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Four dimensional matrix mappings on double summable spaces

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Abstract. In a previous paper [9], some classes of triangular matrix transformations between the series spaces summable by the absolute weighted summability methods were characterized. In the present paper, we extend these classes to four dimensional matrices and double summability methods.

1. Introduction

Consider an infinite single series Σx_v of complex or real numbers with partial sums s_n and let σ_n^α denote the n-th term of the Cesàro mean of order $\alpha > -1$ of the sequence (s_n). The series Σx_v is summable $|C,\alpha|_k, k \geq 1$, in Flett's notation (see [4]), if $\left(n^{1-1/k}\Delta\sigma_n^\alpha\right) \in \ell_k$, where ℓ_k is the set of absolutely k-summable sequences. Further let $\left(\phi_n\right)$ be a sequence of positive numbers and (p_n) be a sequences of positive numbers satisfying

$$P_n = p_0 + p_1 + ... + p_n \to \infty \; as \; n \to \infty, \; P_{-1} = p_{-1} = 0. \tag{1}$$

By T_n, we denote the n- th term of weighted mean $\left(\overline{N}, p_n\right)$ of the sequence of (s_n), *i.e.*

$$T_n = \sum_{v=0}^{n} p_v s_v / P_n.$$

The series Σx_v is said to be summable $\left|\overline{N}, p_n, \phi_n\right|_k, k \geq 1$, if (see [15]) $\left(\phi_n^{1-1/k}\Delta T_n\right) \in \ell_k$, which reduces to the methods $\left|\overline{N}, p_n\right|_k$ and $\left|R, p_n\right|_k$ for $\phi_n = P_n/p_n$ and $\phi_n = n$ (see [2] and [12], respectively).

For $k \geq 1$, the space $\left|\overline{N}_p^\phi\right|_k$, the set of all series summable by the method $\left|\overline{N}, p_n, \phi_n\right|_k$, is a Banach space (see [9], [14]) according to the norm

$$\|x\|_{\left|\overline{N}_p^\phi\right|_k} = \left(|x_0|^k + \sum_{n=1}^{\infty} \phi_n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} x_v \right|^k\right)^{1/k}.$$

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Further, a series Σx_v is summable $\left|\overline{N}, p_n, \phi_n\right|_k$ iff a sequence $x = (x_v) \in \left|\overline{N}_p^{\phi}\right|_k$, and the space $\left|\overline{N}_p^{\phi}\right|_k$ is the same as the spaces $\left|\overline{N}_p\right|_k$ and $\left|R_p\right|_k$ for $\phi_n = P_n/p_n$ and $\gamma_n = n$, (see [14] and [12], respectively).

We denote the set of all infinite triangular matrices which map a single sequence space X to another sequence space Y by (X, Y). The following characterizations of matrix classes are well known (see [9]), which include some known corollaries and applications for particular matrices (see [3, 5, 10-14, 16]).

Throughout the paper k^* will denote the conjugate of k, i.,e., $1/k + 1/k^* = 1$ for $k > 1, 1/k^* = 0$ for $k = 1$.

Theorem 1.1. Let (p_n) and (q_n) be positive sequences satisfying (1). Further, let $A = (a_{nv})$ be an infinite triangular matrix and (ϕ_n) be a sequence of positive numbers. Then, $A \in \left(\left|\overline{N}_p\right|, \left|\overline{N}_q^{\phi}\right|_k\right)$, for the case $1 \leq k < \infty$, if and only if

$$\frac{P_v q_v}{p_v Q_v} \phi_v^{1/k^*} a_{vv} = O(1) \tag{2}$$

$$\sum_{n=v+1}^{\infty} \phi_n^{k-1} \left| \mu_n \sum_{r=v}^{n} Q_{r-1}(a_{rv} - a_{r,v+1}) \right|^k = O\left\{ \left(\frac{p_v}{P_v} \right)^k \right\} \tag{3}$$

$$\sum_{n=v+1}^{\infty} \phi_n^{k-1} \left| \mu_n \sum_{r=v+1}^{n} Q_{r-1} a_{r,v+1} \right|^k = O(1). \tag{4}$$

where

$$\mu_n = \frac{q_n}{Q_n Q_{n-1}}, n \geq 1 \tag{5}$$

Theorem 1.2. Let (p_n) and (q_n) be positive sequences satisfying (1). Further, let $A = (a_{nv})$ be an infinite triangular matrix and (ϕ_n) be a sequence of positive numbers. Then, $A \in \left(\left|\overline{N}_p^{\phi}\right|_k, \left|\overline{N}_q\right|\right)$, for the case $1 < k < \infty$, if and only if

$$\sum_{v=1}^{\infty} \frac{p_v^{-k^*}}{\phi_v} \left(\sum_{n=v}^{\infty} \mu_n \left| \sum_{r=v}^{n} Q_{r-1}(P_r a_{rv} - P_{r-1} a_{r,v+1}) \right| \right)^{k^*} < \infty. \tag{6}$$

where μ_n is defined by (5).

In the present paper we establih Theorem 1.1 and Theorem 1.2 for four dimensional matrices and double summability, which extend earlier factor and inclusion results on absolute weighted summability to double summability.

2. Absolute double weighted summability

For any double sequence (x_{rs}) and four dimentional sequence (y_{mnrs}), we write for $m, n, r, s \geq 0$,

$$\begin{aligned}
\Delta_1 x_{rs} &= x_{rs} - x_{r-1,s} \quad \Delta_2 x_{rs} = x_{rs} - x_{r,s-1} \\
\Delta_{12} x_{rs} &= \Delta_2(\Delta_1 x_{rs}), \quad x_{-1,0} = x_{0,-1} = 0 \\
\Delta_1 y_{mnrs} &= y_{mnrs} - y_{mn,r-1,s} \quad \Delta_2 y_{mnrs} = y_{mnrs} - y_{mn,r,s-1} \\
\Delta_{12} y_{mnrs} &= \Delta_2(\Delta_1 y_{mnrs}), \quad y_{mn,-1,0} = y_{mn,0,-1} = 0,
\end{aligned}$$

We use the notations $\sum_{r,s=0}^{\infty}$ and $\sum_{r,s=0}^{m,n}$ instead of $\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}$ and $\sum_{r=0}^{m}\sum_{s=0}^{n}$, respectively, and also

$$\mu'_{mn}=\begin{cases}\frac{\gamma_{m0}^{1/k^{*}}p'_{m}}{P'_{m-1}P'_{m}}, & n=0,m\geq 1\\ \frac{\gamma_{0n}^{1/k^{*}}q'_{n}}{Q'_{n}Q'_{n-1}}, & m=0,n\geq 1\\ \frac{\gamma_{mn}^{1/k^{*}}p'_{m}q'_{n}}{P'_{m-1}P'_{m}Q'_{n}Q'_{n-1}}, & m\geq 1,n\geq 1.\end{cases} \tag{7}$$

Let $\sum_{r,s=0}^{\infty}x_{rs}$ be an infinite double series with partial sums s_{mn}, *i.e.*,

$$s_{mn}=\sum_{r,s=0}^{m,n}x_{rs}$$

Let us denote the double weighted mean $\left(\overline{N},p_m,p_n\right)$ of the double sequence (s_{mn}) by

$$T_{mn}=\frac{1}{P_mQ_n}\sum_{r,s=0}^{m,n}p_rq_ss_{rs} \tag{8}$$

we shall say that the series $\sum_{r,s=0}^{\infty}x_{rs}$ is called summable $\left|\overline{N},p_m,q_n;\gamma_{mn}\right|_k$, $k\geq 1$, if

$$\sum_{m,n=0}^{\infty}\gamma_{mn}^{k-1}\left|\Delta_{21}T_{m,n}\right|^k<\infty. \tag{9}$$

It may be noticed this method reduces to the methods $\left|\overline{N},p_m,q_n\right|_k$, $\left|R,p_m,q_n\right|_k$ and $|C,1,1|_k$ for $\gamma_{mn}=P_mQ_n/p_mq_n$, $\gamma_{mn}=mn$ and $p_n=q_n=1$, respectively, [8], [6-7].

Now, by $\left|\overline{N}_{pq}^{\phi}\right|_k$, we introduce the set of all double series summable by the method $\left|\overline{N},p_m,q_n;\gamma_{mn}\right|_k$. Then, the double series $\sum_{r,s=0}^{\infty}x_{rs}$ is summable $\left|\overline{N},p_m,q_n;\gamma_{mn}\right|_k$ if and only if a double sequence $x=(x_{rs})\in\left|\overline{N}_{pq}^{\phi}\right|_k$. Further, since, for $m,n\geq 0$

$$\begin{aligned}T_{mn}&=\frac{1}{P_mQ_n}\sum_{r,s=0}^{m,n}p_rq_ss_{rs}=\frac{1}{P_mQ_n}\sum_{v,\mu=0}^{m,n}p_vq_\mu\sum_{r,s=0}^{v,\mu}x_{rs}\\&=\frac{1}{P_mQ_n}\sum_{r,s=0}^{m,n}x_{rs}\sum_{v,\mu=r,s}^{m,n}p_vq_\mu\\&=\frac{1}{P_mQ_n}\sum_{r,s=0}^{m,n}x_{rs}\left(P_m-P_{r-1}\right)\left(Q_n-Q_{s-1}\right)\\&=\sum_{r,s=0}^{m,n}x_{rs}\left(1-\frac{P_{r-1}}{P_m}\right)\left(1-\frac{Q_{s-1}}{Qn}\right),\end{aligned}$$

it is easily seen that $\Delta_1T_{00}=\Delta_2T_{00}=\Delta_{21}T_{00}=x_{00}$ and, for $m,n\geq 1$,

$$\begin{aligned}\Delta_1T_{m0}&=\frac{p_m}{P_mP_{m-1}}\sum_{r=1}^{m}P_{r-1}x_{r0}\\ \Delta_2T_{0n}&=\frac{q_n}{Q_nQ_{n-1}}\sum_{s=1}^{n}Q_{s-1}x_{0s}\\ \Delta_{21}T_{mn}&=\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}\sum_{r,s=1,1}^{m,n}P_{r-1}Q_{s-1}x_{rs}.\end{aligned} \tag{10}$$

Define the following space which plays an impotant role in this paper

$$\pi\left|\overline{N}_{pq}^{\gamma}\right|_k=\left\{x=(x_{rs})\in\left|\overline{N}_{pq}^{\gamma}\right|_k : x_{r0}=x_{0s}=0 \text{ for } r,s\geq 0\right\}$$

Hence it is routine to verify that $\left|\overline{N}_{pq}^{\gamma}\right|_k$ and $\pi\left|\overline{N}_{pq}^{\gamma}\right|_k$ are a Banach space according to the norm

$$\|x\|_{\left|\overline{N}_{pq}^{\gamma}\right|_k}=\left(\sum_{m,n=0}^{\infty}\gamma_{mn}^{k-1}\left|\Delta_{21}T_{mn}\right|^k\right)^{1/k}. \tag{11}$$

Also, there is a close relationship between the spaces $\left|\overline{N}_{pq}^{\gamma}\right|_k$ and $\mathcal{L}_k$, *i.e.*, $(x_{rs})\in\left|\overline{N}_{pq}^{\gamma}\right|_k$ if and only if $\left(\gamma_{mn}^{1/k^*}\Delta_{21}T_{m,n}\right)\in\mathcal{L}_k$, where $\mathcal{L}_k$ is the set of all double sequences (x_{rs}) of complex numbers such that $\sum_{r,s=0}^{\infty}|x_{rs}|^k<\infty$, the case $k=1$ of which reduces to the space $\mathcal{L}$, studied by Zeltser [18]. The space $\mathcal{L}_k, 1\leq k<\infty$, is a Banach space [1] according to the natural norm

$$\|x\|_{\mathcal{L}_k}=\left(\sum_{r,s=0}^{\infty}|x_{rs}|^k\right)^{1/k}$$

and the space $\mathcal{L}_\infty$ of all bounded double sequences is also a Banach space with the norm $\|x\|_\infty=\sup_{r,s}|x_{rs}|$.

Let $x=(x_{rs})$ be a double sequence. If for every $\varepsilon>0$ there exists a natural interger $n_0(\varepsilon)$ and real number l such that $|x_{rs}-l|<\varepsilon$ for all $r,s\geq n_0(\varepsilon)$, then, the double sequence $x=(x_{rs})$ is said to be convergent in the Peringsheim's sense. Also, a double series $\sum_{r,s=0}^{\infty}x_{rs}$ is convergent if and only if the double sequence of partial sums of series is convergent.

Let U and V be double sequence spaces and $A=(a_{mnrs})$ be a four dimensional infinite matrix of complex (or, real) numbers. Then, A defines a matrix transformation from U to V, written $A\in(U,V)$, if for every sequence $x=(x_{rs})\in U$, the A-transform $A(x)=(A_{mn}(x))$ of x is well defined and belongs to V, where

$$A_{mn}(x)=\sum_{r,s=0}^{\infty}a_{mnrs}x_{rs}$$

provided the double series in the right hand side converges for $m,n\geq 0$.

The transpose $A^t=(a_{rsmn})$ of the matrix $A=(a_{mnrs})$ is defined by

$$A_{rs}^t(x)=\sum_{m,n=0}^{\infty}a_{mnrs}x_{mn} \text{ for } m,n\geq 0.$$

The β-dual U^β of the space U is the set of all double sequences (b_{rs}) such that $\sum_{r,s=0}^{\infty}b_{rs}x_{rs}$ converges for all $x\in U$.

An infinite four dimensional matrix $A=(a_{mnrs})$ is called triangular if $a_{mnrs}=0$ for $r>m$ or $s>n$.

We require the following lemmas for the proof of our theorems.

Lemma 2.1. ([18]). If T is a linear mapping from a Banach space X into a Banach space Y, then T is continuous if and only if it is bounded, i.e., there exists a constant L such that

$$\|T(x)\|_Y\leq L\|x\|_X \text{ for all } x\in X$$

Lemma 2.2. Let $1<k<\infty$ and $A=\left(a_{mnij}\right)$ be an infinite four dimentional matrix. Define $W_k(A)$ and $w_k(A)$ by

$$W_k(A)=\sum_{r,s=0}^{\infty}\left(\sum_{m,n=0}^{\infty}|a_{mnrs}|\right)^k,$$

$$w_k(A) = \sup_{MXN} \sum_{r,s=0}^{\infty} \left| \sum_{(m,n)\in MXN} a_{mnrs} \right|^k$$

where the supremum is taken through all finite subsets M and N of the natural numbers. Then, the following statements are equivalent:

(i) $W_{k^*}(A) < \infty$ $\qquad$ (ii) $A \in (\mathcal{L}_k, \mathcal{L})$

(iii) $A^t \in (\mathcal{L}_\infty, \mathcal{L}_{k^*})$ $\qquad$ (iv) $w_{k^*}(A) < \infty$.

Proof. To prove the lemma, it is enough to show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$.
$(i) \Rightarrow (ii)$. Assume (i) holds. Then, for all $x \in \mathcal{L}_k$, it follows from Hölder's inequality that

$$\begin{aligned} \|A(x)\|_{\mathcal{L}} &= \sum_{m,n=0}^{\infty} \left| \sum_{r,s=0}^{\infty} a_{mnrs} x_{rs} \right| \le \sum_{r,s=0}^{\infty} \sum_{m,n=0}^{\infty} |a_{mnrs} x_{rs}| \\ &\le \left\{ \sum_{r,s=0}^{\infty} \left(\sum_{m,n=0}^{\infty} |a_{mnrs}| \right)^{k^*} \right\}^{1/k^*} \|x\|_{\mathcal{L}_k} \\ &\le (W_{k^*}(A))^{1/k^*} \|x\|_{\mathcal{L}_k} < \infty, \end{aligned}$$

which gives (ii).

$(ii) \Rightarrow (iii)$. Suppose $A \in (\mathcal{L}_k, \mathcal{L})$. Then, since $\mathcal{L}_k$ is a Banach space for $k \ge 1$, by Lemma 2.1, there exists a constant L such that

$$\|A(x)\|_{\mathcal{L}} = \sum_{m,n=0}^{\infty} \left| \sum_{r,s=0}^{\infty} a_{mnrs} x_{rs} \right| \le L \|x\|_{\mathcal{L}_k} \tag{12}$$

for all $x \in \mathcal{L}_k$. Also, it is observed by putting $x_{rs} sgn a_{mnrs}$ instead of x_{rs} that

$$\sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} |a_{mnrs} x_{rs}| \le L \|x\|_{\mathcal{L}_k}$$

Now, let $u \in \mathcal{L}_\infty$ be given. Then, by (12),

$$\begin{aligned} \left| \sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} u_{mn} a_{mnrs} x_{rs} \right| &\le \|u\|_{\mathcal{L}_\infty} \sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} |a_{mnrs} x_{rs}| \\ &\le L \|u\|_{\mathcal{L}_\infty} \|x\|_{\mathcal{L}_k} \end{aligned} \tag{13}$$

In (13), taking $x_{rs} = 1$ for $(r, s) = (i, j)$, and zero otherwise, it is easily seen that

$$\left| \sum_{m,n=0}^{\infty} a_{mnrs} u_{mn} \right| \le \sum_{m,n=0}^{\infty} |a_{mnrs} u_{mn}| \le L \|u\|_{\mathcal{L}_\infty},$$

which gives that $A^t(u)$ is defined for all $r, s \ge 0$, where the double sequence $A^t(u) = (A^t_{rs}(u))$ is given by

$$A^t_{rs}(u) = \sum_{m,n=0}^{\infty} a_{mnrs} u_{mn} : m, n \ge 0 \tag{14}$$

Again, it follows by considering (13) that

$$\left|\sum_{r,s=0}^{\infty} A_{rs}^{t}(u)x_{rs}\right| \leq L \left\|u\right\|_{\mathcal{L}\infty} \left\|x\right\|_{\mathcal{L}_k} \tag{15}$$

which implies that the series in the left side hand of (14) converges. Therefore, since the dual of space $\mathcal{L}_k$ is the space $\mathcal{L}_{k^*}$(see [1]), we obtain $A^t(u) \in \mathcal{L}_{k^*}$, *i.e.*, $A^t \in (\mathcal{L}_\infty, \mathcal{L}_{k^*})$.

$(iii) \Rightarrow (iv)$. If $A^t \in (\mathcal{L}_\infty, \mathcal{L}_{k^*})$, then, by Lemma 2.1, there exists a constant K such that $\left\|A^t(x)\right\|_{\mathcal{L}_{k^*}} \leq K \left\|x\right\|_{\mathcal{L}_\infty}$ for all $x \in \mathcal{L}_\infty$, *i.e.*,

$$\left(\sum_{r,s=0}^{\infty}\left|\sum_{m,n=0}^{\infty} a_{mnrs}x_{mn}\right|^{k^*}\right)^{1/k^*} \leq K \left\|x\right\|_{\mathcal{L}_\infty} . \tag{16}$$

Let M and N be any finite subsets of all nature numbers. Take a sequence $x = (x_{mn})$ as $x_{mn} = 1$ for $(r, s) \in MXN$, and zero otherwise. Then, (16) is reduced to.

$$\left(\sum_{r,s=0}^{\infty}\left|\sum_{(m,n)\in MXN} a_{mnrs}\right|^{k^*}\right)^{1/k^*} \leq K$$

which proves $w_{k^*}(A) < \infty$.

$(iii) \Rightarrow (iv)$. Suppose (iii) is satisfied and a_{mnrs} are real numbers. Then, for every finite subsets M and N of nature numbers,

$$\sum_{r,s=0}^{\infty}\left|\sum_{(m,n)\in MXN} a_{mnrs}\right|^{k^*} \leq w_{k^*}(A).$$

Let $H^+ = \{(m, n) \in MXN : a_{mnrs} \geq 0\}$ and $H^- = \{(m, n) \in MXN : a_{mnrs} < 0\}$. Then, by considering the inequality $|a + b|^{k^*} \leq 2^{k^*}\left(|a|^{k^*} + |b|^{k^*}\right)$, where a and b are complex numbers, we have

$$\begin{aligned} W_{k^*}(A) &= \sum_{r,s=0}^{\infty}\left(\sum_{m,n=0}^{\infty} |a_{mnrs}|\right)^{k^*} \\ &= \sum_{r,s=0}^{\infty}\left\{\sum_{(m,n)\in H^+}^{\infty} a_{mnrs} + \sum_{(m,n)\in H^-}^{\infty} -a_{mnrs}\right\}^{k^*} \\ &\leq 2^{k^*}\sum_{r,s=0}^{\infty}\left\{\left(\sum_{(m,n)\in H^+}^{\infty} a_{mnrs}\right)^{k^*} + \left(\sum_{(m,n)\in H^-}^{\infty} -a_{mnrs}\right)^{k^*}\right\} \\ &\leq 2^{k^*+1} w_k(A). \end{aligned}$$

If a_{mnrs} is complex number for $m, n, r, s \geq 0$, it is easily seen that $W_{k^*}(A) \leq 2^{2k^*+3} w_k(A) < \infty$, which implies (iv) .

This step ends the proof.

3. Main Results

In this section we prove the following theorems.

Theorem 3.1. Let $(p_n), (q_n), (p'_n)$ and (q'_n) be sequences of positive numbers satisfying (1). Further, let $\gamma = (\gamma_{rs})$ be a double sequence of positive numbers and $A = (a_{mnrs})$ be a four dimensional triangle matrix and define the matrix B by

$$b_{mnrs} = \begin{cases} \sum_{i,j=r,s}^{m,n} P'_{i-1}Q'_{j-1}a_{ijrs}, 1 \leq r \leq m, 1 \leq s \leq n \\ 0, \qquad r > m, \text{ or } s > n. \end{cases} \tag{17}$$

Then, $A \in \left(\left|\overline{N}_{pq}\right|, \pi\left|\overline{N}^{\gamma}_{p'q'}\right|_k\right)$, $1 \leq k < \infty$, if and only if

$$\sum_{m,n=r,s}^{\infty} \left|\mu'_{mn} b_{mn,r+1,s+1}\right|^k = O(1) \tag{18}$$

$$\sum_{m,n=r,s}^{\infty} \left|\mu'_{mn} \Delta_2 b_{mn,r+1,s+1}\right|^k = O\left\{\left(\frac{q_s}{Q_s}\right)^k\right\} \tag{19}$$

$$\sum_{m,n=r,s}^{\infty} \left|\mu'_{mn} \Delta_1 b_{mn,r+1,s+1}\right|^k = O\left\{\left(\frac{p_r}{P_r}\right)^k\right\} \tag{20}$$

$$\sum_{m=r,n=s}^{\infty} \left|\mu'_{mn} \Delta_{12} b_{mn,r+1,s}\right|^k = O\left\{\left(\frac{P_r Q_s}{p_r q_s}\right)^k\right\} \tag{21}$$

where μ'_{mn} is defined by (7).

Proof. Necessity. Let $A \in \left(\left|\overline{N}_{p,q}\right|, \pi\left|\overline{N}^{\gamma}_{p'q'}\right|_k\right)$. Then, since $\left|\overline{N}_{p,q}\right|$ and $\pi\left|\overline{N}_{p'q'}\right|_k$ are Banach spaces, it is seen from Lemma 2.1 that $A : \left|\overline{N}_{p,q}\right| \to \pi\left|\overline{N}^{\gamma}_{p'q'}\right|_{kk}$ defined by

$$A_{mn}(x) = \sum_{r,s=0}^{m,n} a_{mnrs} x_{rs} \tag{22}$$

is a bounded linear operator. So, there exists a constant M such that

$$\|A(x)\|_{\pi\left|\overline{N}^{\gamma}_{p'q'}\right|_k} \leq M \|x\|_{\left|\overline{N}_{pq}\right|} \tag{23}$$

for all $x = (x_{rs}) \in \left|\overline{N}_{p,q}\right|$. Put $t_{mn} = \Delta_{21} T_{mn}$ for $m, n \geq 0$, where $\Delta_{21} T_{mn}$ is defined by (9). Then, $t = (t_{mn}) \in \mathcal{L}$. Also, $A(x) = (A_{rs}(x)) \in \pi\left|\overline{N}^{\gamma}_{p'q'}\right|_k$ if and only if $L'(x) = (L'_{mn}(x) \in \mathcal{L}_k$, i.e.,

$$\|A(x)\|_{\pi\left|\overline{N}^{\gamma}_{p'q'}\right|_k} = \|L'(x)\|_{\mathcal{L}_k} = \left(\sum_{m,n=1}^{\infty} \left|L'_{mn}(x)\right|^k\right)^{1/k} < \infty \tag{24}$$

where

$$L'_{mn}(x) = \mu'_{mn} \sum_{r,s=1}^{m,n} P'_{r-1} Q'_{s-1} A_{rs}(x). \tag{25}$$

Choose a sequence $x = (x_{ij}) \in \left|\overline{N}_{p,q}\right|$ such that $x_{rs} = 1$, $x_{ij} = 0$ for $i \neq r, j \neq s$. Then, using (9), we have, for $m, n \geq 1$,

$$t_{mn} = \begin{cases} 0, & m < r, n < s \\ \frac{p_m q_n P_{r-1} Q_{s-1}}{P_m P_{m-1} Q_n Q_{n-1}}, & m \geq r, n \geq s. \end{cases}, \ \|x\|_{\left|\overline{N}_{p,q}\right|} = \|t\|_{\mathcal{L}} = 1 \tag{26}$$

Also, it is easily seen that

$$A_{mn}(x) = \begin{cases} 0, & m < r, n < s \\ a_{mnrs}, & m \geq r, n \geq s \end{cases}$$

which gives, by (24),

$$L'_{mn}(x) = \begin{cases} 0, & m < r, n < s \\ \mu'_{mn} b_{mnrs}, & m \geq r, n \geq s. \end{cases}$$

and so

$$\|A(x)\|_{\left|\overline{N}_{p'q'}\right|_k} = \left(\sum_{m,n=r,s}^{\infty} \left| \mu'_{mn} b_{mnrs} \right|^k \right)^{1/k}. \tag{27}$$

Now, it follows by applying (26) and (27) to the inequality (23) that, for $r, s \geq 1$,

$$\sum_{m,n=r,s}^{\infty} \left| \mu'_{mn} b_{mnvu} \right|^k \leq M^k$$

which is equivalent to (18).

Now take $x_{rs} = 1$, $x_{r,s+1} = -1$, and zero, otherwise. Then, by (10), we get

$$t_{mn} = \begin{cases} 0, & m < r, n < s \\ \frac{p_m q_s P_{r-1}}{P_m P_{m-1} Q_s}, & m \geq r, n = s \\ -\frac{p_m q_n P_{r-1} q_s}{P_m P_{m-1} Q_n Q_{n-1}}, & m \geq r, n > s \end{cases}, \quad \|x\|_{\left|\overline{N}_{pq}\right|} = \|t\|_{\mathcal{L}} = \frac{2q_s}{Q_s}. \tag{28}$$

Further, we obtain

$$A_{mn}(x) = \sum_{i,j=r,s}^{m,n} a_{mnij} x_{ij} = \begin{cases} 0, & n < s, m < r \\ -\Delta_2 a_{mnr,s+1}, & n \geq s, m \geq r \end{cases}$$

which implies, by (25),

$$L'_{mn}(x) = \begin{cases} 0, & m < r, n < s \\ -\mu'_{mn} \Delta_2 b_{mnr,s+1}, & n \geq s, m \geq r \end{cases}$$

and

$$\|A(x)\|_{\left|\overline{N}_{p'q'}\right|_k} = \left(\sum_{m,n=r,s}^{\infty} \left| \mu'_{mn} \Delta_2 b_{mnr,s+1} \right|^k \right)^{1/k}. \tag{29}$$

So, using (28) and (29), we have from (23) that (19) holds. Also, by taking $x_{rs} = 1$, $x_{r+1,s} = -1$, and zero, otherwise, then, similarly, (20) holds.

Finally, put $x_{rs}=1, x_{r,s+1}=-1, x_{r+1,s}=-1, x_{r+1,s+1}=1$, and zero, otherwise. Then,

$$t_{mn}=\begin{cases} 0, & m<r, n<s \\ \frac{p_r q_s}{P_r Q_s}, & n=s, m=r \\ -\frac{p_r q_n q_s}{P_r Q_n Q_{n-1}}, & n>s, m=r \\ -\frac{p_r p_m q_s}{P_m P_{m-1} Q_s}, & n=s, m>r \\ \frac{p_m q_n p_r q_s}{P_m P_{m-1} Q_n Q_{n-1}}, & n>s, m>r \end{cases}, \quad \|x\|_{|\overline{N}_{pq}|}=\frac{4p_r q_s}{P_r Q_s} \tag{30}$$

and

$$A_{mn}(x)=\begin{cases} \Delta_{21} a_{mn,r+1,s+1}, & r\leq m, s\leq n \\ 0, & r>m, s>n. \end{cases}$$

This verifies

$$L'_{mn}(x)=\begin{cases} \mu'_{mn}\Delta_{12} b_{mnr+1s+1}, & r\leq m, s\leq n \\ 0, & r>m, s>n \end{cases}$$

and

$$\|x\|_{|\overline{N}_{p'q'}|_k}=\left(\sum_{m,n=r,s}^{\infty}\left|\mu'_{mn}\Delta_{12} b_{mnr+1s+1}\right|^k\right)^{1/k}. \tag{31}$$

Therefore, considering (30) and (31), it follows from (23) that (21) holds.

Sufficiency. Given $x=(x_{rs})\in\left|\overline{N}_{p,q}\right|$. Then, $t=(t_{mn})\in\mathcal{L}$, where $t_{mn}=\Delta_{21}T_{mn}$ for $m,n\geq 0$, as above. Now, we should show that $A(x)=(A_{rs}(x))\in\pi\left|\overline{N}_{p'q'}\right|_k$, *i.e.*,

$$\sum_{m,n=1}^{\infty}\left|L'_{mn}(x)\right|^k<\infty$$

where $L'(x)=(L'_{mn}(x))$ is defined by (25). To achieve this, by solving (10) for x_{mn}, we obtain, for $m,n\geq 1$,

$$\begin{aligned} x_{mn} &= \frac{P_m Q_n}{p_m q_n} t_{mn}-\frac{P_{m-2} Q_n}{p_{m-1} q_n} t_{m-1,n} \\ &\quad -\frac{Q_{n-2} P_m}{q_{n-1} p_m} t_{m,n-1}+\frac{P_{m-2} Q_{n-2}}{p_{m-1} q_{n-1}} t_{m-1,n-1}. \end{aligned} \tag{32}$$

Hence, since B is a triangular matrix, a few calculations reveal that

$$\begin{aligned} L'_{mn}(x) &= \mu'_{mn}\sum_{i,j=1}^{m,n} P'_{i-1} Q'_{j-1} A_{ij}(x) \\ &= \mu'_{mn}\sum_{r,s=1}^{m,n} x_{rs}\sum_{i,j=r,s}^{m,n} P'_{i-1} Q'_{j-1} a_{ijrs}=\mu'_{mn}\sum_{r,s=1}^{m,n} b_{mnrs} x_{rs} \end{aligned}$$

$$
\begin{aligned}
&= \mu'_{mn}\sum_{r,s=1}^{m,n} b_{mnrs}\left(\frac{P_rQ_s}{p_rq_s}t_{rs}-\frac{P_{r-2}Q_s}{p_{r-1}q_s}t_{r-1,s}\right.\\
&\left.-\frac{P_rQ_{s-2}}{p_rq_{s-1}}t_{r,s-1}+\frac{P_{r-2}Q_{s-2}}{p_{r-1}q_{s-1}}t_{r-1,s-1}\right)\\
&= \mu'_{mn}\left\{\sum_{r,s=1}^{m,n} b_{mnrs}\frac{P_rQ_s}{p_rq_s}t_{rs}-\sum_{r,s=1}^{m-1,n} b_{mn,r+1,s}\frac{P_{r-1}Q_s}{p_rq_s}t_{rs}\right.\\
&\left.-\sum_{r,s=1}^{m,n-1} b_{mn,r,s+1}\frac{P_rQ_{s-1}}{p_rq_s}t_{rs}+\sum_{r,s=1}^{m-1,n-1} b_{mn,r+1,s+1}\frac{P_{r-1}Q_{s-1}}{p_rq_s}t_{rs}\right\}\\
&= \mu'_{mn}\sum_{r,s=1}^{m,n}\left(b_{mnrs}\frac{P_rQ_s}{p_rq_s}-b_{mn,r+1,s}\frac{P_{r-1}Q_s}{p_rq_s}-\right.\\
&\left. b_{mn,r,s+1}\frac{P_rQ_{s-1}}{p_rq_s}+b_{mn,r+1,s+1}\frac{P_{r-1}Q_{s-1}}{p_rq_s}\right)t_{rs}\\
&= \mu'_{mn}\sum_{r,s=1}^{m,n} c_{mnrs}t_{rs},
\end{aligned}
$$

where

$$
\begin{aligned}
c_{mnrs} &= \left(b_{mnrs}\frac{P_r}{p_r}-b_{mn,r+1,s}\frac{P_{r-1}}{p_r}\right)\frac{Q_s}{q_s}\\
&-\left(b_{mn,r,s+1}\frac{P_r}{p_r}-b_{mn,r+1,s+1}\frac{P_{r-1}}{p_r}\right)\frac{Q_{s-1}}{q_s}\\
&= \frac{P_rQ_s}{p_rq_s}\Delta_{12}b_{mn,r+1,s+1}-\frac{P_r}{p_r}\Delta_1 b_{mn,r+1,s+1}\\
&-\frac{Q_s}{q_s}\Delta_2 b_{mn,r+1,s+1}+b_{mn,r+1,s+1}.
\end{aligned}
\tag{33}
$$

Also, since

$$
\begin{aligned}
|c_{mnrs}|^k &\le 3^k\left\{\left|\frac{P_rQ_s}{p_rq_s}\Delta_{12}b_{mn,r+1,s+1}\right|^k+\left|\frac{P_r}{p_r}\Delta_1 b_{mn,r+1,s+1}\right|^k\right.\\
&\left.+\left|\frac{Q_s}{q_s}\Delta_2 b_{mn,r+1,s+1}\right|^k+\left|b_{mn,r+1,s+1}\right|^k\right\},
\end{aligned}
$$

we get by Minkowski's inequality and the hypohese that

$$
\begin{aligned}
\left(\sum_{m,n=1}^{\infty}\left|L'_{mn}(x)\right|^k\right)^{1/k} &\le \left\{\sum_{m,n=1}^{\infty}\left(\sum_{r,s=1}^{m,n}\left|\mu'_{mn}c_{mnrs}t_{rs}\right|\right)^k\right\}^{1/k}\\
&\le \sum_{r,s=1}^{\infty,\infty}|t_{rs}|\left(\sum_{m,n=r,s}^{\infty,\infty}\left|\mu'_{mn}c_{mnrs}\right|^k\right)^{1/k}\\
&= O(1)\sum_{r,s=1}^{\infty,\infty}|t_{rs}|<\infty
\end{aligned}
$$

which completes the proof of the sufficiency.

Now, it is obvious that $A(x)=(A_{rs}(x))\in\left|\overline{N}^{\gamma}_{p'q'}\right|_k$, *i.e.*,

$$\sum_{m,n=0}^{\infty}\left|L'_{mn}(x)\right|^k=\sum_{m=0}^{\infty}\left|L'_{m0}(x)\right|^k+\sum_{n=1}^{\infty}\left|L'_{0n}(x)\right|^k+\sum_{m,n=1}^{\infty}\left|L'_{mn}(x)\right|^k<\infty \tag{34}$$

if and only if

$$\sum_{m=0}^{\infty}\left|L'_{m0}(x)\right|^k<\infty,\sum_{n=1}^{\infty}\left|L'_{0n}(x)\right|^k<\infty,\sum_{m,n=1}^{\infty,\infty}\left|L'_{mn}(x)\right|^k<\infty,$$

where

$$\begin{aligned}L'_{m0}(x)&=\mu'_{m0}\sum_{r=1}^{m}P'_{r-1}A_{r0}(x)\\ L'_{0n}(x)&=\mu'_{0n}\sum_{s=1}^{n}Q'_{s-1}A_{0s}(x)\\ L'_{mn}(x)&=\mu'_{mn}\sum_{r,s=1}^{m,n}P'_{r-1}Q'_{s-1}A_{rs}(x).\end{aligned}$$

So, if we define the matrices A_1,A_2 and A_3 by

$$A_1=(a_{m0r0}),A_2=(a_{0n0s}),\ A_3=(a_{mnrs})\text{ for }m,n\geq 1 \tag{35}$$

then, $A\in\left(\left|\overline{N}_{p,q}\right|,\left|\overline{N}^{\gamma}_{p'q'}\right|_k\right)$ if and only if $A_1\in\left(\left|\overline{N}_{p}\right|,\left|\overline{N}^{\gamma_1}_{p'}\right|_k\right)$, $A_2\in\left(\left|\overline{N}_{q}\right|,\left|\overline{N}^{\gamma_2}_{q'}\right|_k\right)$ and $A_3\in\left(\left|\overline{N}_{p,q}\right|,\pi\left|\overline{N}^{\gamma}_{p'q'}\right|_k\right)$, where $A=(a_{mnrs})$ is a triangle matrix for $m,n\geq 0$, $\gamma_1=(\gamma_{m0})$ and $\gamma_2=(\gamma_{0n})$.

By identifying $A_1=(a_{m0r0})\equiv(a_{mr})$, $A_2=(a_{0n0s})\equiv(a_{ns})$, the main theorem is immediately deduced by Theorem 1.1 and Theorem 2.1 as follows.

Theorem 3.2. Let the sequences (p_n), (q_n), (p'_n), (q'_n), (γ_{mn}), and the matrices A, B be as in Theorem 3.1. Then, $A\in\left(\left|\overline{N}_{p,q}\right|,\left|\overline{N}^{\gamma}_{p'q'}\right|_k\right)$, $k\geq 1$, if and only if conditions (18) − (21) and the following conditions are satisfied:

$$\frac{P_rp'_r}{p_rP'_r}\gamma_{r0}^{1/k^*}a_{r0r0}=O(1)$$

$$\sum_{n=r+1}^{\infty}\left|\mu'_{n0}\sum_{v=r}^{n}P'_{v-1}\Delta_1a_{v,0,r+1,0}\right|^k=O\left\{\left(\frac{p_r}{P_r}\right)^k\right\}$$

$$\sum_{n=r+1}^{\infty}\left|\mu'_{n0}\sum_{v=r+1}^{n}P'_{v-1}a_{v,0,r+1,0}\right|^k=O(1)$$

$$\frac{Q_vq'_v}{q_vQ'_v}\gamma_{0v}^{1/k^*}a_{0v0v}=O(1)$$

$$\sum_{n=v+1}^{\infty}\left|\mu'_{0n}\sum_{r=v}^{n}Q'_{r-1}\Delta_2a_{0,r,0,v+1)}\right|^k=O\left\{\left(\frac{q_v}{Q_v}\right)^k\right\}$$

$$\sum_{n=v+1}^{\infty}\left|\mu'_{0n}\sum_{r=v+1}^{n}Q'_{r-1}a_{0,r,0,v+1}\right|^k=O(1)$$

where μ'_{n0} and μ'_{0n} are defined by (7). Now we qualify the converse of the matrix class in Theorem 3.2, which, althout is similar to the previous one, has a very different character.

Theorem 3.3. Let $(p_n), (q_n), (p'_n), (q'_n)$ and (γ_{mn}) be as in Theorem 3.1. Further, let $A = (a_{mnrs})$ be a four dimensional triangle matrix and define the matrix B by (17) for $m, n \geq 1$, and

$$b_{mnrs} = \begin{cases} \sum_{j=s}^{n} Q'_{j-1} a_{0j0s}, & 1 \leq s \leq n, \quad m = 0 \\ \sum_{j=r}^{m} P'_{j-1} a_{j0r0}, & 1 \leq r \leq m, \quad n = 0 \end{cases}$$

Then, $A \in \left(\left|\overline{N}_{pq}\right|_k, \left|\overline{N}^{\gamma}_{p'q'}\right|\right), 1 < k < \infty$, if and only if

$$\sum_{s=1}^{\infty} \frac{1}{\gamma_{0s}} \left(\sum_{n=s}^{\infty} \mu'_{0n} \left| \frac{Q_s}{q_s} \Delta_2 b_{0n0,s+1} - b_{0n0,s+1} \right| \right)^{k^*} < \infty \tag{36}$$

$$\sum_{r=1}^{\infty} \frac{1}{\gamma_{r0}} \left(\sum_{m=r}^{\infty} \mu'_{m0} \left| \frac{P_r}{p_r} \Delta_1 b_{m0,r+1,0} - b_{m0,r+1,0} \right| \right)^{k^*} < \infty \tag{37}$$

$$\sum_{r,s=1}^{\infty} \frac{1}{\gamma_{rs}} \left(\sum_{m,n=r,s}^{\infty} \left| \mu'_{mn} c_{mnrs} \right| \right)^{k^*} < \infty \tag{38}$$

where μ'_{mn} and c_{mnrs} are given by (7) with $k = 1$ and (21), respectively.

Proof. Assume that $x = (x_{rs}) \in \left|\overline{N}_{p,q}\right|_k$ and $A(x)$ is A-transform sequence of x. Let $t_{m0} = \Delta_1 T_{m0}$, $t_{0n} = \Delta_2 T_{0n}$ and $t_{mn} = \Delta_{21} T_{mn}$ for $m, n \geq 1$, where $\Delta_1 T_{m0}, \Delta_2 T_{0n}$ and $\Delta_{21} T_{mn}$ are defined by (10). Further, put $u_{m0} = \gamma_{m0}^{1/k^*} t_{m0}$, $u_{0n} = \gamma_{0n}^{1/k^*} t_{0n}$ and $u_{mn} = \gamma_{mn}^{1/k^*} t_{mn}$. Then, $u = (u_{mn}) \in \mathcal{L}_k$, or, equivalently, $(u_{m0}), (u_{0n}) \in \ell_k$ and $(u_{mn}) \in \mathcal{L}_k$.Also, $A(x) \in \left|\overline{N}_{p'q'}\right|$, iff $L'(x) = (L'_{mn}(x)) \in \mathcal{L}$, or, equivalently, as in (34), $(L'_{0n}(x), \left(L'_{m0}(x)\right) \in \ell$, and $(L'_{mn}(x)) \in \mathcal{L}$, where

$$\begin{aligned} L'_{0n}(x) &= \mu'_{0n} \sum_{s=1}^{n} Q'_{s-1} A_{0s}(x) \\ L'_{m0}(x) &= \mu'_{m0} \sum_{r=1}^{m} P'_{r-1} A_{r0}(x) \\ L'_{mn}(x) &= \mu'_{mn} \sum_{r,s=1}^{m,n} Q'_{r-1} Q'_{s-1} A_{rs}(x) \end{aligned}$$

It follows by solving (10) for x_{m0} and x_{0n} that

$$x_{m0} = \frac{P_m}{p_m} t_{m0} - \frac{P_{m-2}}{p_{m-1}} t_{m-1,0}, \; x_{0n} = \frac{Q_n}{q_n} t_{0n} - \frac{Q_{n-2}}{q_{n-1}} t_{0,n-1} \tag{39}$$

Since A and B are triangular matrix and $P_{-1} = Q_{-1} = 0$, it is easily written from (39) and (32) that, for $m, n \geq 1$,

$$\begin{aligned} L'_{0n}(x) &= \mu'_{0n} \sum_{s=1}^{n} Q'_{s-1} A_{0s}(x) = \mu'_{0n} \sum_{j=0}^{n} b_{0n0j} x_{0j} \\ &= \mu'_{0n} \sum_{j=0}^{n} \left(\frac{Q_j}{q_j} \Delta_2 b_{0n0,j+1} - b_{0n0,j+1} \right) \gamma_{0j}^{-1/k^*} u_{0j}, \end{aligned}$$

$$\begin{aligned} L'_{m0}(x) &= \mu'_{m0}\sum_{r=1}^{m}P'_{r-1}A_{0r}(x)=\mu'_{m0}\sum_{j=0}^{m}b_{m0j0}x_{j0} \\ &= \mu'_{m0}\sum_{j=0}^{m}\left(\frac{P_j}{p_j}\Delta_1 b_{m0,j+1,0}-b_{m0,j+1,0}\right)\gamma_{j0}^{-1/k^*}u_{j0} \end{aligned}$$

$$\begin{aligned} L'_{mn}(x) &= \mu'_{mn}\sum_{i,j=1}^{m,n}P'_{i-1}Q'_{j-1}A_{ij}(x)=\mu'_{mn}\sum_{i,j=1}^{m,n}P'_{i-1}Q'_{j-1}\sum_{r,s=1}^{i,j}a_{ijrs}x_{rs} \\ &= \mu'_{mn}\sum_{r,s=1}^{m,n}x_{rs}\sum_{i,j=r,s}^{m,n}P'_{i-1}Q'_{j-1}a_{ijrs}=\mu'_{mn}\sum_{r,s=1}^{m,n}b_{mnrs}x_{rs} \\ &= \mu'_{mn}\sum_{r,s=1}^{m,n}b_{mnrs}\left(\frac{P_rQ_s}{p_rq_s}t_{rs}-\frac{P_{r-2}Q_s}{p_{r-1}q_s}t_{r-1,s}-\frac{P_rQ_{s-2}}{p_rq_{s-1}}t_{r,s-1}+\frac{P_{r-2}Q_{s-2}}{p_{r-1}q_{s-1}}t_{r-1,s-1}\right) \\ &= \mu'_{mn}\sum_{r,s=1}^{m,n}\left(b_{mnrs}\frac{P_rQ_s}{p_rq_s}-b_{mn,r+1,s}\frac{P_{r-1}Q_s}{p_rq_s}-b_{mn,r,s+1}\frac{P_rQ_{s-1}}{p_rq_s}+b_{mn,r+1,s+1}\frac{P_{r-1}Q_{s-1}}{p_rq_s}\right)\frac{u_{rs}}{\gamma_{rs}^{1/k^*}} \\ &= \mu'_{mn}\sum_{r,s=1}^{m,n}\left(\frac{P_rQ_s}{p_rq_s}\Delta_{12}b_{mn,r+1,s+1}-\frac{P_r}{p_r}\Delta_1 b_{mn,r+1,s+1}-\frac{Q_s}{q_s}\Delta_2 b_{mn,r+1,s+1}+b_{mn,r+1,s+1}\right)\frac{u_{rs}}{\gamma_{rs}^{1/k^*}} \\ &= \mu'_{mn}\sum_{r,s=1}^{m,n}c_{mnrs}u_{rs}. \end{aligned}$$

Hence, it can be expressed that

$$L'_{mn}(u)=\sum_{r,s=0}^{m,n}d_{mnrs}u_{rs}$$

where

$$d_{mnrs}=\begin{cases} \mu'_{0n}\left(\frac{Q_s}{q_s}\Delta_2 b_{0n0,s+1}-b_{0n0,s+1}\right)\gamma_{0s}^{-1/k^*}, & 0\le s\le n, m=r=0 \\ \mu'_{m0}\left(\frac{P_r}{p_r}\Delta_1 b_{m0,r+1,0}-b_{m0,r+1,0}\right)\gamma_{r0}^{-1/k^*}, & 0\le r\le m, n=s=0 \\ \mu'_{mn}c_{mnrs}\gamma_{rs}^{-1/k^*}, & 1\le r\le m, 1\le s\le n \\ 0, & \text{otherwise.} \end{cases}$$

This gives that $A\in\left(\left|\overline{N}_{p,q}^{\gamma}\right|_k,\left|\overline{N}_{p'q'}\right|\right)$ if and only if $D\in(\mathcal{L}_k,\mathcal{L})$. Therefore, it follows from Lemma 2.2 that the conclusion of the theorem is valid if and only if $W_{k^*}(A)<\infty$, or, equivalently,

$$\sum_{s=0}^{\infty}\left(\sum_{n=s}^{\infty}|d_{0n0s}|\right)^{k^*}<\infty,\ \sum_{r=0}^{\infty}\left(\sum_{m=r}^{\infty}|d_{m0r0}|\right)^{k^*}<\infty$$

and

$$\sum_{r,s=1,1}^{\infty}\left(\sum_{m,n=r,s}^{\infty}|d_{mnrs}|\right)^{k^*}<\infty,$$

which gives (36), (37) and (38).

This completes the proof.

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