω to $T^2(M_r)$ satisfies the following formulas:*

$$\begin{aligned} d^I\omega &= (d\omega)^1 + (dH)^0, \\ d^C\omega &= (d\omega)^2 + (dG)^1 + (dH)^0. \end{aligned}$$

References

[1] K. Yano, S. Ishihara, Differential geometry of tangent bundles of order 2, Kodai Math. Sem. Rep. 20 (1968) 318–354.
[2] K. Yano, S. Ishihara, Tangent and Cotangent Bundles, New York, Marcel Dekker, 423 pp., 1973.
[3] T. Sultanova, A. Salimov, On holomorphic metrics of 2−jet bundles, Mediterr. J. Math. 19 (2022) 1–12.
[4] S. Aslanci, T. Sultanova, Deformed lifts in the bundle of 2−jets, Tbilisi Math. J. 14 (2021) 155–162.
[5] A. Salimov, On structure-preserving connections, Period. Math. Hungar. 77 (2018) 69–76.
[6] V.V. Vishnevskii, A.P. Shirokov, V.V. Shurygin, Spaces over Algebras, Kazan, Kazanskii Gosudarstvennyi Universitet., 1985.
[7] S. Tachibana, Analytic tensor and its generalization, Tohoku Math. J. 12 (1960) 208–221.
[8] A. Salimov, N. Cengiz, M. B. Asl, On holomorphic hypercomplex connections, Adv. Appl. Clifford Algebr. 23 (2013) 179–207.