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A note on $\mathcal{I}$-convergence in quasi-metric spaces

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Abstract. In this paper, we define several ideal versions of Cauchy sequences and completeness in quasi-metric spaces. Some examples are constructed to clarify their relationships. We also show that: (1) if a quasi-metric space (X, ρ) is $\mathcal{I}$-sequentially complete, for each decreasing sequence $\{F_n\}$ of nonempty $\mathcal{I}$-closed sets with $\operatorname{diam}\{F_n\} \to 0$ as $n \to \infty$, then $\bigcap_{n\in\mathbb{N}} F_n$ is a single-point set; (2) let $\mathcal{I}$ be a P-ideal, then every precompact left $\mathcal{I}$-sequentially complete quasi-metric space is compact.

1. Introduction

Statistical convergence is a generalization of usual convergence, which was first proposed by Zygmund in the first edition of his monograph [37] in 1935. In 1951, Fast [18] and Steinhaus [31] independently gave the concept of statistical convergence of real number sequences based on the asymptotic density of subsets of positive integers. After Fridy's papers [19, 20], active researches on statistical convergence started and appeared in many mathematical fields. Statistical convergence has many applications in different fields of mathematics, see [2–6, 8, 15, 17, 23, 26, 32] etc.

The idea of statistical convergence had been extended to $\mathcal{I}$-convergence by Kostyrko et al. in [24] with the help of ideals. $\mathcal{I}$-convergence includes ordinary convergence and statistical convergence when $\mathcal{I}$ is the ideal of all finite subsets of the set of natural numbers and all subsets of the set of natural numbers of natural density zero, respectively. Over the last 20 years a lot of work has been done on this convergence and associated topics, and it has turned out to be one of the most active research areas in Topology and Analysis, for more details see [1, 9–12, 21, 27, 33, 35, 36] etc.

The term quasi-metric was proposed as early as 1931 by Wilson [34]. Quasi-metric spaces were considered also by Niemytzki [28] in connection with the axioms defining a metric space and metrizability. A quasi-metric of a set X is a map $d : X \times X \to \mathbb{R}_{\geq 0}$ that satisfies all the axioms of a metric but the symmetry. This modification of the axioms of a metric space drastically changes the whole theory, mainly with respect to completeness, compactness and total boundedness [7]. There are a lot of completeness notions in quasi-metric and quasi-uniform spaces, all agreeing with the usual notion of completeness in the case of metric or uniform spaces, each of them having its advantages and weaknesses [7, 29, 30]. In 2013, Das et

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al. considered the basic properties of ideal convergence and K-$\mathcal{I}$-Cauchy sequences in quasi-metric spaces [13]. Recently, Kara et al. discussed statistical convergence and statistical Cauchy sequences in quasi-metric spaces [22].

In this paper, we mainly discuss ideal versions of Cauchy sequences and completeness in quasi-metric spaces. The paper is organized as follows. In Section 3, we define ideal versions of Cauchy sequences in quasi-metric spaces. Some examples are constructed to clarify their relationships. In Section 4, some ideal versions of completeness related to ideal versions of Cauchy sequences in quasi-metric spaces are considered. We prove that: (1) if a quasi-metric space (X, ρ) is $\mathcal{I}$-sequentially complete, for each decreasing sequence $\{F_n\}$ of nonempty $\mathcal{I}$-closed sets with $\operatorname{diam}\{F_n\} \to 0$ as $n \to \infty$, then $\bigcap_{n\in\mathbb{N}} F_n$ is a single-point set; (2) let $\mathcal{I}$ be a P-ideal, then every precompact left $\mathcal{I}$-sequentially complete quasi-metric space is compact.

2. Preliminaries

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers.

We state our results mainly for left structures and similar results can be obtained for right structures.

Definition 2.1. ([7]) Let X be a set. A non-negative real-valued function ρ on X^2 is called a *quasi-metric* on X if it satisfies the following axioms:

(1) $\rho(x, y) = \rho(y, x) = 0$ if and only if $x = y$,

(2) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, for all $x, y, z \in X$.

In this case, (X, ρ), or simply X, is called a *quasi-metric space* (also known as an *asymmetric metric space*).

The function $\overline{\rho}$ defined by $\overline{\rho}(x, y) = \rho(y, x)$ for all $x, y \in X$ is also a quasi-metric on X, and is called the *conjugate quasi-metric* of ρ. Also, the mapping $\rho^s(x, y) = \max\{\rho(x, y), \overline{\rho}(x, y)\}$ is a metric on X. We call the structures obtained using the original quasi-metric as the *left structures* and the structures obtained using the conjugate quasi-metric as the *right structures*. There are two natural topologies on a quasi-metric space (X, ρ). Each quasi-metric ρ naturally induces a topology τ_ρ whose base consists of all left ρ-open balls

$$B_\rho(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\},\ \varepsilon > 0,\ x \in X.$$

Similarly, the topology $\tau_{\overline{\rho}}$ induced by the conjugate quasi-metric is also defined.

A sequence $\{x_n\}$ in a quasi-metric space (X, ρ) is said to be *left (right) convergent* to x, if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\rho(x, x_n) < \varepsilon$ ($\rho(x_n, x) < \varepsilon$) for all $n \geq k$. Actually, the statement that a sequence $\{x_n\}$ in a quasi-metric space (X, ρ) left converges to x is equivalent to the sequence $\{x_n\}$ converges to x respect to the topology τ_ρ.

Definition 2.2. ([29]) A sequence $\{x_n\}$ in a quasi-metric space (X, ρ) is said to be:

(1) ρ^s-*Cauchy* if it is a Cauchy sequence in the metric space (X, ρ^s);

(2) *left (right) Cauchy* if for each $\varepsilon > 0$ there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that $\rho(x, x_n) < \varepsilon$ ($\rho(x_n, x) < \varepsilon$) for all $n \geq n_0$;

(3) *left (right) K-Cauchy* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\rho(x_m, x_n) < \varepsilon$ ($\rho(x_n, x_m) < \varepsilon$) for all $n \geq m \geq n_0$;

(4) *weakly left (right) K-Cauchy* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\rho(x_{n_0}, x_n) < \varepsilon$ ($\rho(x_n, x_{n_0}) < \varepsilon$) for all $n \geq n_0$.

It seems that K in Definition 2.2 of a left K-Cauchy comes from Kelly who was the first to consider this notion. By Definition 2.2, the following implications are clear. However, No one of those implications is reversible [29].

Fig. 1

Corresponding to Definition 2.2 of Cauchy sequences in quasi-metric spaces, we have some versions of completeness.

Definition 2.3. ([29]) A quasi-metric space (X, ρ) is said to be:

(1) *ρ-sequentially complete* if every ρ^s-Cauchy sequence is left convergent;

(2) *left sequentially complete* if every left Cauchy sequence is left convergent;

(3) *left K-sequentially complete* if every left K-Cauchy sequence is left convergent;

(4) *weakly left K-sequentially complete* if every weakly left K-Cauchy sequence is left convergent.

Let $\mathcal{I}$ be a family of non-empty subsets on $\mathbb{N}$, $\mathcal{I}$ is said to be an *ideal* if

(i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,

(ii) $A \in \mathcal{I}$, $B \subseteq A$ imply $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is said to be *non-trivial* if $\mathbb{N} \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$. The family of sets $\mathcal{F}(\mathcal{I}) = \{\mathbb{N} - A : A \in \mathcal{I}\}$ is a filter called the *associated filter* of $\mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called *admissible* if $\mathcal{I} \supseteq \{\{x\} : x \in \mathbb{N}\}$.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and X be a topological space. A sequence $\{x_n\}$ in X is said to be *$\mathcal{I}$-convergent* to a point $x \in X$ if for every neighborhood U of x, we have the set $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$, which is denoted by $x_n \xrightarrow{\mathcal{I}} x$ or $x = \mathcal{I}\text{-}\lim x_n$ [24]. Especially, if $\mathcal{I}$ is the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$, then $\mathcal{I}_f$ is an admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence of sequences; if $\mathcal{I}_d$ is the class of all $A \subseteq \mathbb{N}$ with $d(A) = 0$, where $d(A)$ denotes the asymptotic density of a set A, then $\mathcal{I}_d$ is an admissible ideal and $\mathcal{I}_d$-convergence coincides with the statistical convergence. A set $P \subseteq X$ is said to an *$\mathcal{I}$-closed set* of X if whenever a sequence $\{x_n\}$ in P with $x_n \xrightarrow{\mathcal{I}} x$ in X, the $\mathcal{I}$-limit point $x \in P$ [27].

Using Zorn's lemma, we can show that in the family of all admissible ideals of $\mathbb{N}$, there exists a maximal ideal (with respect to inclusion).

Lemma 2.4. ([11]) *Let $\mathcal{I}_0$ be an admissible ideal on $\mathbb{N}$. Then $\mathcal{I}_0$ is maximal if and only if*

$$(A \in \mathcal{I}_0) \vee (\mathbb{N} \setminus A \in \mathcal{I}_0)$$

holds for each $A \subseteq \mathbb{N}$.

Definition 2.5. ([11]) An admissible ideal $\mathcal{I}$ is said to satisfy the *condition (AP)* (or is called a *P-ideal* or sometimes *an AP-ideal*) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ from $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j$ is finite for each $j \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} B_k \in \mathcal{I}$. It is clear that $B_j \in \mathcal{I}$ for each $j \in \mathbb{N}$.

Lemma 2.6. ([25, Theorem 8 (i)]) *If $\mathcal{I}$ is a P-ideal and (X, τ) a first-countable space, then for an arbitrary sequence $\{x_n\}$ in X, $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = x$ implies $\mathcal{I}^*\text{-}\lim_{n\to\infty} x_n = x$, i.e., there is an $K \in \mathcal{F}(I)$ such that $\{x_n\}_{n\in K}$ converges to x.*

In this paper, $\mathcal{I}$ denotes an admissible ideal on $\mathbb{N}$ unless stated otherwise. Readers may consult [7, 11, 16] for notation and terminology not given here.

3. Ideal versions of Cauchy sequences in quasi-metric spaces

In this section, we define some ideal versions of Cauchy sequences in quasi-metric spaces. Many examples are constructed to clarify their relationships.

Definition 3.1. ([13]) A sequence $\{x_n\}$ in a quasi-metric space (X, ρ) is called *left $\mathcal{I}$-convergent to a point $x \in X$* if for each $\varepsilon > 0$, $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x, x_n) \geq \varepsilon\} \in \mathcal{I}$.

The right $\mathcal{I}$-convergent of a sequence can be defined similarly.

Proposition 3.2. *If a sequence $\{x_n\}$ in a quasi-metric space (X,ρ) is $\mathcal{I}$-convergent to a point $x \in X$ with respect to the metric ρ^s, then it is both left and right $\mathcal{I}$-convergent to x.*

Proof. By the definition of ρ^s, we have the inclusions $\{n \in \mathbb{N} : \rho(x,x_n) \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : \rho^s(x,x_n) \geq \varepsilon\}$ and $\{n \in \mathbb{N} : \overline{\rho}(x,x_n) \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : \rho^s(x,x_n) \geq \varepsilon\}$ for any $\varepsilon > 0$. Since the sequence $\{x_n\}$ is $\mathcal{I}$-convergent to x with respect to the metric ρ^s, it follows that $\{n \in \mathbb{N} : \rho^s(x,x_n) \geq \varepsilon\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : \rho(x,x_n) \geq \varepsilon\} \in \mathcal{I}$ and $\{n \in \mathbb{N} : \overline{\rho}(x,x_n) \geq \varepsilon\} \in \mathcal{I}$, which show that the sequence $\{x_n\}$ is both left and right $\mathcal{I}$-convergent to x, respectively. □

By [22, Example 2.2], the converse is not true for the ideal $\mathcal{I}_d$. Actually, the converse is not true for any non-maximal ideal.

Example 3.3. For each non-maximal ideal $\mathcal{I}$, there is a quasi-metric space (X,ρ) and a sequence $\{x_n\}$ in X such that the sequence $\{x_n\}$ is both left and right $\mathcal{I}$-convergent, but it is not $\mathcal{I}$-convergent with respect to the metric ρ^s.

Proof. Let $X = \mathbb{R}$ and ρ be the quasi-metric on X defined by

$$\rho(x,y) = \begin{cases} y - x, & \text{if } x < y; \\ 0, & \text{if } x \geq y. \end{cases}$$

Since $\mathcal{I}$ is a non-maximal ideal, it follows from Lemma 2.4 that there is an infinite set $A \subseteq \mathbb{N}$ with $A \notin \mathcal{I}$ and $\mathbb{N} \setminus A \notin \mathcal{I}$. Consider the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} 1, & \text{if } n \in A; \\ -1, & \text{if } n \notin A. \end{cases}$$

Thus $\rho(1,x_n) = 0$ and $\rho(x_n,-1) = 0$ for all $n \in \mathbb{N}$. Hence, the sequence $\{x_n\}$ is left $\mathcal{I}$-convergent to 1 and right $\mathcal{I}$-convergent to -1, respectively. However, since ρ^s is the absolute value metric on $\mathbb{R}$, we have

$$\{n \in \mathbb{N} : \rho^s(x,x_n) < 1\} = \begin{cases} A \text{ or } \mathbb{N} \setminus A, & \text{if } |x| < 2 \text{ and } x \neq 0; \\ \emptyset, & \text{otherwise.} \end{cases}$$

This implies that the sequence $\{x_n\}$ is not $\mathcal{I}$-convergent respect to ρ^s. □

Since quasi-metric spaces are first-countable spaces, the following result is obvious by Lemma 2.6.

Lemma 3.4. *If $\mathcal{I}$ is a P-ideal. A sequence $\{x_n\}$ in a quasi-metric space (X,ρ) is left $\mathcal{I}$-convergent if and only if it is left $\mathcal{I}^*$-convergent, i.e. there is an $F \in \mathcal{F}(\mathcal{I})$ such that $\{x_n\}_{n\in F}$ is left convergent.*

A sequence $\{x_n\}$ in a metric space (X,ρ) is said to be *$\mathcal{I}$-Cauchy* if for each $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ such that $\{n \in \mathbb{N} : \rho(x_k,x_n) \geq \varepsilon\} \in \mathcal{I}$ [11].

Definition 3.5. A sequence $\{x_n\}$ in a quasi-metric space (X,ρ) is said to be:

(1) *ρ^s-$\mathcal{I}$-Cauchy* if it is an $\mathcal{I}$-Cauchy sequence in the metric space (X,ρ^s);

(2) *left $\mathcal{I}$-Cauchy* if for each $\varepsilon > 0$ there exists $x \in X$ such that $\{n \in \mathbb{N} : \rho(x,x_n) \geq \varepsilon\} \in \mathcal{I}$;

(3) *left K-$\mathcal{I}$-Cauchy* if for every $\varepsilon > 0$ there exists $M \in \mathcal{F}(\mathcal{I})$ such that for any $k \in M$, $\{n \in \mathbb{N} : \rho(x_k,x_n) \geq \varepsilon\} \in \mathcal{I}$ [13];

(4) *weakly left K-$\mathcal{I}$-Cauchy* if for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $\{n \in \mathbb{N} : \rho(x_{n_0},x_n) \geq \varepsilon\} \in \mathcal{I}$.

Remark 3.6. It is obvious that the left convergence of a sequence implies that it is left $\mathcal{I}$-convergent to the same point. Also, if a sequence is left Cauchy or weakly left K-Cauchy, then it is a left $\mathcal{I}$-Cauchy sequence or a weakly left K-$\mathcal{I}$-Cauchy sequence, respectively. Furthermore, a left $\mathcal{I}$-convergent sequence and a weakly left K-$\mathcal{I}$-Cauchy sequence are left $\mathcal{I}$-Cauchy.

Proposition 3.7. *Every left K-Cauchy sequence in a quasi-metric space is left K-$\mathcal{I}$-Cauchy.*

Proof. Suppose that (X, ρ) is a quasi-metric space, and $\{x_n\}$ is a left K-Cauchy sequence in X. Then for each $\varepsilon > 0$ there exits an $n_0 \in \mathbb{N}$ such that $\rho(x_m, x_n) < \varepsilon$ whenever $n \geq m \geq n_0$. Put $M = \{n_0, n_0 + 1, ...\}$, it is clear that $M \in \mathcal{F}(\mathcal{I})$. Then

$$\{n \in \mathbb{N} : \rho(x_k, x_n) \geq \varepsilon\} \subseteq \{1, 2, ..., k\}$$

for any $k \in M$. Since $\mathcal{I}$ is admissible, we have $\{1, 2, ..., k\} \in \mathcal{I}$. Therefore, $\{n \in \mathbb{N} : \rho(x_k, x_n) \geq \varepsilon\} \in \mathcal{I}$, which shows that the sequence $\{x_n\}$ is left K-$\mathcal{I}$-Cauchy. □

From Definitions 2.2, 3.1, 3.5, Remark 3.6, Proposition 3.7 and properties of metric spaces, we have the following implications for sequences in quasi-metric spaces.

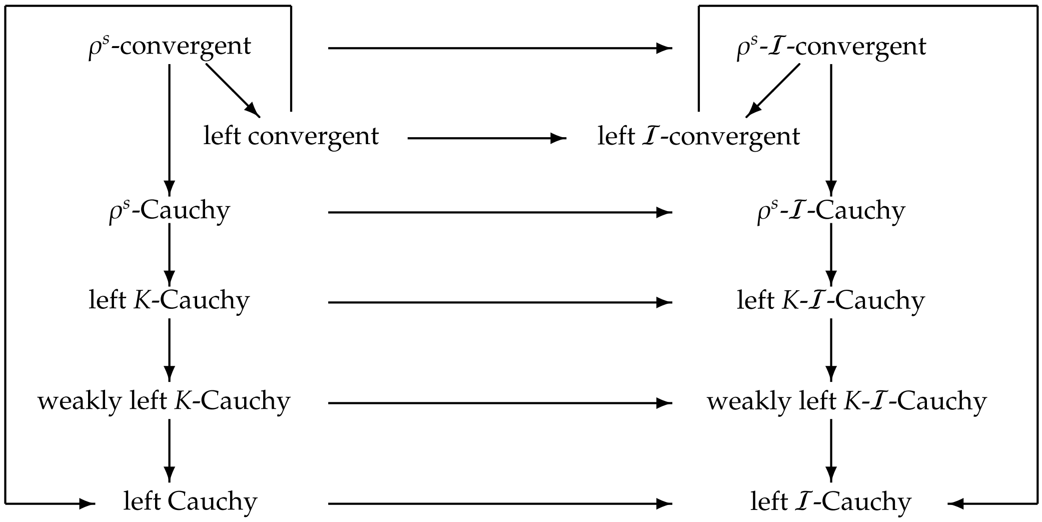

Fig. 2

Theorem 3.8. *Let $\{x_n\}$ be a sequence in a quasi-metric space (X, ρ). If there is an $F \in \mathcal{F}(\mathcal{I})$ such that $\{x_n\}_{n \in F}$ is left Cauchy (left K-$\mathcal{I}$-Cauchy, weakly left K-Cauchy), then the sequence $\{x_n\}$ is left $\mathcal{I}$-Cauchy (left K-$\mathcal{I}$-Cauchy, weakly left K-$\mathcal{I}$-Cauchy).*

Proof. We prove the theorem for left Cauchy sequences. The other statements can be proved in a similar way. Suppose that there is an $F \in \mathcal{F}(\mathcal{I})$ such that $\{x_n\}_{n \in F}$ is left Cauchy. Then for every $\varepsilon > 0$ there is $x \in X$ and $n_0 \in \mathbb{N}$ such that for every $n \in F$ with $n \geq n_0$, we have $\rho(x, x_n) < \varepsilon$. Thus

$$\{n \in \mathbb{N} : \rho(x, x_n) \geq \varepsilon\} \subseteq (\mathbb{N} \setminus F) \cup \{1, 2, \ldots, n_0 - 1\}.$$

Since $\mathbb{N} \setminus F \in \mathcal{I}$ and $\mathcal{I}$ is admissible, we can conclude that $\{n \in \mathbb{N} : \rho(x, x_n) \geq \varepsilon\} \in \mathcal{I}$. Consequently, the sequence $\{x_n\}$ is left $\mathcal{I}$-Cauchy. □

We do not know if the converse of Theorem 3.8 is true for left $\mathcal{I}$-Cauchy sequences or weakly left K-$\mathcal{I}$-Cauchy sequences. The following result has been proved in [13].

Theorem 3.9. ([13]) *Let $\mathcal{I}$ be a P-ideal. A sequence $\{x_n\}$ in a quasi-metric space (X, ρ) is left K-$\mathcal{I}$-Cauchy if and only if there is an $F \in \mathcal{F}(\mathcal{I})$ such that $\{x_n\}_{n \in F}$ is left K-Cauchy.*

Thus we have the following question.

Question 3.10. *Let $\{x_n\}$ be a sequence in a quasi-metric space (X, ρ). If the sequence $\{x_n\}$ is left $\mathcal{I}$-Cauchy (weakly left K-$\mathcal{I}$-Cauchy), is there $F \in \mathcal{F}(\mathcal{I})$ such that $\{x_n\}_{n \in F}$ is left Cauchy (weakly left K-Cauchy)? What if $\mathcal{I}$ is additionally a P-ideal?*

The following example shows that the converse of the Proposition 3.7 is not true for any ideal $\mathcal{I}$ with $\mathcal{I} \neq \mathcal{I}_f$.

Example 3.11. For each ideal $\mathcal{I}$ with $\mathcal{I} \neq \mathcal{I}_f$, there is a quasi-metric space (X, ρ) and a left K-$\mathcal{I}$-Cauchy sequence $\{x_n\}$ in X such that it is neither weakly left K-Cauchy nor ρ^s-$\mathcal{I}$-Cauchy.

Proof. Let $X = \mathbb{R}$ and ρ be the quasi-metric on X defined by

$$\rho(x, y) = \begin{cases} x - y, & \text{if } x \geq y; \\ 1, & \text{if } x < y. \end{cases}$$

Then (X, ρ^s) is the discrete topology of X. Since $\mathcal{I} \neq \mathcal{I}_f$, there is an infinite set $A \subseteq \mathbb{N}$ with $A \in \mathcal{I}$. Consider the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} 1, & \text{if } n \in A; \\ 1/n, & \text{if } n \notin A. \end{cases}$$

Thus $\rho(x_n, x_m) = 1$ for $m \in A$, $n \notin A$ and $n < m$; and $\rho(x_n, x_m) = 1 - \frac{1}{m}$ for $m \notin A$, $n \in A$. Therefore, the sequence $\{x_n\}$ is not weakly left K-Cauchy. On the other hand, put $F = \mathbb{N} \setminus A$, then $F \in \mathcal{F}(\mathcal{I})$. For each $\varepsilon > 0$, there is a number $n_0 \in F$ such that $1/n_0 < \varepsilon$. Hence $\rho(x_n, x_m) = 1/n - 1/m < \varepsilon$ for $m \geq n \geq n_0$ and $n, m \in F$. It follows that the subsequence $\{1/n\}_{n \in F}$ of $\{x_n\}$ is a left K-Cauchy sequence. By Theorem 3.8, $\{x_n\}$ is a left K-$\mathcal{I}$-Cauchy sequence. Since (X, ρ^s) is discrete and $\mathcal{I}$ is non-trivial, the sequence is not ρ^s-$\mathcal{I}$-Cauchy. □

Example 3.11 also shows that there is a quasi-metric space (X, ρ) and a left K-$\mathcal{I}$-Cauchy sequence $\{x_n\}$ in X such that it is not left K-Cauchy, and there is a quasi-metric space (X, ρ) and a weakly left K-$\mathcal{I}$-Cauchy sequence $\{x_n\}$ in X such that it is not weakly left K-Cauchy.

The following example shows that the converse of the implication ρ^s-$\mathcal{I}$-Cauchy $\Rightarrow$ left $\mathcal{I}$-convergent is not true in general.

Example 3.12. There is a quasi-metric space (X, ρ) and a left $\mathcal{I}$-convergent sequence $\{x_n\}$ in X such that it is not weakly left K-$\mathcal{I}$-Cauchy.

Proof. Let $X = [0, 1]$ and ρ be the quasi-metric on X defined by

$$\rho(x, y) = \begin{cases} 0, & \text{if } x \leq y; \\ 1, & \text{if } x > y. \end{cases}$$

If $\mathcal{I} = \mathcal{I}_f$, there is a left $\mathcal{I}_f$-convergent sequence $\{x_n\}$ in X such that it is not a weakly left K-$\mathcal{I}_f$-Cauchy sequence [29, Example 1]. If $\mathcal{I} \neq \mathcal{I}_f$, there is an infinite set $A \subseteq \mathbb{N}$ with $A \in \mathcal{I}$. Consider the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} \frac{1}{2} + \frac{1}{2^n}, & \text{if } n \in A; \\ \\ \frac{1}{6} + \frac{1}{6^n}, & \text{if } n \notin A. \end{cases}$$

It is obvious that $\rho(\frac{1}{6}, x_n) = 0$, thus $\{x_n\}$ left converges to $\frac{1}{6}$. Therefore, $\{x_n\}$ is left $\mathcal{I}$-convergent. However, $\{x_n\}$ is not a weakly left K-$\mathcal{I}$-Cauchy sequence. In fact, for any $N \in \mathbb{N}$, if $N \in A$, then

$$\{n \in \mathbb{N} : \rho(x_N, x_n) < \frac{1}{2}\} \subseteq \{1, \ldots, N\}.$$

If $N \notin A$, then

$$\{n \in \mathbb{N} : \rho(x_N, x_n) < \frac{1}{2}\} = A \cup \{1, \ldots, N\}.$$

Since $\mathcal{I}$ is admissible and $A \in \mathcal{I}$, we conclude that $A \cup \{1, \ldots, N\} \in \mathcal{I}$. Therefore, $\{n \in \mathbb{N} : \rho(x_N, x_n) < \frac{1}{2}\} \in \mathcal{I}$ for each $N \in \mathbb{N}$. Note that $\mathcal{I}$ is non-trival, it follows that $\{n \in \mathbb{N} : \rho(x_N, x_n) \geq \frac{1}{2}\} = \mathbb{N} \setminus (A \cup \{1, \ldots, N\}) \notin \mathcal{I}$. Thus the sequence $\{x_n\}$ is not weakly left K-$\mathcal{I}$-Cauchy. □

Since every left $\mathcal{I}$-convergent sequence is left $\mathcal{I}$-Cauchy, Example 3.12 also shows that there is a left $\mathcal{I}$-Cauchy sequence which is not weakly left K-$\mathcal{I}$-Cauchy.

Example 3.13. There is a quasi-metric space (X, ρ) and a left $\mathcal{I}$-Cauchy sequence $\{x_n\}$ in X such that it is not left $\mathcal{I}$-convergent.

Proof. Since $\mathcal{I}$ is a non-trivial ideal, there is an infinite set $A \subseteq \mathbb{N}$ with $A \notin \mathcal{I}$. Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ and ρ be the quasi-metric on X defined by

$$\rho(x,y) = \begin{cases} 1, & \text{if } n \notin A,\ y \neq 1/n \text{ and } x = 1/n; \\ 1/n, & \text{if } n \in A,\ y \neq 1/n \text{ and } x = 1/n; \\ 1/n, & \text{if } n \notin A,\ y = 1/n \text{ and } x = 0; \\ 1, & \text{if } n \in A,\ y = 1/n \text{ and } x = 0; \\ 0, & \text{if } x = y. \end{cases}$$

Consider the sequence $\{x_n\}$ defined by $x_n = 1/n$ for each $n \in \mathbb{N}$. For each $\varepsilon > 0$, there is an $n_0 \in A$ such that $x_{n_0} = 1/n_0 < \varepsilon$. Thus $\{n \in \mathbb{N} : \rho(x_{n_0}, x_n) \geq \varepsilon\} = \emptyset$. This implies that $\{x_n\}$ is a weakly left K-$\mathcal{I}$-Cauchy sequence, hence it is a left $\mathcal{I}$-Cauchy sequence.

We will show that the sequence $\{x_n\}$ is not left $\mathcal{I}$-convergent. If $x = 0$, then $\{n \in \mathbb{N} : \rho(0, x_n) \geq 1\} = A \notin \mathcal{I}$. It follows that $\{x_n\}$ is not left $\mathcal{I}$-convergent to 0. If $x = x_N$ for some $N \in \mathbb{N}$, then $\{n \in \mathbb{N} : \rho(x_N, x_n) \geq 1/N\} = \mathbb{N} \setminus \{N\} \notin \mathcal{I}$, which shows that $\{x_n\}$ is not left $\mathcal{I}$-convergent to $x = x_N$ for any $N \in \mathbb{N}$. Therefore, the sequence is not left $\mathcal{I}$-convergent. □

Example 3.14. For each ideal $\mathcal{I}$ with $\mathcal{I} \neq \mathcal{I}_f$, there is a quasi-metric space (X, ρ) and a left $\mathcal{I}$-convergent sequence $\{x_n\}$ in X such that it is not a left Cauchy sequence.

Proof. Let $X = \mathbb{R}$ and ρ be the quasi-metric on X defined by

$$\rho(x,y) = \begin{cases} y - x, & \text{if } x \leq y; \\ 1, & \text{if } x > y. \end{cases}$$

Since $\mathcal{I} \neq \mathcal{I}_f$, there is an infinite set $A \subseteq \mathbb{N}$ with $A \in \mathcal{I}$. Consider the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} n, & \text{if } n \in A; \\ 0, & \text{if } n \notin A. \end{cases}$$

For each $\varepsilon > 0$, we have $\{n \in \mathbb{N} : \rho(0, x_n) \geq \varepsilon\} \subseteq A$. It follows that $\{x_n\}$ is left $\mathcal{I}$-convergent to 0. Since $\mathbb{N} \setminus A \in \mathcal{F}(\mathcal{I})$ and $\{x_n\}_{n \in \mathbb{N} \setminus A}$ is left K-Cauchy, we can conclude that $\{x_n\}$ is left K-$\mathcal{I}$-Cauchy by Theorem 3.8. However, the sequence $\{x_n\}$ is not left Cauchy. In fact, for any $x \in \mathbb{R}$, there exists a number $n_0 \in \mathbb{N}$ such that $x + 1 < n_0$. Therefore, $\rho(x, x_n) \geq 1$ for every $n \geq n_0$ and $n \in A$, which shows that the sequence $\{x_n\}$ is not left Cauchy. □

Example 3.14 also shows that there is a left $\mathcal{I}$-Cauchy sequence which is not left Cauchy for each ideal $\mathcal{I}$ with $\mathcal{I} \neq \mathcal{I}_f$.

Example 3.15. There is a quasi-metric space (X, ρ) and a weakly left K-$\mathcal{I}$-Cauchy sequence $\{x_n\}$ in X such that it is not left K-$\mathcal{I}$-Cauchy.

Proof. Let (X, ρ) be the quasi-metric space in Example 3.12. Consider the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} 0, & \text{if } n = 1; \\ \frac{1}{n}, & \text{if } n \neq 1. \end{cases}$$

It is clear that the sequence $\{x_n\}$ is weakly left K-Cauchy, thus it is weakly left K-$\mathcal{I}$-Cauchy. Since $\mathcal{I}$ is a non-trivial ideal, every $M \in \mathcal{F}(\mathcal{I})$ is infinite. Note that $\rho(x_n, x_m) = 1$ for all $m > n > 1$, therefore, the sequence $\{x_n\}$ is not left K-$\mathcal{I}$-Cauchy. □

4. Completeness and compactness in quasi-metric spaces

In this section, we define some ideal versions of completeness of quasi-metric spaces. Some properties are considered. Corresponding to the Definition 3.5, we have the following definition.

Definition 4.1. A quasi-metric space (X, ρ) is said to be:

(1) *$\mathcal{I}$-sequentially complete* if every ρ^s-$\mathcal{I}$-Cauchy sequence is left $\mathcal{I}$-convergent;

(2) *left $\mathcal{I}$-sequentially complete* if every left $\mathcal{I}$-Cauchy sequence is left $\mathcal{I}$-convergent;

(3) *left K-$\mathcal{I}$-sequentially complete* if every left K-$\mathcal{I}$-Cauchy sequence is left $\mathcal{I}$-convergent;

(4) *weakly left K-$\mathcal{I}$-sequentially complete* if every weakly left K-$\mathcal{I}$-Cauchy sequence is left $\mathcal{I}$-convergent.

It follows that the implications between these $\mathcal{I}$-sequentially completeness notions are obtained by reversing the implications between the corresponding notions of $\mathcal{I}$-Cauchy sequences, which is the following proposition.

Proposition 4.2. *These notions of $\mathcal{I}$-sequentially completeness are related in the following way: left $\mathcal{I}$-sequentially complete $\Rightarrow$ weakly left K-$\mathcal{I}$-sequentially complete $\Rightarrow$ left K-$\mathcal{I}$-sequentially complete $\Rightarrow$ $\mathcal{I}$-sequentially complete.*

It is known that left K-sequentially complete and weakly left K-sequentially complete are equivalent in a quasi-metric space [22]. Thus we have the following question:

Question 4.3. *Which one of the converse of the implications in Proposition 4.2 is true?*

The diameter of a subset A of a quasi-metric space (X, ρ) is defined by

$$\text{diam}(A) = \sup\{\rho(x, y) : x, y \in A\}.$$

It is clear that the diameter, as defined, is in fact the diameter with respect to the associated metric ρ^s.

Concerning Baire's characterization of completeness in terms of descending sequences of closed sets we have the following result.

Theorem 4.4. *If a quasi-metric space (X, ρ) is $\mathcal{I}$-sequentially complete, for each decreasing sequence $\{F_n\}$ of nonempty $\mathcal{I}$-closed sets with diam$\{F_n\} \to 0$ as $n \to \infty$, then $\bigcap_{n\in\mathbb{N}} F_n$ is a single-point set.*

Proof. Let $\{F_n\}$ be a decreasing sequence of non-empty $\mathcal{I}$-closed with diam$\{F_n\} \to 0$ as $n \to \infty$. Choosing $x_n \in F_n$ for each $n \in \mathbb{N}$, then $n, k \geq n_0$ implies $x_n, x_k \in F_{n_0}$, thus the sequence $\{x_n\}$ is ρ^s-Cauchy, and hence $\{x_n\}$ is ρ^s-$\mathcal{I}$-Cauchy. Since the space (X, ρ) is $\mathcal{I}$-sequentially complete, it follows that the sequence $\{x_n\}$ is left $\mathcal{I}$-convergent to some $x \in X$. It is clear that $\{x_{n+k} : k \in \mathbb{N}\} \subseteq F_n$, then the sequence $\{x_{n+k}\}$ is left $\mathcal{I}$-convergent to x as $k \to \infty$. Taking into account that $\{F_n\}$ is $\mathcal{I}$-closed, it follows that $x \in F_n$. Therefore, $x \in \bigcap_{n\in\mathbb{N}} F_n$. Since ρ^s is a metric, the hypothesis diam$\{F_n\} \to 0$ implies that $\bigcap_{n\in\mathbb{N}} F_n$ can contain at most one element. Therefore, $\bigcap_{n\in\mathbb{N}} F_n$ is a single-point set. □

However, the following question is unknown.

Question 4.5. *Is the converse of Theorem 4.4 true?*

Definition 4.6. A quasi-metric space (X, ρ) is said to be $\mathcal{I}$-sequentially compact if any sequence in X has a $\mathcal{I}$-convergent subsequence.

Theorem 4.7. *Let $\mathcal{I}$ be a P-ideal, a quasi-metric space (X, ρ) is $\mathcal{I}$-sequentially compact if and only if it is sequentially compact.*

Proof. Let X be a $\mathcal{I}$-sequentially compact quasi-metric space and $\{x_n\}$ a sequence in X. Then it has a $\mathcal{I}$-convergent subsequence $\{x_{n_i}\}$. By Lemma 3.4, the subsequence $\{x_{n_i}\}$ has a convergent subsequence. This proves sequential compactness of X. The converse is clear. □

A set Y in a quasi-metric space (X, ρ) is said to be *precompact* if for every $\varepsilon > 0$, there exists a finite subset F of Y such that $Y \subseteq \bigcup\{B_\rho(x, \varepsilon) : x \in F\}$.

Lemma 4.8. ([7]) *A precompact countably compact quasi-metric space is compact.*

Theorem 4.9. *Let $\mathcal{I}$ be a P-ideal, then every precompact left $\mathcal{I}$-sequentially complete quasi-metric space is compact.*

Proof. Let (X, ρ) be a precompact left $\mathcal{I}$-sequentially complete quasi-metric space and $\{x_n\}$ a sequence in X. Since X is precompact, the sequence $\{x_n\}$ has a left Cauchy subsequence $\{x_{n_i}\}$ [30]. By Theorem 3.8, the subsequence $\{x_{n_i}\}$ is left $\mathcal{I}$-Cauchy. Since (X, ρ) is left $\mathcal{I}$-sequentially complete, the sequence $\{x_{n_i}\}$ is left $\mathcal{I}$-convergent. According to Lemma 3.4, the subsequence $\{x_{n_i}\}$ has a convergent subsequence. This shows that X is sequentially compact. Note that (X, ρ) is first-countable, thus (X, ρ) is countably compact. Hence, by Lemma 4.8, X is compact. □

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