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Hurewicz and Hurewicz-type star selection principles for hit-and-miss topology

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Abstract. In this paper we continue the study of the characterization of selection principles in the hyperspaces $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ and $\mathbb{CS}(X)$, endowed with the hit-and-miss topology, by using $\pi_\Delta(\Lambda)$-networks and $c_\Delta(\Lambda)$-covers of a topological space X. Specifically, we prove theorems which characterize the covering properties Hurewicz, strongly star Hurewicz, star Hurewicz and absolutely strongly star Hurewicz in these hyperspaces.

1. Introduction and preliminaries

The hyperspace theory started with the works of D. Pompeiu [21], F. Hausdorff [11], L. Vietoris [29] and E. Michael [20]. We recall that given a topological space X, $CL(X)$ denotes the set of all nonempty closed subsets of X. The set $CL(X)$, endowed with some topology, is known as hyperspace of X. Numerous relations between properties of the space X and their hyperspaces have been widely studied. On the other hand, the study of selection principles started in [3, 13, 19, 22]. Furthermore, Scheepers [25], initiated a systematic investigation on selection principles, which led to a great deal of research into selection principles and their applications. One of the lines of research generated by the selection principles is concerning to selection principles type star, which was started by Kočinac in [14]. Nowadays, many authors have worked with these concepts to obtain results in hyperspaces for example [5–9, 16, 24]. Furthermore, this theory has now applications on several fields of mathematics, for example, set theory, function spaces and hyperspaces.

The study of relationships between selection principles and hyperspaces has been improved. For instance, in [8] the authors used π-networks to characterize topological spaces whose hyperspace, endowed with the upper Fell topology, satisfies the Rothberger property. Later, Li [18] defined the concepts of π_F-network, π_V-network, k_F-cover and c_V-cover and uses these notions to study the $S_1(\mathscr{A},\mathscr{B})$ and $S_{fin}(\mathscr{A},\mathscr{B})$ principles in $CL(X)$ endowed with the Fell and Vietoris topologies, for different families $\mathscr{A}$ and $\mathscr{B}$. Then,

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in [5] the authors introduce the generic notions of $\pi_\Delta(\Lambda)$-networks (and $c_\Delta(\Lambda)$-cover), which are a generalization of π_F-networks and π_V-networks (and of k_F-cover and c_V-cover, respectively), to characterize Menger-type star selection principles in hyperspaces endowed with the hit-and-miss topology. Also, using $c_\Delta(\Lambda)$-covers, are shown characterizations of star and strong star-type versions of Rothberger and Menger principles in hyperspaces [6].

Following this line, in this paper we use $\pi_\Delta(\Lambda)$-networks and $c_\Delta(\Lambda)$-covers of X, to characterize the covering properties Hurewicz (Theorem 2.4), strongly star Hurewicz (Theorem 3.4), star Hurewicz (Theorem 4.4) and absolutely strongly star Hurewicz (Theorem 5.4) in hyperspaces endowed with the hit-and-miss topology.

Next, we recall two known concepts both defined in 1996 by M. Scheepers [25]. Given an infinite set X, let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of X.

- $\mathbf{S}_1(\mathscr{A},\mathscr{B})$ denotes the principle: For any sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $B_n \in \mathcal{A}_n$ and $\{B_n : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.
- $\mathbf{S}_{fin}(\mathscr{A},\mathscr{B})$ denotes the principle: for each sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ such that $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n \in \mathscr{B}$.
- $\mathbf{U}_{fin}(\mathscr{A},\mathscr{B})$ denotes the principle: for each sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\{\bigcup \mathcal{B}_n : n \in \mathbb{N}\} \in \mathscr{B}$.

Remember that an infinite open cover C of X is a γ-cover of X if for any $x \in X$, the set $\{U \in C : x \notin U\}$ is finite. We denote by $\mathscr{O}$ and Γ the collections of every open covers and all γ-covers, respectively, of a topological space X. When we have $\mathbf{S}_1(\mathscr{O},\mathscr{O})$ and $\mathbf{S}_{fin}(\mathscr{O},\mathscr{O})$, we get the well known *Rothberger property* [22] and the *Menger property* [13, 19], respectively. On the other hand, the principle $\mathbf{U}_{fin}(\mathscr{O},\Gamma)$ is the *Hurewicz property* [13].

Now, for every set $A \subseteq X$ and any collection $\mathcal{U}$ of subsets of X, the star of A with respect to $\mathcal{U}$ is denoted by $St(A,\mathcal{U})$ and defined as $\bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. We write $St(x,\mathcal{U})$ instead of $St(\{x\},\mathcal{U})$, for each $x \in X$. In [14], Kočinac introduced some selection principles using the star operator St. As an interesting example, we remember the star versions of $\mathbf{S}_1(\mathscr{A},\mathscr{B})$, $\mathbf{S}_{fin}(\mathscr{A},\mathscr{B})$ and $\mathbf{U}_{fin}(\mathscr{A},\mathscr{B})$, which are defined as follows.

- $\mathbf{S}^*_1(\mathscr{A},\mathscr{B})$, if for any sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $B_n \in \mathcal{A}_n$ and $\{St(B_n,\mathcal{A}_n) : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.
- $\mathbf{S}^*_{fin}(\mathscr{A},\mathscr{B})$, if for each sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\bigcup_{n\in\mathbb{N}} \{St(B,\mathcal{A}_n) : B \in \mathcal{B}_n\}$ is an element of $\mathscr{B}$.
- $\mathbf{U}^*_{fin}(\mathscr{A},\mathscr{B})$, if for each sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\{St(\bigcup \mathcal{B}_n,\mathcal{A}_n) : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.
- $\mathbf{SS}^*_1(\mathscr{A},\mathscr{B})$, if for every sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(x_n : n \in \mathbb{N})$ of elements of X such that $\{St(x_n,\mathcal{A}_n) : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.
- $\mathbf{SS}^*_{fin}(\mathscr{A},\mathscr{B})$, if for any sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(K_n : n \in \mathbb{N})$ of finite subsets of X such that $\{St(K_n,\mathcal{A}_n) : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.

The particular cases $\mathbf{S}^*_1(\mathscr{O},\mathscr{O})$, $\mathbf{S}^*_{fin}(\mathscr{O},\mathscr{O})$, $\mathbf{SS}^*_1(\mathscr{O},\mathscr{O})$ and $\mathbf{SS}^*_{fin}(\mathscr{O},\mathscr{O})$ are known as the star-Rothberger property (*SR*), the star-Menger property (*SM*), the strongly star-Rothberger property (*SSR*) and the strongly star-Menger property (*SSM*), respectively (see [14]). On the other hand, $\mathbf{U}^*_{fin}(\mathscr{O},\Gamma)$ is the star-Hurewicz property (*SH*) and $\mathbf{SS}^*_{fin}(\mathscr{O},\Gamma)$ is the strongly star-Hurewicz property (*SSH*) (see [2]). Finally, in [4] is defined the absolute version of the strongly star-Hurewicz property (*aSSH*).

Diagram 1 provides relationships between the properties defined previously. These follow immediately from the definitions and it is known that they are not reversible. However, when X is a paracompact Hausdorff space, the properties Menger, star-Menger and strongly star-Menger are equivalent and the properties

Rothberger, star-Rothberger and strongly star-Rothberger are equivalent too (see [14]). Also, under the same hypothesis, the properties Hurewicz, star-Hurewicz and strongly star-Hurewicz are equivalent (see [2]). The authors recommend the excellent work in [17] to readers interested about these and some more results on star selection principles.

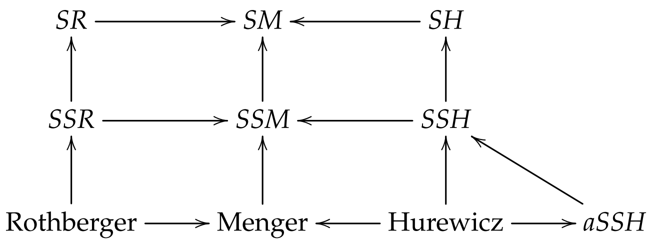

Diagram 1: Relationships between classic selection principles.

Many authors have made investigations on selection principles and star selection principles and interesting results have been obtained. See [1, 2, 4, 14–17, 23, 26–28], among other works.

Next, we present some basic concepts and notations about the theory of hyperspaces. To avoid trivialities, all spaces are assumed to be Hausdorff noncompact and, even, nonparacompact.

Given a topological space (X, τ), it is denoted by $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ and $\mathbb{CS}(X)$ the family of all nonempty closed subsets, the family of all nonempty compact subsets, the family of all nonempty finite subsets of X and the family of all convergent sequences of X, respectively.

For every subset $U \subseteq X$ and any family $\mathcal{U}$ of subsets of X, we write:

$$\begin{array}{lcl} U^- & = & \{A \in CL(X) : A \cap U \neq \emptyset\}; \\ U^+ & = & \{A \in CL(X) : A \subseteq U\}; \\ U^c & = & X \setminus U; \\ \mathcal{U}^c & = & \{U^c : U \in \mathcal{U}\}. \end{array}$$

Consider $\Delta \subseteq CL(X)$, closed under finite unions and containing all singletons. Then, the *hit-and-miss topology on $CL(X)$ respect to* Δ, and denoted by τ^+_Δ, is defined by means of the base

$$\left\{\left(\bigcap_{i=1}^m V_i^-\right) \cap (B^c)^+ : B \in \Delta \text{ and } V_i \in \tau \text{ for } i \in \{1, \ldots, m\}\right\}.$$

Following [30], the basic element $(\bigcap_{i=1}^m V_i^-) \cap (B^c)^+$ will be denoted by $(V_1, \ldots, V_m)^+_B$.

Two well known important particular cases of the hit-and-miss topology are the *Vietoris topology*, τ_V, when $\Delta = CL(X)$ (see [20, 29]), and the *Fell topology*, τ_F, when $\Delta = \mathbb{K}(X)$ (see [10]). Although in the literature there are several topologies that can be defined on $\mathbb{K}(X)$, $\mathbb{F}(X)$ and $\mathbb{CS}(X)$, throughout this work we will consider them as subspaces of $(CL(X), \tau^+_\Delta)$.

Along this paper, unless we say the opposite, we will consider a family $\Lambda \subseteq CL(X)$ such that it is closed under finite unions. Recall that $[A]^{<\omega}$ denotes the collection of all finite subsets of any set A.

Now, we reproduce the definitions of $\pi_\Delta(\Lambda)$-network and $c_\Delta(\Lambda)$-cover of a space X and a couple of lemmas which will be used along this work (see [5]).

Let $\mathcal{J}$ denote the family:

$$\mathcal{J} = \{(B; V_1, \ldots, V_n) : B \in \Delta \text{ and } V_1, \ldots, V_n \text{ are open subsets of } X \text{ with } V_i \cap B^c \neq \emptyset \ (1 \leq i \leq n),\ n \in \mathbb{N}\}.$$

Definition 1.1. *A family $\mathcal{J}$ is called a $\pi_\Delta(\Lambda)$-*network *of X, if for each $U \in \Lambda^c$, there exist $(B; V_1, \ldots, V_n) \in \mathcal{J}$ with $B \subseteq U$ and $F \in [X]^{<\omega}$ such that $F \cap U = \emptyset$ and for each $i \in \{1, \ldots, n\}$, $F \cap V_i \neq \emptyset$. The family of all $\pi_\Delta(\Lambda)$-networks is denoted by $\Pi_\Delta(\Lambda)$.*

Lemma 1.2. *Let (X, τ) be a topological space. Suppose that $\mathcal{J} = \{(B_s; V_{1,s}, \ldots, V_{m_s,s}) : s \in S\}$ and $\mathcal{U} = \left\{(V_{1,s}, \ldots, V_{m_s,s})^+_{B_s} : (B_s; V_{1,s}, \ldots, V_{m_s,s}) \in \mathcal{J}\right\}$. Then, $\mathcal{J}$ is a $\pi_\Delta(\Lambda)$-network of X if and only if $\mathcal{U}$ is an open cover of (Λ, τ^+_Δ).*

Definition 1.3. *Let (X, τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is called a* $c_\Delta(\Lambda)$-cover of X, *if for any $B \in \Delta$ and open subsets $V_1, \ldots, V_m$ of X, with $B^c \cap V_i \neq \emptyset$ for any $i \in \{1, \ldots, m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $B \subseteq U$, $F \cap U = \emptyset$ and for each $i \in \{1, \ldots, m\}$, $F \cap V_i \neq \emptyset$. We denote by $\mathbb{C}_\Delta(\Lambda)$ the family of all $c_\Delta(\Lambda)$-covers of X.*

Lemma 1.4. *Let (X, τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is a $c_\Delta(\Lambda)$-cover of X if and only if the family $\mathcal{U}^c$ is a dense subset of (Λ, τ^+_Δ).*

2. Characterization of H in hyperspaces

Definition 2.1. *A space X satisfies the* Hurewicz property *if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \mathcal{U}_n$, and each $x \in X$ belongs to $\bigcup \mathcal{V}_n$, for all but finitely many n.*

Definition 2.2. *Let (X, τ) be a topological space. We say that a family $\mathcal{J}$ is a $j\pi^\gamma_\Delta(\Lambda)$*-network of X induced by the sequences $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{J}'_n : n \in \mathbb{N})$, *where for any $n \in \mathbb{N}$, $\mathcal{J}_n$ is a $\pi_\Delta(\Lambda)$-network and $\mathcal{J}'_n \in [\mathcal{J}_n]^{<\omega}$, if $\mathcal{J} = \bigcup\{\mathcal{J}'_n : n \in \mathbb{N}\}$ and for any $U \in \Lambda^c$, there exists $(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}) \in \mathcal{J}'_n$ such that $B^n_s \subseteq U$ and $V^n_{i,s} \nsubseteq U$ $(1 \leq i \leq m_s)$, for all but finitely many n.*

Remark 2.3. *In the previous definition, $\mathcal{J}$ turns out to be a $\pi_\Delta(\Lambda)$-network of X.*

Theorem 2.4. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) (Λ, τ^+_Δ) has the Hurewicz property;

(2) (X, τ) satisfies the property: for any sequence $(\mathcal{J}_n : n \in \mathbb{N})$ of $\pi_\Delta(\Lambda)$-networks of X, there exists a sequence $(\mathcal{J}'_n : n \in \mathbb{N})$, with $\mathcal{J}'_n \in [\mathcal{J}_n]^{<\omega}$ for each n, such that $\mathcal{J}$ given as in Definition 2.2, is a $j\pi^\gamma_\Delta(\Lambda)$-network of X induced by the sequences $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{J}'_n : n \in \mathbb{N})$.

Proof. (1) $\Rightarrow$ (2): Let $(\mathcal{J}_n : n \in \mathbb{N})$ be a sequence of $\pi_\Delta(\Lambda)$-networks of X. We put, for each $n \in \mathbb{N}$, $\mathcal{J}_n = \{(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}) : s \in S_n\}$. From Lemma 1.2, for each $n \in \mathbb{N}$, the collection $\mathscr{U}_n = \left\{\left(V^n_{1,s}, \ldots, V^n_{m_s,s}\right)^+_{B^n_s} : s \in S_n\right\}$ is an open cover of (Λ, τ^+_Δ).

Now, applying (1) to the sequence $(\mathscr{U}_n : n \in \mathbb{N})$, there is $R_n \in [S_n]^{<\omega}$, for each $n \in \mathbb{N}$, such that $\mathscr{W}_n = \left\{\left(V^n_{1,r}, \ldots, V^n_{m_r,r}\right)^+_{B^n_r} : r \in R_n\right\}$ satisfies that for each $A \in \Lambda$, $A \in \bigcup \mathscr{W}_n$, for all but finitely many n. We put, for each $n \in \mathbb{N}$, $\mathcal{J}'_n = \left\{(B^n_r; V^n_{1,r}, \ldots, V^n_{m_r,r}) : r \in R_n\right\}$. Then, $\mathcal{J}'_n \in [\mathcal{J}_n]^{<\omega}$.

Let us show that $\mathcal{J} = \bigcup_{n \in \mathbb{N}} \mathcal{J}'_n$ is a $j\pi^\gamma_\Delta(\Lambda)$-network of X induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{J}'_n : n \in \mathbb{N})$. Let $U \in \Lambda^c$, then $U^c \in \Lambda$. Hence, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $U^c \in \bigcup \mathscr{W}_n$. Fix $n \geq n_0$, then $U^c \in \left(V^n_{1,r}, \ldots, V^n_{m_r,r}\right)^+_{B^n_r}$, for some element in $\mathcal{W}_n$. It follows that $B^n_r \subseteq U$ and $V^n_{i,r} \nsubseteq U$, for any $i \in \{1, \ldots, m_r\}$. Given that $(B^n_r; V^n_{1,r}, \ldots, V^n_{m_r,r}) \in \mathcal{J}'_n$, (2) holds.

(2) $\Rightarrow$ (1): Let $\mathscr{U}_n = \left\{\left(V^n_{1,s}, \ldots, V^n_{m_s,s}\right)^+_{B^n_s} : s \in S_n\right\}$ be a sequence of open covers of (Λ, τ^+_Δ), where $B^n_s \in \Delta$ and $V^n_{i,s}$ is an open subset of X with $V^n_{i,s} \cap (B^n_s)^c \neq \emptyset$, for each $i \in \{1, \ldots, m_s\}$. For any $n \in \mathbb{N}$, let $\mathcal{J}_n = \left\{\left(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}\right) : s \in S_n\right\}$. So, from Lemma 1.2, we have that each $\mathcal{J}_n$ is a $\pi_\Delta(\Lambda)$-network of X.

We apply condition (2) to the sequence $(\mathcal{J}_n : n \in \mathbb{N})$ to obtain a sequence $(\mathcal{J}'_n : n \in \mathbb{N})$, with $\mathcal{J}'_n \in [\mathcal{J}_n]^{<\omega}$ for each n, such that $\mathcal{J} = \bigcup_{n \in \mathbb{N}} \mathcal{J}'_n$ is a $j\pi^\gamma_\Delta(\Lambda)$-network of X induced by the sequences $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{J}'_n : n \in \mathbb{N})$. Suppose that $\mathcal{J}'_n = \left\{(B^n_r; V^n_{1,r}, \ldots, V^n_{m_r,r}) : r \in R_n\right\}$, where $R_n \in [S_n]^{<\omega}$. We have $\mathscr{W}_n = \left\{\left(V^n_{1,r}, \ldots, V^n_{m_r,r}\right)^+_{B^n_r} : r \in R_n\right\} \subseteq \mathscr{U}_n$.

Let $A \in \Lambda$, as $A^c \in \Lambda^c$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, there is $(B^n_r; V^n_{1,r}, \ldots, V^n_{m_r,r}) \in \mathcal{J}'_n$ such that $B^n_r \subseteq A^c$ and $V^n_{i,r} \nsubseteq A^c$ $(1 \leq i \leq m_r)$. Hence, we get $A \in \left(V^n_{1,r}, \ldots, V^n_{m_r,r}\right)^+_{B^n_r} \in \mathscr{W}_n$. Therefore, (Λ, τ^+_Δ) satisfies the Hurewicz property. □

As some immediate consequences of Theorem 2.4, we get the following.

Remark 2.5. *Let (X,τ) be a topological space. If Λ is any of the hyperspaces $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then (Λ,τ^+_Δ) has the Hurewicz property if and only if X satisfies the property: for any sequence $(\mathcal{J}_n : n\in\mathbb{N})$ of $\pi_\Delta(\Lambda)$-networks of X, there exists a sequence $(\mathcal{J}'_n : n\in\mathbb{N})$, with $\mathcal{J}'_n\in[\mathcal{J}_n]^{<\omega}$ for each n, such that $\mathcal{J}$ given as in Definition 2.2, is a $j\pi^\gamma_\Delta(\Lambda)$-network of X induced by the sequences $(\mathcal{J}_n : n\in\mathbb{N})$ and $(\mathcal{J}'_n : n\in\mathbb{N})$.*

When Λ is endowed with Fell or Vietoris topology, we get the next particular cases.

Corollary 2.6. *Let (X,τ) be a topological space and let Λ be any of the hyperspaces $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then*

1. *(Λ,τ_F) has the Hurewicz property if and only if X satisfies the property: for any sequence $(\mathcal{J}_n : n\in\mathbb{N})$ of $\pi_{\mathbb{K}(X)}(\Lambda)$-networks of X, there exists a sequence $(\mathcal{J}'_n : n\in\mathbb{N})$, with $\mathcal{J}'_n\in[\mathcal{J}_n]^{<\omega}$ for each n, such that $\mathcal{J}$ given as in Definition 2.2, is a $j\pi^\gamma_{\mathbb{K}(X)}(\Lambda)$-network of X induced by the sequences $(\mathcal{J}_n : n\in\mathbb{N})$ and $(\mathcal{J}'_n : n\in\mathbb{N})$.*
2. *(Λ,τ_V) has the Hurewicz property if and only if X satisfies the property: for any sequence $(\mathcal{J}_n : n\in\mathbb{N})$ of $\pi_{CL(X)}(\Lambda)$-networks of X, there exists a sequence $(\mathcal{J}'_n : n\in\mathbb{N})$, with $\mathcal{J}'_n\in[\mathcal{J}_n]^{<\omega}$ for each n, such that $\mathcal{J}$ given as in Definition 2.2, is a $j\pi^\gamma_{CL(X)}(\Lambda)$-network of X induced by the sequences $(\mathcal{J}_n : n\in\mathbb{N})$ and $(\mathcal{J}'_n : n\in\mathbb{N})$.*

3. Characterization of *SSH* in hyperspaces

Definition 3.1. *A space (X,τ) satisfies the* strongly star-Hurewicz property (SSH) *if for all sequence $(\mathcal{U}_n : n\in\mathbb{N})$ of open covers of X, there is a sequence $(A_n : n\in\mathbb{N})$ of finite subsets of X such that each $x\in X$ belongs to $St(A_n,\mathcal{U}_n)$ for all but finitely many n.*

Definition 3.2. *Let (X,τ) be a topological space. We say that a family $\mathcal{J}$ is a $k\pi^\gamma_\Delta(\Lambda)$*-network of X induced by the sequences $(\mathcal{J}_n : n\in\mathbb{N})$ and $(\mathcal{V}_n : n\in\mathbb{N})$, *where for any $n\in\mathbb{N}$, $\mathcal{J}_n$ is a $\pi_\Delta(\Lambda)$-network and $\emptyset\neq\mathcal{V}_n\subseteq\Lambda^c$, if are satisfied:*

(a) *$\mathcal{J}=\bigcup_{n\in\mathbb{N}}\mathcal{K}_n$, where*

$$\mathcal{K}_n=\Big\{(B^n_s;V^n_{1,s},\ldots,V^n_{m_s,s})\in\mathcal{J}_n:\ \text{there exists}\ \ V\in\mathcal{V}_n \text{ with } B^n_s\subseteq V, V^n_{i,s}\nsubseteq V\ (1\leq i\leq m_s)\Big\}.$$

(b) *For any $U\in\Lambda^c$, there exists $(B^n_s;V^n_{1,s},\ldots,V^n_{m_s,s})\in\mathcal{K}_n$ such that $B^n_s\subseteq U$ and $V^n_{i,s}\nsubseteq U$ $(1\leq i\leq m_s)$, for all but finitely many n.*

Remark 3.3. *In the previous definition, $\mathcal{J}$ turns out to be a $\pi_\Delta(\Lambda)$-network of X.*

Next, we use the $k\pi^\gamma_\Delta(\Lambda)$-networks of X to characterize, as a particular case of the next generic theorem, the strong star Hurewicz property for $CL(X)$ and its subspaces endowed with the hit-and-miss topology.

Theorem 3.4. *Let (X,τ) be a topological space. The following conditions are equivalent:*

(1) (Λ,τ^+_Δ) *is* SSH;

(2) *(X,τ) satisfies the property: for any sequence $(\mathcal{J}_n : n\in\mathbb{N})$ of $\pi_\Delta(\Lambda)$-networks, there exists a sequence $(\mathcal{V}_n : n\in\mathbb{N})$ in $[\Lambda^c]^{<\omega}$, with $\mathcal{V}_n\neq\emptyset$ for each $n\in\mathbb{N}$, such that $\mathcal{J}$, defined as (a) in Definition 3.2 is a $k\pi^\gamma_\Delta(\Lambda)$-network induced by $(\mathcal{J}_n : n\in\mathbb{N})$ and $(\mathcal{V}_n : n\in\mathbb{N})$.*

Proof. (1) ⇒ (2): Let $(\mathcal{J}_n : n\in\mathbb{N})$ be a sequence of $\pi_\Delta(\Lambda)$-networks of X. We put, for each $n\in\mathbb{N}$, $\mathcal{J}_n=\{(B^n_s;V^n_{1,s},\ldots,V^n_{m_s,s}): s\in S_n\}$. From Lemma 1.2, for each $n\in\mathbb{N}$, the collection $\mathscr{U}_n=\Big\{\big(V^n_{1,s},\ldots,V^n_{m_s,s}\big)^+_{B^n_s}: s\in S_n\Big\}$ is an open cover of (Λ,τ^+_Δ).

Now, applying (1) to the sequence $(\mathscr{U}_n : n\in\mathbb{N})$, we choose $\mathscr{A}_n\in[\Lambda]^{<\omega}$, for any $n\in\mathbb{N}$, such that for each $A\in\Lambda$, $A\in St(\mathscr{A}_n,\mathscr{U}_n)$, for all but finitely many n. We put, for each $n\in\mathbb{N}$, $\mathcal{V}_n=\mathscr{A}^c_n$. Then, $(\mathcal{V}_n : n\in\mathbb{N})$ is a sequence in $[\Lambda^c]^{<\omega}$.

Let us show that $\mathcal{J}=\bigcup_{n\in\mathbb{N}}\mathcal{K}_n$ is a $k\pi^\gamma_\Delta(\Lambda)$-network of X induced by the sequences $(\mathcal{J}_n : n\in\mathbb{N})$ and $(\mathcal{V}_n : n\in\mathbb{N})$, where

$$\mathcal{K}_n = \left\{(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}) \in \mathcal{J}_n : \text{ there exists } V \in \mathcal{V}_n \text{ with } B^n_s \subseteq V,\ \ V^n_{i,s} \nsubseteq V\ (1 \le i \le m_s)\right\}.$$

It is enough to show (b) of Definition 3.2. Let $U \in \Lambda^c$, then $U^c \in \Lambda$. Hence, there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$, $U^c \in St(\mathcal{A}_n, \mathcal{U}_n)$. Fix $n \ge n_0$, then there is $A \in \mathcal{A}_n$ and $\left(V^n_{1,s}, \dots, V^n_{m_s,s}\right)^+_{B^n_s} \in \mathcal{U}_n$ so that $\{U^c, A\} \subseteq \left(V^n_{1,s}, \dots, V^n_{m_s,s}\right)^+_{B^n_s}$. Since $U^c \in \left(V^n_{1,s}, \dots, V^n_{m_s,s}\right)^+_{B^n_s}$, it follows that $B^n_s \subseteq U$ and for any $i \in \{1, \dots, m_s\}$, $V^n_{i,s} \nsubseteq U$. On the other hand, $A^c \in \mathcal{V}_n$; as $A \in \left(V^n_{1,s}, \dots, V^n_{m_s,s}\right)^+_{B^n_s}$, $B^n_s \subseteq A^c$ and $V^n_{i,s} \nsubseteq A^c$. So, A^c witnesses that $\left(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}\right) \in \mathcal{K}_n$. Thus, we obtain $\left(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}\right) \in \mathcal{K}_n$ for all but finitely many $n \in \mathbb{N}$, such that $B^n_s \subseteq U$ and $V^n_{i,s} \nsubseteq U$ $(1 \le i \le m_s)$.

(2) ⇒ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of (Λ, τ^+_Δ). Without loss of generality, we can suppose that every open cover $\mathcal{U}_n$ consists of basic open sets and $\mathcal{U}_n = \left\{\left(V^n_{1,s}, \dots, V^n_{m_s,s}\right)^+_{B^n_s} : s \in S_n\right\}$. For each $n \in \mathbb{N}$, let $\mathcal{J}_n = \left\{\left(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}\right) : s \in S_n\right\}$. So, from Lemma 1.2, we have that each $\mathcal{J}_n$ is a $\pi_\Delta(\Lambda)$-network of X.

We apply condition (2) to the sequence $(\mathcal{J}_n : n \in \mathbb{N})$ to obtain a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ in $[\Lambda^c]^{<\omega}$, with $\mathcal{V}_n \neq \emptyset$ for each $n \in \mathbb{N}$, such that $\mathcal{J} = \bigcup_{n\in\mathbb{N}} \mathcal{K}_n$, where

$$\mathcal{K}_n = \left\{(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}) \in \mathcal{J}_n : \text{ there exists } V \in \mathcal{V}_n \text{ with } B^n_s \subseteq V,\ \ V^n_{i,s} \nsubseteq V\ (1 \le i \le m_s)\right\},$$

satisfies that for any $U \in \Lambda^c$, there exists $(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}) \in \mathcal{K}_n$ such that $B^n_s \subseteq U$, $V^n_{i,s} \nsubseteq U$ $(1 \le i \le m_s)$ for all but finitely many n.

For each $n \in \mathbb{N}$, put $\mathcal{A}_n = \mathcal{V}^c_n$. Thus, $\{\mathcal{A}_n : n \in \mathbb{N}\} \subseteq [\Lambda]^{<\omega}$. Let us show that the sequence $(St(\mathcal{A}_n, \mathcal{U}_n) : n \in \mathbb{N})$ is a γ-cover of (Λ, τ^+_Δ). Given $A \in \Lambda$, as $A^c \in \Lambda^c$, there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$, there is $\left(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}\right) \in \mathcal{K}_n$ such that $B^n_s \subseteq A^c$, $V^n_{i,s} \nsubseteq A^c$ $(1 \le i \le m_s)$. Fix $n \ge n_0$, thus $A \in \left(V^n_{1,s}, \dots, V^n_{m_s,s}\right)^+_{B^n_s}$. As $\left(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}\right) \in \mathcal{K}_n$, there is $V_n \in \mathcal{V}_n$ with $B^n_s \subseteq V_n$, $V^n_{i,s} \nsubseteq V_n$ $(1 \le i \le m_s)$. So, $V^c_n \in \mathcal{A}_n$ and $V^c_n \in \left(V^n_{1,s}, \dots, V^n_{m_s,s}\right)^+_{B^n_s}$. It follows that $A \in St(\mathcal{A}_n, \mathcal{U}_n)$. We conclude that $A \in St(\mathcal{A}_n, \mathcal{U}_n)$, for all but finitely many n. □

As some immediate consequences of Theorem 3.4, we get the following.

Remark 3.5. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then (Λ, τ^+_Δ) is* SSH *if and only if X satisfies the property: for any sequence $(\mathcal{J}_n : n \in \mathbb{N})$ of $\pi_\Delta(\Lambda)$-networks, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ in $[\Lambda^c]^{<\omega}$, with $\mathcal{V}_n \neq \emptyset$ for each $n \in \mathbb{N}$, such that $\mathcal{J}$, defined as (a) in Definition 3.2 is a $k\pi^\gamma_\Delta(\Lambda)$-network induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{V}_n : n \in \mathbb{N})$.*

If Λ is endowed with Fell or Vietoris topology, we get the next particular cases.

Corollary 3.6. *Let (X, τ) be a topological space and let Λ be any of the hyperspaces $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then*

1. *(Λ, τ_F) is* SSH *if and only if X satisfies the property: for any sequence $(\mathcal{J}_n : n \in \mathbb{N})$ of $\pi_{\mathbb{K}(X)}(\Lambda)$-networks, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ in $[\Lambda^c]^{<\omega}$, with $\mathcal{V}_n \neq \emptyset$ for each $n \in \mathbb{N}$, such that $\mathcal{J}$, defined as (a) in Definition 3.2 is a $k\pi^\gamma_{\mathbb{K}(X)}(\Lambda)$-network induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{V}_n : n \in \mathbb{N})$.*
2. *(Λ, τ_V) is* SSH *if and only if X satisfies the property: for any sequence $(\mathcal{J}_n : n \in \mathbb{N})$ of $\pi_{CL(X)}(\Lambda)$-networks, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ in $[\Lambda^c]^{<\omega}$, with $\mathcal{V}_n \neq \emptyset$ for each $n \in \mathbb{N}$, such that $\mathcal{J}$, defined as (a) in Definition 3.2 is a $k\pi^\gamma_{CL(X)}(\Lambda)$-network induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{V}_n : n \in \mathbb{N})$.*

4. Characterization of *SH* in hyperspaces

Definition 4.1. *A space X satisfies the* star-Hurewicz property (SH) *if for each sequence* $(\mathcal{U}_n : n \in \mathbb{N})$ *of open covers of X, there is a sequence* $(\mathcal{V}_n : n \in \mathbb{N})$ *such that for each* $n \in \mathbb{N}$, $\mathcal{V}_n$ *is a finite subset of* $\mathcal{U}_n$ *and each* $x \in X$ *belongs to* $St(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ *for all but finitely many n.*

Definition 4.2. *Let* (X, τ) *be a topological space. We say that a family* $\mathcal{J}$ *is a* $J\pi^{\gamma}_{\Delta}(\Lambda)$-network of X induced by the sequences $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{J}'_n : n \in \mathbb{N})$, *where for any* $n \in \mathbb{N}$, $\mathcal{J}_n$ *is a* $\pi_\Delta(\Lambda)$*-network and* $\mathcal{J}'_n \in [\mathcal{J}_n]^{<\omega}$, *if are satisfied:*

(a) $\mathcal{J} = \bigcup_{n\in\mathbb{N}} \mathcal{K}_n$, *where*

$$\mathcal{K}_n = \Big\{(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}) \in \mathcal{J}_n : \textit{there exist } (B^n_r; V^n_{1,r}, \ldots, V^n_{m_r,r}) \in \mathcal{J}'_n \textit{ and } F \in [X]^{<\omega} \textit{ with } F \subseteq (B^n_r \cup B^n_s)^c \textit{ and } F \cap V^n_{i,r} \neq \emptyset \neq F \cap V^n_{j,s}, \textit{ for any } i, j\Big\},$$

(b) *For any* $U \in \Lambda^c$, *there exists* $(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}) \in \mathcal{K}_n$ *such that* $B^n_s \subseteq U$ *and* $V^n_{i,s} \not\subseteq U$ $(1 \le i \le m_s)$, *for all but finitely many n.*

Remark 4.3. *In the previous definition,* $\mathcal{J}$ *turns out to be a* $\pi_\Delta(\Lambda)$*-network of X.*

Theorem 4.4. *Let* (X, τ) *be a topological space. Suppose furthermore that for every* $x \in X$, $\{x\} \in \Gamma$. *The following conditions are equivalent:*

(1) (Λ, τ^+_Δ) *is* SH;
(2) (X, τ) *satisfies the property: for any sequence* $(\mathcal{J}_n : n \in \mathbb{N})$ *of* $\pi_\Delta(\Lambda)$*-networks of X, there exists a sequence* $(\mathcal{J}'_n : n \in \mathbb{N})$, *with* $\mathcal{J}'_n \in [\mathcal{J}_n]^{<\omega}$ *for each n, such that* $\mathcal{J}$ *given as in (a) of Definition 4.2, is a* $J\pi^{\gamma}_{\Delta}(\Lambda)$*-network of X induced by the sequences* $(\mathcal{J}_n : n \in \mathbb{N})$ *and* $(\mathcal{J}'_n : n \in \mathbb{N})$.

Proof. (1) ⇒ (2): Let $(\mathcal{J}_n : n \in \mathbb{N})$ be a sequence of $\pi_\Delta(\Lambda)$-networks of X. We put, for each $n \in \mathbb{N}$, $\mathcal{J}_n = \{(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}) : s \in S_n\}$. From Lemma 1.2, for each $n \in \mathbb{N}$, the collection $\mathscr{U}_n = \Big\{\Big(V^n_{1,s}, \ldots, V^n_{m_s,s}\Big)^+_{B^n_s} : s \in S_n\Big\}$ is an open cover of (Λ, τ^+_Δ).

Now, applying (1) to the sequence $(\mathscr{U}_n : n \in \mathbb{N})$, there is $R_n \in [S_n]^{<\omega}$, for each $n \in \mathbb{N}$, such that $\mathscr{W}_n = \Big\{\Big(V^n_{1,r}, \ldots, V^n_{m_r,r}\Big)^+_{B^n_r} : r \in R_n\Big\}$ satisfies that for each $A \in \Lambda$, $A \in St(\bigcup \mathscr{W}_n, \mathscr{U}_n)$, for all but finitely many n. We put, for each $n \in \mathbb{N}$, $\mathcal{J}'_n = \Big\{(B^n_r; V^n_{1,r}, \ldots, V^n_{m_r,r}) : r \in R_n\Big\}$. Then, $\mathcal{J}'_n \in [\mathcal{J}_n]^{<\omega}$.

Let us show that $\mathcal{J} = \bigcup_{n\in\mathbb{N}} \mathcal{K}_n$ is a $J\pi^{\gamma}_{\Delta}(\Lambda)$-network of X induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{J}'_n : n \in \mathbb{N})$, where

$$\mathcal{K}_n = \Big\{(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}) \in \mathcal{J}_n : \text{there exist } (B^n_r; V^n_{1,r}, \ldots, V^n_{m_r,r}) \in \mathcal{J}'_n \text{ and } F \in [X]^{<\omega} \text{ with } F \subseteq (B^n_r \cup B^n_s)^c \text{ and } F \cap V^n_{i,r} \neq \emptyset \neq F \cap V^n_{j,s}, \text{ for any } i, j\Big\}.$$

Let $U \in \Lambda^c$, then $U^c \in \Lambda$. Hence, there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$, $U^c \in St(\bigcup \mathscr{W}_n, \mathscr{U}_n)$.

Fix $n \ge n_0$, then there is $A \in \bigcup \mathscr{W}_n$ and $\Big(V^n_{1,s}, \ldots, V^n_{m_s,s}\Big)^+_{B^n_s} \in \mathscr{U}_n$ so that $\{U^c, A\} \subseteq \Big(V^n_{1,s}, \ldots, V^n_{m_s,s}\Big)^+_{B^n_s}$. Furthermore, there exists $r \in R_n$ such that $A \in \Big(V^n_{1,r}, \ldots, V^n_{m_r,r}\Big)^+_{B^n_r}$.

We have $(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}) \in \mathcal{K}_n$. Indeed, $(B^n_r; V^n_{1,r}, \ldots, V^n_{m_r,r}) \in \mathcal{J}'_n$. Since $A \in \Big(V^n_{1,s}, \ldots, V^n_{m_s,s}\Big)^+_{B^n_s}$, it follows that $B^n_s \subseteq A^c$ and for any $j \in \{1, \ldots, m_s\}$, $V^n_{j,s} \not\subseteq A^c$. So, for any $j \in \{1, \ldots, m_s\}$, we can choose $x_j \in A \cap V^n_{j,s}$. On the other hand, as $A \in \Big(V^n_{1,r}, \ldots, V^n_{m_r,r}\Big)^+_{B^n_r}$, it follows that $B^n_r \subseteq A^c$ and for any $i \in \{1, \ldots, m_r\}$, $V^n_{i,r} \not\subseteq A^c$. So, for any $i \in \{1, \ldots, m_r\}$, there is $y_i \in A \cap V^n_{i,r}$. It can be shown that $F = \{x_j : j \in \{1, \ldots, m_s\}\} \cup \{y_i : i \in \{1, \ldots, m_r\}\}$ satisfies that $F \in [X]^{<\omega}$, $F \subseteq (B^n_r \cup B^n_s)^c$ and $F \cap V^n_{i,r} \neq \emptyset \neq F \cap V^n_{j,s}$, for any i and j.

Now, since $U^c \in \left(V^n_{1,s}, \ldots, V^n_{m_s,s}\right)^+_{B^n_s}$, it follows that $B^n_s \subseteq U$ and for any $j \in \{1, \ldots, m_s\}$, $V^n_{j,s} \nsubseteq U$. It proves (b) of Definition 4.2.

(2) ⇒ (1): Let $\mathscr{U}_n = \left\{\left(V^n_{1,s}, \ldots, V^n_{m_s,s}\right)^+_{B^n_s} : s \in S_n\right\}$ be a sequence of open covers of (Λ, τ^+_Δ), where $B^n_s \in \Delta$ and $V^n_{i,s}$ is an open subset of X with $V^n_{i,s} \cap (B^n_s)^c \neq \emptyset$, for each $i \in \{1, \ldots, m_s\}$. For any $n \in \mathbb{N}$, let $\mathcal{J}_n = \left\{\left(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}\right) : s \in S_n\right\}$. So, from Lemma 1.2, we have that each $\mathcal{J}_n$ is a $\pi_\Delta(\Lambda)$-network of X.

We apply condition (2) to the sequence $(\mathcal{J}_n : n \in \mathbb{N})$ to obtain a sequence $(\mathcal{J}'_n : n \in \mathbb{N})$, with $\mathcal{J}'_n \in [\mathcal{J}_n]^{<\omega}$ for each n, such that $\mathcal{J} = \bigcup_{n \in \mathbb{N}} \mathcal{K}_n$ is a $J\pi^\gamma_\Delta(\Lambda)$-network of X induced by the sequences $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{J}'_n : n \in \mathbb{N})$, where

$$\mathcal{K}_n = \Big\{(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}) \in \mathcal{J}_n : \text{there exist } (B^n_r; V^n_{1,r}, \ldots, V^n_{m_r,r}) \in \mathcal{J}'_n \text{ and } F \in [X]^{<\omega} \text{ with } F \subseteq (B^n_r \cup B^n_s)^c \text{ and } F \cap V^n_{i,r} \neq \emptyset \neq F \cap V^n_{j,s}, \text{ for any } i, j\Big\}.$$

Suppose that $\mathcal{J}'_n = \left\{(B^n_r; V^n_{1,r}, \ldots, V^n_{m_r,r}) : r \in R_n\right\}$, where $R_n \in [S_n]^{<\omega}$. Let $\mathscr{W}_n = \left\{\left(V^n_{1,r}, \ldots, V^n_{m_r,r}\right)^+_{B^n_r} : r \in R_n\right\}$.

Now, given $A \in \Lambda$, as $A^c \in \Lambda^c$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, there is $(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}) \in \mathcal{K}_n$ such that $B^n_s \subseteq A^c$, $V^n_{i,s} \nsubseteq A^c$ $(1 \leq i \leq m_s)$. Fix $n \geq n_0$, hence, we get $A \in \left(V^n_{1,s}, \ldots, V^n_{m_s,s}\right)^+_{B^n_s} \in \mathscr{U}_n$. On the other hand, as $(B^n_s; V^n_{1,s}, \ldots, V^n_{m_s,s}) \in \mathcal{K}_n$, there exist $(B^n_r; V^n_{1,r}, \ldots, V^n_{m_r,r}) \in \mathcal{J}'_n$ and $F \in [X]^{<\omega}$, with $F \subseteq (B^n_r \cup B^n_s)^c$ and $F \cap V^n_{i,r} \neq \emptyset \neq F \cap V^n_{j,s}$, for any i and j, with $1 \leq i \leq m_r$ and $1 \leq j \leq m_s$. It implies that $F \in \left(V^n_{1,s}, \ldots, V^n_{m_s,s}\right)^+_{B^n_s} \cap \left(V^n_{1,s}, \ldots, V^n_{m_r,r}\right)^+_{B^n_r}$. So, $A \in St(\bigcup \mathscr{W}_n, \mathscr{U}_n)$. Therefore, (Λ, τ^+_Δ) is *SH*. □

As some immediate consequences of Theorem 4.4, we get the following.

Remark 4.5. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then (Λ, τ^+_Δ) is* SH *if and only if X satisfies the property: for any sequence $(\mathcal{J}_n : n \in \mathbb{N})$ of $\pi_\Delta(\Lambda)$-networks of X, there exists a sequence $(\mathcal{J}'_n : n \in \mathbb{N})$, with $\mathcal{J}'_n \in [\mathcal{J}_n]^{<\omega}$ for each n, such that $\mathcal{J}$ given as in (a) of Definition 4.2, is a $J\pi^\gamma_\Delta(\Lambda)$-network of X induced by the sequences $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{J}'_n : n \in \mathbb{N})$.*

Whenever that Λ is endowed with Fell or Vietoris topology, we get the next particular cases.

Corollary 4.6. *Let (X, τ) be a topological space and let Λ be any of the hyperspaces $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then*

1. *(Λ, τ_F) is* SH *if and only if X satisfies the property: for any sequence $(\mathcal{J}_n : n \in \mathbb{N})$ of $\pi_{\mathbb{K}(X)}(\Lambda)$-networks of X, there exists a sequence $(\mathcal{J}'_n : n \in \mathbb{N})$, with $\mathcal{J}'_n \in [\mathcal{J}_n]^{<\omega}$ for each n, such that $\mathcal{J}$ given as in (a) of Definition 4.2, is a $J\pi^\gamma_{\mathbb{K}(X)}(\Lambda)$-network of X induced by the sequences $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{J}'_n : n \in \mathbb{N})$.*
2. *(Λ, τ_V) is* SH *if and only if X satisfies the property: for any sequence $(\mathcal{J}_n : n \in \mathbb{N})$ of $\pi_{CL(X)}(\Lambda)$-networks of X, there exists a sequence $(\mathcal{J}'_n : n \in \mathbb{N})$, with $\mathcal{J}'_n \in [\mathcal{J}_n]^{<\omega}$ for each n, such that $\mathcal{J}$ given as in (a) of Definition 4.2, is a $J\pi^\gamma_{CL(X)}(\Lambda)$-network of X induced by the sequences $(\mathcal{J}_n : n \in \mathbb{N})$ and $(\mathcal{J}'_n : n \in \mathbb{N})$.*

5. Characterization of *aSSH* in hyperspaces

Definition 5.1. *A topological space X is* absolutely strongly star-Hurewicz *(*aSSH*) if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, and each dense subset $D \subseteq X$, there exists a sequence $\{F_n : n \in \mathbb{N}\}$ so that for each $n \in \mathbb{N}$, $F_n \in [D]^{<\omega}$ and $\{St(F_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a γ-cover of X.*

Definition 5.2. *Let (X, τ) be a topological space. We say that $\mathcal{J}$ is a* $c\pi^{\gamma}_{\Delta}(\Lambda)$-network of X induced by a sequence $(\mathcal{J}_n : n \in \mathbb{N})$ of $\pi_{\Delta}(\Lambda)$-networks and by $C \in \mathbb{C}_{\Delta}(\Lambda)$, *if are satisfied:*

(a) *There exist finite subsets $\mathcal{V}_n \subseteq C$, such that $\mathcal{J} = \bigcup\{\mathcal{J}'_n : n \in \mathbb{N}\}$, where $\mathcal{J}'_n = \{(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}) \in \mathcal{J}_n :$ there exists $V \in \mathcal{V}_n$ such that $B^n_s \subseteq V$, $V^n_{i,s} \not\subseteq V$ $(1 \le i \le m_s)\}$.*

(b) *For any $U \in \Lambda^c$, there is $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$, there are $(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}) \in \mathcal{J}'_n$ and $F_n \in [X]^{<\omega}$ such that $B^n_s \subseteq U$, $F_n \cap V^n_{i,s} \neq \emptyset$ $(1 \le i \le m_s)$ and $F_n \cap U = \emptyset$.*

Remark 5.3. *In the previous definition, it follows that $\mathcal{J}$ is a $\pi_{\Delta}(\Lambda)$-network of X.*

Theorem 5.4. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) $(\Lambda, \tau^+_{\Delta})$ *is* aSSH;

(2) (X, τ) *satisfies the property: for each sequence $(\mathcal{J}_n : n \in \mathbb{N})$ in $\Pi_{\Delta}(\Lambda)$ and each $C \in \mathbb{C}_{\Delta}(\Lambda)$, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite subsets of C such that $\mathcal{J}$ defined as in (a) of Definition 5.2 is a $c\pi^{\gamma}_{\Delta}(\Lambda)$-network of X induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and C.*

Proof. (1) $\Rightarrow$ (2): Let $(\mathcal{J}_n : n \in \mathbb{N})$ be a sequence of $\pi_{\Delta}(\Lambda)$-networks of X and let $C \in \mathbb{C}_{\Delta}(\Lambda)$. By Lemma 1.4, $\mathcal{D} = C^c$ is a dense subset of Λ. Even more, from Lemma 1.2, for each $n \in \mathbb{N}$, the collection $\mathcal{U}_n = \left\{(V^n_1, \dots, V^n_m)^+_{B^n} : (B^n; V^n_1, \dots, V^n_m) \in \mathcal{J}_n\right\}$ is an open cover of $(\Lambda, \tau^+_{\Delta})$.

Now, applying (1) to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ and to $\mathcal{D}$, there exists a sequence $(\mathcal{D}_n : n \in \mathbb{N})$ so that for each $n \in \mathbb{N}$, $\mathcal{D}_n \in [\mathcal{D}]^{<\omega}$ and $\{St(\mathcal{D}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a γ-cover of Λ.

We put, for each $n \in \mathbb{N}$, $\mathcal{V}_n = \mathcal{D}^c_n$. Clearly $\mathcal{V}_n \in [C]^{<\omega}$, for any $n \in \mathbb{N}$. Let us show that $\mathcal{J} = \bigcup\{\mathcal{J}'_n : n \in \mathbb{N}\}$, where $\mathcal{J}'_n = \{(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}) \in \mathcal{J}_n :$ there exists $U_n \in \mathcal{V}_n$ such that $B^n_s \subseteq U_n$, $V^n_{i,s} \not\subseteq U_n$ $(1 \le i \le m_s)\}$, is a $c\pi^{\gamma}_{\Delta}(\Lambda)$-network of X induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and C.

Indeed, by construction (a) of Definition 5.2 is satisfied. Now let $U \in \Lambda^c$. Thus, $U^c \in \Lambda$ and therefore, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $U^c \in St(\mathcal{D}_n, \mathcal{U}_n)$. Then, for each $n \ge n_0$, there are $(V^n_{1,s}, \dots, V^n_{m_s,s})^+_{B^n_s} \in \mathcal{U}_n$ and $D_n \in \mathcal{D}_n$ so that $\{U^c, D_n\} \subseteq (V^n_{1,s}, \dots, V^n_{m_s,s})^+_{B^n_s}$. Let $U_n = D^c_n \in \mathcal{D}^c_n = \mathcal{V}_n$. It is not difficult to show that $B^n_s \subseteq U_n$ and $V^n_{i,s} \not\subseteq U_n$ $(1 \le i \le m_s)$. Hence, $(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}) \in \mathcal{J}'_n$, for each $n \ge n_0$. Furthermore, let $n \ge n_0$; for each $i \in \{1, \dots, m_s\}$, we take $x^n_i \in U^c \cap V^n_{i,s}$. We put $F_n = \{x^n_i : i \in \{1, \dots, m_s\}\}$. Hence, $F_n \in [X]^{<\omega}$ with $F_n \cap V^n_{i,s} \neq \emptyset$, for each $n \ge n_0$. Moreover, since $U^c \subseteq (B^n_s)^c$, for each $n \ge n_0$, then we obtain that $B^n_s \subseteq U$. Finally, $F_n \cap U = \emptyset$. We conclude that $\mathcal{J}$ is a $c\pi^{\gamma}_{\Delta}(\Lambda)$-network induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and C.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $(\Lambda, \tau^+_{\Delta})$ and let $\mathcal{D}$ be a dense subset of Λ. Without loss of generality, we can suppose that, for any $n \in \mathbb{N}$, the open cover $\mathcal{U}_n$ consists of basic open subsets in Λ, that is, $\mathcal{U}_n = \left\{(V^n_{1,s}, \dots, V^n_{m_s,s})^+_{B^n_s} : s \in S_n\right\}$, where $B^n_s \in \Delta$ and $V^n_{i,s}$ is an open subset of X, for every $i \in \{1, \dots, m_s\}$. For each $n \in \mathbb{N}$, let $\mathcal{J}_n = \{(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}) : s \in S_n\}$. Notice that, by Lemma 1.2, for each $n \in \mathbb{N}$, $\mathcal{J}_n$ is a $\pi_{\Delta}(\Lambda)$-network of X. Let $C = \mathcal{D}^c$, by Lemma 1.4, we have $C \in \mathbb{C}_{\Delta}(\Lambda)$.

We apply the condition (2) to the sequence $(\mathcal{J}_n : n \in \mathbb{N})$ and to C to obtain $\{\mathcal{V}_n : n \in \mathbb{N}\} \subseteq [C]^{<\omega}$ such that $\mathcal{J} = \bigcup\{\mathcal{J}'_n : n \in \mathbb{N}\}$, where $\mathcal{J}'_n = \{(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}) \in \mathcal{J}_n :$ there exists $U_n \in \mathcal{V}_n$ such that $B^n_s \subseteq U_n$, $V^n_{i,s} \not\subseteq U_n$ $(1 \le i \le m_s)\}$, is a $c\pi^{\gamma}_{\Delta}(\Lambda)$-network of X induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and C.

Let us show that the collection $\{St(\mathcal{D}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$, where $\mathcal{D}_n = \mathcal{V}^c_n$ satisfies that for each $A \in \Lambda$, there exists $n_0 \in \mathbb{N}$ such that $A \in St(\mathcal{D}_n, \mathcal{U}_n)$, for each $n \ge n_0$. Indeed, let $A \in \Lambda$. Since $\mathcal{J}$ is a $c\pi^{\gamma}_{\Delta}(\Lambda)$-network of X induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and C and $A^c \in \Lambda^c$, there exists $n_0 \in \mathbb{N}$ such that for each $n \ge n_0$, there exist $(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}) \in \mathcal{J}'_n$ and a finite set $F_n \subseteq X$ such that for every $i \in \{1, \dots, m_s\}$, $F_n \cap V^n_{i,s} \neq \emptyset$, $B^n_s \subseteq A^c$ and $F_n \cap A^c = \emptyset$. As $\left(B^n_s; V^n_{1,s}, \dots, V^n_{m_s,s}\right) \in \mathcal{J}'_n$, there is $U_n \in \mathcal{V}_n$ such that $B^n_s \subseteq U_n$ and for each $i \in \{1, \dots, m_s\}$, $V^n_{i,s} \not\subseteq U_n$ (for each $n \ge n_0$). It means that $\{A, A_n\} \subseteq (V^n_{1,s}, \dots, V^n_{m_s,s})^+_{B^n_s} \in \mathcal{U}_n$, where $A_n = U^c_n$. Hence $A \in St(\mathcal{D}_n, \mathcal{U}_n)$. This shows that the collection $\{St(\mathcal{D}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a γ-cover of $(\Lambda, \tau^+_{\Delta})$. □

As some immediate consequences of Theorem 5.4, we get the following.

Remark 5.5. *Let (X,τ) be a topological space. If Λ is any of the hyperspaces $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then $(\Lambda,\tau^{+}_{\Delta})$ is* aSSH *if and only if X satisfies the property: for each sequence $(\mathcal{J}_n : n \in \mathbb{N})$ of $\pi_{\Delta}(\Lambda)$-networks and each $C \in \mathbb{C}_{\Delta}(\Lambda)$, there is $\{\mathcal{V}_n : n \in \mathbb{N}\} \subseteq [C]^{<\omega}$ such that $\mathcal{J}$ defined as in (a) of Definition 5.2 is a $c\pi^{\gamma}_{\Delta}(\Lambda)$-network of X induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and C.*

When Λ is endowed with Fell or Vietoris topology, we get the next particular cases.

Corollary 5.6. *Let (X,τ) be a topological space and let Λ be any of the hyperspaces $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then*

1. (Λ,τ_F) *is* aSSH *if and only if X satisfies the property: for each sequence $(\mathcal{J}_n : n \in \mathbb{N})$ of $\pi_{\mathbb{K}(X)}(\Lambda)$-networks and each $C \in \mathbb{C}(\Lambda)$, there is $\{\mathcal{V}_n : n \in \mathbb{N}\} \subseteq [C]^{<\omega}$ such that $\mathcal{J}$ defined as in (a) of Definition 5.2 is a $c\pi^{\gamma}_{\mathbb{K}(X)}(\Lambda)$-network of X induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and C.*
2. (Λ,τ_V) *is* aSSH *if and only if X satisfies the property: for each sequence $(\mathcal{J}_n : n \in \mathbb{N})$ of $\pi_{CL(X)}(\Lambda)$-networks and each $C \in \mathbb{C}_{CL(X)}(\Lambda)$, there is $\{\mathcal{V}_n : n \in \mathbb{N}\} \subseteq [C]^{<\omega}$ such that $\mathcal{J}$ defined as in (a) of Definition 5.2 is a $c\pi^{\gamma}_{CL(X)}(\Lambda)$-network of X induced by $(\mathcal{J}_n : n \in \mathbb{N})$ and C.*

We conclude this paper with a couple of comments. First, it is known that every σ-compact space is Hurewicz; even more, in [12] has been shown that $(CL(X),\tau_F)$ is σ-compact, whenever that X is Hausdorff and σ-compact. Then, taking $\mathbb{R}$ as the set of the real numbers endowed with the usual topology, we have that $(CL(\mathbb{R}),\tau_F)$ is a Hurewicz space, so the Theorems 2.4, 3.4, 4.4 and 5.4 can be applied. Second, even though the families involved in (2) of the Theorems 2.4, 3.4, 4.4 and 5.4 turn out to be $\pi_{\Delta}(\Lambda)$-networks, the authors do not know if it is possible to characterize the covering properties Hurewicz, strongly star Hurewicz, star Hurewicz and absolutely strongly star Hurewicz in hyperspaces endowed with the hit-and-miss topology, by means of selection principles of the space X involving $\pi_{\Delta}(\Lambda)$-networks or $c_{\Delta}(\Lambda)$-covers.

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