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Matrix transforms between sequence spaces defined by speeds of convergence

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Abstract.
Let X, Y be two sequence spaces defined by speeds of the convergence, i.e.; by monotonically increasing positive sequences. In this paper, we give necessary and sufficient conditions for a matrix A (with real or complex entries) to map X into Y. Also the analogue of the well known result of Steinhaus, which states that a regular matrix cannot transform each bounded sequence into convergent sequence, for the sequence spaces defined by the speeds of convergence has been proved.

1. Introduction

Let X, Y be two sequence spaces and $A = (a_{nk})$ be a matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. If for each $x = (x_k) \in X$ the series

$$A_n x = \sum_k a_{nk} x_k$$

converge and the sequence $Ax = (A_n x)$ belongs to Y, we say that the matrix A transforms X into Y. By (X, Y) we denote the set of all matrices which transform X into Y.

Let throughout this paper $\lambda = (\lambda_k)$ be a positive monotonically increasing sequence, i.e.; the speed of convergence. Following Kangro [3], [5] a convergent sequence $x = (x_k)$ with

$$\lim_k x_k := \xi(x) \text{ and } l_k(x) = \lambda_k (x_k - \xi(x)) \tag{1.1}$$

is called bounded with the speed λ (shortly, λ-bounded) if $l_k(x) = O_x(1)$, and convergent with the speed λ (shortly, λ-convergent) if there the finite limit

$$\lim_k l_k(x) := b(x) \tag{1.2}$$

exists. We denote the set of all λ-bounded sequences by m^λ, and the set of all λ-convergent sequences by c^λ. It is not difficult to see that $c^\lambda \subset m^\lambda \subset c$, where c is the space of all convergent sequences. In addition to

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it, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O(1)$ we get $c^\lambda = m^\lambda = c$. Let $\mu = (\mu_k)$ be another speed of convergence, and

$$c_0^\lambda := \{x \in c \,:\, l(x) = (l_k(x)) \in c_0\},$$

$$m_0^\lambda = \{x = (x_k) \,:\, x \in m^\lambda \cap c_0\},$$

where c_0 is the space of all sequences converging to zero. In [3]-[5] necessary and sufficient conditions for $A \in \left(m^\lambda, m^\mu\right)$, $A \in \left(c^\lambda, c^\mu\right)$ and $A \in \left(c^\lambda, m^\mu\right)$ were found (see also [1] and [6]).

In this paper we find necessary and sufficient conditions for $A \in \left(m^\lambda, c^\mu\right)$, and also for $A \in \left(m^\lambda, m_0^\mu\right)$, $A \in \left(m^\lambda, c_0^\mu\right)$, $A \in \left(c^\lambda, m_0^\mu\right)$, $A \in \left(c^\lambda, c_0^\mu\right)$, $A \in \left(m_0^\lambda, m^\mu\right)$, $A \in \left(m_0^\lambda, m_0^\mu\right)$, $A \in \left(m_0^\lambda, c^\mu\right)$, $A \in \left(m_0^\lambda, c_0^\mu\right)$, $A \in \left(c_0^\lambda, m^\mu\right)$, $A \in \left(c_0^\lambda, m_0^\mu\right)$, $A \in \left(c_0^\lambda, c^\mu\right)$ and $A \in \left(c_0^\lambda, c_0^\mu\right)$.

Let c_A be the summability domain of A, i.e.; the set of sequences x (with real or complex entries), for which the finite limit $\lim_n A_n x$ exists, and let

$$d_n(x) := \lambda_n \left(A_n x - \lim_n A_n x\right)$$

for every $x \in c_A$. Let

$$c_A^0 := \{x \in c_A \,:\, \lim_n A_n x = 0\},$$

$$c_A^\lambda = \{x = (x_k) \,:\, Ax \in c^\lambda\},$$

$$c_{0A}^\lambda := \{x \in c_A \,:\, d(x) = (d_n(x)) \in c_0\},$$

$$z_A^\lambda := \{x \in c_A^0 \,:\, d(x) \in c\},$$

$$n_A^\lambda := \{x \in c_A^0 \,:\, d(x) \in c_0\},$$

$$z^\lambda := \{x \in c_0 \,:\, l(x) = (l_k(x)) \in c\},$$

$$n^\lambda = \{x = (x_k) \,:\, x \in c^\lambda \text{ and } b(x) = \xi(x) = 0\}.$$

It is easy to see, that $n^\lambda \subset z^\lambda \subset c^\lambda \subset m^\lambda \subset c$, $n^\lambda \subset c_0^\lambda \subset c^\lambda$, $z^\lambda \subset m_0^\lambda \subset m^\lambda$, $n_A^\lambda \subset c_{0A}^\lambda$, and $n_A^\lambda \subset z_A^\lambda$. In addition to it, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O(1)$ we get $c_0^\lambda = c^\lambda = m^\lambda = c$, $z^\lambda = n^\lambda = m_0^\lambda = c_0$ and $z_A^\lambda = n_A^\lambda = c_{0A}^\lambda = c_A$.

Let $e = (1, 1, ...)$, $e^k = (0, ..., 0, 1, 0, ...)$, where 1 is in the k-th position, and $\lambda^{-1} = (1/\lambda_k)$. Then $e \in c_0^\lambda \subset c^\lambda \subset m^\lambda$, $e^k \in n^\lambda$, $\lambda^{-1} \in c^\lambda \subset m^\lambda$ and λ^{-1} does not belong into n^λ. In addition, if $\lambda_k \neq O(1)$ we get $\lambda^{-1} \in z^\lambda \subset m_0^\lambda$ and λ^{-1} does not belong into c_0^λ. For $\lambda_k = O(1)$ we get $\lambda^{-1} \in c_0^\lambda$ and λ^{-1} does not belong into m_0^λ.

A matrix A is said to be regular if $\lim_n A_n x = \lim_n x_n$ for every $x = (x_n) \in c$, and λ-conservative if $A \in (c^\lambda, c^\lambda)$. A λ-conservative matrix A is said to be λ-regular (see [7]) if

$$c_{0A}^\lambda \cap c^\lambda = c_0^\lambda \text{ and } z_A^\lambda \cap c^\lambda = z^\lambda.$$

It is well known the Steinhaus result, which asserts that a regular matrix A cannot transform all bounded sequences into c (see, for example, [2], p. 51 or [1], p. 11). We prove the analogue of this result showing that a λ-regular matrix A cannot transform all λ-bounded sequences into c^λ.

2. Auxiliary results

For the proof of the main results we need some auxiliary results.

Lemma 2.1 ([2], p. 44, see also [8], Proposition 12). *A matrix* $A = (a_{nk}) \in (c_0, c)$ *if and only if conditions*

(I) $\lim_n a_{nk} := a_k$ *for all* k

and

(II) $\sum_k |a_{nk}| = O(1)$

are satisfied. Moreover,

$$\lim_n A_n x = \sum_k a_k x_k. \tag{2.1}$$

Lemma 2.2 ([2], pp. 44-45, see also [8], Proposition 23). *We have* $A = (a_{nk}) \in (c_0, c_0)$ *if and only if conditions* (I) *and* (II) *with* $a_k = 0$ *are satisfied.*

Lemma 2.3 ([2], p. 51, see also [8], Proposition 10). The following statements are equivalent:

(a) $A = (a_{nk}) \in (m, c)$.

(b) *The conditions* (I), (II) *are satisfied and*

$$\lim_n \sum_k |a_{nk} - a_k| = 0. \tag{2.2}$$

(c) *The condition in* (I) *holds and*

$$\text{the series } \sum_k |a_{nk}| \text{ converges uniformly in } n.$$

Moreover, if one of statements (a)-(c) is satisfied, then the equation in (2.1) *holds.*

Lemma 2.4 ([8], Proposition 21). *A matrix* $A = (a_{nk}) \in (m, c_0)$ *if and only if equation* (2.2) *with* $a_k = 0$ *holds.*

Lemma 2.5 ([2], p. 42, see also [8], Proposition 1). *A matrix* $A = (a_{nk}) \in (m, m) = (c, m) = (c_0, m)$ *if and only if condition* (II) *holds.*

Lemma 2.6 ([2], p. 46, see also [8], Proposition 11). *A matrix* $A = (a_{nk}) \in (c, c)$ *if and only if conditions* (I), (II) *are satisfied and*

(III) *there exists a finite limit* $\lim_n \sum_k a_{nk} = \tau$.

Moreover, if $A \in (c, c)$, then

$$\lim_n A_n x = \xi\tau + \sum_k (x_k - \xi) a_k.$$

Lemma 2.7 ([8], Proposition 22). *A matrix* $A = (a_{nk}) \in (c, c_0)$ *if and only if conditions* (I) *with* $a_k = 0$, (II) *and* (III) *with* $\tau = 0$ *are satisfied.*

A conservative matrix A is said to be τ-multiplicative, if there exist a number τ such that

$$\lim_n A_n x = \tau \lim_n x_n.$$

Lemma 2.8 ([2], p. 419, see also [6], p.20-21). *A matrix* $A = (a_{nk})$ *is* τ-multiplicative, if and only if conditions (I), (II) *with* $a_k = 0$ *and* (III) *with* $\tau \neq 0$ *are satisfied.*

Lemma 2.9 ([1], Proposition 9.2). *Every element $x := (x_k) \in c^\lambda$ can be represented in the form*

$$x = \xi(x)e + b(x)\lambda^{-1} + \sum_k \frac{l_k(x) - b(x)}{\lambda_k} e^k;$$

i.e., $c^\lambda = \{x = z + \alpha\lambda^{-1} + \beta e : z \in n^\lambda; \alpha, \beta \in \mathbb{C}\}$.

Now we give the necessary and sufficient conditions for λ-regularity of A, which was proved in [6], pp. 141-142. As this proof is available only in Estonian, we present this result with the proof.

Lemma 2.10. *Let λ be unbounded sequence. A matrix $A = (a_{nk})$ is λ-regular, if and only if condition* (I) *with $a_k = 0$ holds, and*

(IV) *$Ae \in c^\lambda \backslash z^\lambda$,*

(V) *The matrix $B = (b_{nk})$ defined by*

$$b_{nk} := \left(\lambda_n \frac{a_{nk} - a_k}{\lambda_k}\right)$$

is τ-multiplicative ($\tau \neq 0$), *where* τ is defined by

$$\tau = \lim_n \lambda_n \sum_k \frac{a_{nk} - a_k}{\lambda_k}. \tag{2.3}$$

Proof. Necessity. Let A be λ-regular. Then A is also λ-conservative. Hence B is conservative (see [1], Theorem 8.3). It follows from the definition of λ-regularity that

$$n_A^\lambda \cap c^\lambda = n^\lambda;$$

i.e., $A \in \left(n^\lambda, n^\lambda\right)$. Since $Ae^k \in n^\lambda$, we have $a_k = \beta_k$, where

$$\beta_k := \lim_n \beta_{nk}, \;\; \beta_{nk} := \lambda_n(a_{nk} - a_k).$$

Since $\lambda^{-1} \in z^\lambda \backslash n^\lambda$, we have $A\lambda^{-1} \in z^\lambda \backslash n^\lambda$. Therefore

$$\tau = \lim_n \lambda_n \sum_k \frac{a_{nk}}{\lambda_k}.$$

Thus, B is τ-multiplicative.

The validity of condition (IV) immediately follows from the definition of λ-regularity.

Sufficiency. Let conditions (I) (with $a_k = 0$), (IV) and (V) be satisfied. Since $c^\lambda = \{x = z + \alpha\lambda^{-1} + \beta e : z \in n^\lambda; \alpha, \beta \in \mathbb{C}\}$ (see Lemma 2.9), we have $c_A^\lambda = \{x = z + \alpha\lambda^{-1} + \beta e : z \in n_A^\lambda; \alpha, \beta \in \mathbb{C}\}$. Hence by the definition of λ-regularity it is sufficient to show that $A\lambda^{-1} \in z^\lambda \backslash n^\lambda$ and $A \in \left(n^\lambda, n^\lambda\right)$. Indeed, in this case from (2.3) we get $\lim_n A_n\lambda^{-1} = 0$. Hence

$$\lim_n d_n(\lambda^{-1}) = \tau \neq 0.$$

It means that $\lambda^{-1} \in z_A^\lambda \backslash n_A^\lambda$, and then $A\lambda^{-1} \in z^\lambda \backslash n^\lambda$. If $x \in n^\lambda$, then $\lambda x := (\lambda_k x_k) \in c_0$ and

$$\lim_n d_n(x) = \lim_n \lambda_n \sum_k a_{nk}x_k = \lim_n \lambda_n \sum_k \frac{a_{nk}}{\lambda_k}\lambda_k x_k = \lim_n B_n(\lambda x) = 0,$$

since τ-multiplicative matrix $B \in (c_0, c_0)$ by Lemma 2.2. Thus $A \in \left(n^\lambda, n^\lambda\right)$.

3. Main results

Now we are able to prove the main results of the paper.

Theorem 3.1. *Let $\lambda_n \neq O(1)$. We have $A = (a_{nk}) \in \left(m^\lambda, c^\mu\right)$ if and only if*

(VI) $Ae^k \in c^\mu$,

(VII) $Ae \in c^\mu$,

(VIII) $\sum\limits_k \frac{|a_{nk}|}{\lambda_k}$ *converges uniformly in* n,

(IX) $\mu_n \sum\limits_k \frac{|a_{nk}-a_k|}{\lambda_k}$ *converges uniformly in* n.

Proof. Necessity. Assume that $A \in \left(m^\lambda, c^\mu\right)$. It is easy to see that $e^k \in m^\lambda$ and $e \in m^\lambda$. Hence conditions (VI) and (VII) hold. Since, from (2.1) we have

$$x_k = \frac{l_k(x)}{\lambda_k} + \xi;\ \xi := \lim_k x_k,\ (l_k(x)) \in m$$

for every $x := (x_k) \in m^\lambda$, it follows that

$$A_n x = \sum_k \frac{a_{nk}}{\lambda_k} l_k(x) + \xi \mathfrak{A}_n,\ \text{where } \mathfrak{A}_n := \sum_k a_{nk}. \tag{3.1}$$

By (VII) we have $(\mathfrak{A}_n) \in c^\mu$, and from (3.1) we conclude that the matrix

$$A_\lambda := \left(\frac{a_{nk}}{\lambda_k}\right)$$

transforms this bounded sequence (l_k) into c. On the contrary, for every sequence $(l_k) \in m$, the sequence $(l_k/\lambda_k) \in c_0$. There exists a convergent sequence $x := (x_k)$ with $\xi := \lim_k x_k$, such that $l_k/\lambda_k = x_k - \xi$. Thus, we have proved that, for every sequence $(l_k) \in m$ there exists a sequence $(x_k) \in m^\lambda$ such that $l_k = \lambda_k (x_k - \xi)$. Hence $A_\lambda \in (m, c)$. This implies by Lemma 2.3 ((a) and (c)) that condition (VIII) is satisfied and for every $x := (x_k) \in m^\lambda$ we get

$$\phi(x) := \lim_n A_n x = \sum_k \frac{a_k}{\lambda_k} l_k(x) + \xi \lim_n \mathfrak{A}_n. \tag{3.2}$$

Hence for every $x := (x_k) \in m^\lambda$ we can write

$$\gamma_n(x) := \mu_n(A_n x - \phi(x)) = \mu_n \sum_k \frac{a_{nk} - a_k}{\lambda_k} l_k(x) + \xi \mu_n(\mathfrak{A}_n - \lim_n \mathfrak{A}_n). \tag{3.3}$$

Since $(\gamma_n(x)) \in c$ for every $x := (x_k) \in m^\lambda$, from (3.3) we conclude, by (VII), that the matrix $A_{\lambda,\mu} \in (m, c)$, where

$$A_{\lambda,\mu} := \left(\mu_n \frac{a_{nk} - a_k}{\lambda_k}\right).$$

Therefore condition (IX) is satisfied by Lemma 2.3 ((a) and (c)).

Sufficiency. Let conditions (VI) - (IX) be satisfied. Then relation (3.1) also holds for every $x \in m^\lambda$, condition (I) holds by (VI) and $(\mathfrak{A}_n) \in m^\mu$ by (VII). Hence by Lemma 2.3 ((a) and (c)), we obtain $A_\lambda \in (m, c)$ by (IX). Then also the limit $\phi(x)$ exists for every $x \in m^\lambda$ and is finite, and therefore relation (3.2) holds for every $x \in m^\lambda$. Consequently, using again Lemma 2.3 ((a) and (c)), we have $A_{\lambda,\mu} \in (m, c)$ by (VI) and (IX). Therefore from (3.2) we can conclude that $A \in \left(m^\lambda, c^\mu\right)$ by (VII).

Remark 3.1. Theorem 3.1 holds also for $\lambda_n = O(1)$. Indeed, in this case $(l_k(x)) \in c_0$ for every $x := (x_k) \in m^\lambda$. Therefore instead of $A_\lambda \in (m, c)$ and $A_{\lambda,\lambda} \in (m, c)$ we get $A_\lambda \in (c_0, c)$ and $A_{\lambda,\lambda} \in (c_0, c)$. These inclusions hold by Lemma 2.1 if and only if condition (VI) holds and

(X) $\sum_k \frac{|a_{nk}|}{\lambda_k} = O(1)$,

(XI) $\mu_n \sum_k \frac{|a_{nk}-a_k|}{\lambda_k} = O(1)$.

Besides, the validity of conditions (X) and (XI) follows from (VI), (VIII) and (IX) (see also Lemma 2.3).

Remark 3.2. Conditions (VIII) and (IX) in Theorem 3.1 can be replaced by conditions (X), (XI) and

(XII) $\lim_n \sum_k \frac{|\mu_n(a_{nk}-a_k)-s_k|}{\lambda_k} = 0$, $s_k := \lim_n \mu_n(a_{nk} - a_k)$.

Besides, if $\mu_n = O(1)$, then it is necessary to replace $O(1)$ by $o(1)$ in (XI); i.e., condition (XI) is equivalent to condition

(XIII) $\lim_n \sum_k \frac{|a_{nk}-a_k|}{\lambda_k} = 0$.

Indeed, in the proof of Theorem 3.1, instead of Lemma 2.3 ((a) and (c)) it is necessary to use Lemma 2.3 ((a) and (b)). The limit s_k exists and is finite by condition (VI) and the validity of condition (XIII) follows from (XI) for $\mu_n = O(1)$.

Now we prove the analogue of the well known theorem of Steinhaus (see, for example, [2], p. 51 or [1], p. 11).

Theorem 3.2. *Let $\lambda_n \neq O(1)$ and a matrix $A = (a_{nk})$ be λ-regular. Then A does not belong to $\left(m^\lambda, c^\lambda\right)$.*

Proof. Assume that A is λ-regular and $A \in \left(m^\lambda, c^\lambda\right)$. Then, by Lemma 2.10, A is τ-multiplicative ($\tau \neq 0$), where τ is defined by (2.3), where $a_k = 0$. Also the sequence

$$\left(\lambda_n \sum_k \frac{|a_{nk}|}{\lambda_k}\right)$$

converges uniformly in n by Theorem 3.1. Hence

$$\tau = \sum_k \frac{\lim_n \lambda_n a_{nk}}{\lambda_k}.$$

Since $Ae^k \in c^\lambda$ by Theorem 3.1 and A is λ-regular, then

$$\lim_n \lambda_n a_{nk} = 0.$$

Therefore $\tau = 0$. This leads to a contradiction. Thus, a λ-regular matrix A cannot belong to $\left(m^\lambda, c^\lambda\right)$.

Theorem 3.3. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(c^\lambda, c_0^\mu\right)$ if and only if conditions* (X) *and* (XI) *are satisfied, and*

(XIV) $Ae^k \in c_0^\mu$,

(XV) $Ae \in c_0^\mu$,

(XVI) $A\lambda^{-1} \in c^\mu$.

Proof. Necessity. Assume that $A \in \left(c^\lambda, c_0^\mu\right)$. It is easy to see that e^k, e, $\lambda^{-1} \in c^\lambda$. Hence conditions (XIV) - (XVI) hold. Since the equality in (3.1) holds for every $x := (x_k) \in c^\lambda$, and the finite limit $\tau = \lim_n \mathfrak{A}_n$ exists by (XV), then the matrix A_λ transforms this convergent sequence $(l_k(x))$ into c. By similar arguments as those used in the proof of the necessity in Theorem 3.1, we may show that, for every sequence $(l_k) \in c$, there exists a sequence $(x_k) \in c^\lambda$ such that $l_k = \lambda_k (x_k - \xi)$. Hence $A_\lambda \in (c, c)$. This implies by Lemma 2.6 that the finite limits a_k and

$$a^\lambda := \lim_n \sum_k \frac{a_{nk}}{\lambda_k}$$

exist, and that condition (X) is satisfied. Using the statement in (3.1), for every $x \in c^\lambda$, we can write

$$\phi(x) = a^\lambda b(x) + \sum_k \frac{a_k}{\lambda_k} (l_k(x) - b(x)) + \tau\xi. \tag{3.4}$$

Now, using (3.1) and (3.4), we obtain

$$\mu_n(A_n x - \phi(x)) = \mu_n \sum_k \frac{a_{nk} - a_k}{\lambda_k} (l_k(x) - b(x)) + \mu_n (\mathfrak{A}_n - \tau)\, \xi + \mu_n \left(\sum_k \frac{a_{nk}}{\lambda_k} - a^\lambda \right) b(x) \tag{3.5}$$

for every $x \in c^\lambda$. Since

$$\lim_n \mu_n (\mathfrak{A}_n - \tau)\, \xi = 0 \ \text{ and } \ \lim_n \mu_n \left(\sum_k \frac{a_{nk}}{\lambda_k} - a^\lambda \right) b(x) = 0 \tag{3.6}$$

by (XV) and (XVI), then $A_{\lambda,\mu} \in (c_0, c_0)$. Hence condition (XI) is satisfied by Lemma 2.1.
Sufficiency. Let conditions (X), (XI) and (XIV) - (XVI) be satisfied. First we note that relation (3.1) holds for every $x \in c^\lambda$ and the finite limits a_k, τ and a^λ exist correspondingly by (XIV), (XV) and (XVI). As (X) also holds, then $A_\lambda \in (c, c)$ by Lemma 2.6, and therefore relations (3.4) and (3.5) hold for every $x \in c^\lambda$. The relations (3.6) also hold by (XIV) and (XV). In addition, using conditions (XI) and (XIV), we can assert that $A_{\lambda,\mu} \in (c_0, c_0)$ by Lemma 2.2. Thus, $A \in \left(c^\lambda, c_0^\mu\right)$.

The proof of the next results is similar to the proof of Theorem 3.1 (see also the following Remark 3.2). Therefore we give for these results only short outlines of the proofs. The general role in the proof of Theorem 3.1 had the matrices A_λ and $A_{\lambda,\lambda}$. Therefore we always show the role of these matrices in the next results.

Theorem 3.4. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(m^\lambda, c_0^\mu\right)$ if and only if conditions* (X). (XIV) *and* (XV) *are satisfied, and*

(XVII) $\lim_n \mu_n \sum_k \frac{|a_{nk} - a_k|}{\lambda_k} = 0$.

Outline of the proof. It is easy to see that now $A_\lambda \in (m, c)$. Since $(\gamma_n(x)) \in c_0$ for every $x := (x_k) \in m^\lambda$, we have $A_{\lambda,\mu} \in (m, c_0)$. We complete the proof using Lemmas 2.3 and 2.4.

Theorem 3.5. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(m^\lambda, m_0^\mu\right)$ if and only if conditions* (XI) *and* (XIII) *are satisfied, and*

(XVIII) $Ae^k \in m_0^\mu$,

(XIX) $Ae \in m_0^\mu$.

Outline of the proof. In this case $\phi(x) = 0$ for every $x \in m^\lambda$. Hence, due to $(l(x)) \in m$ and $(\gamma_n(x)) \in m$, we have $A_\lambda \in (m, c_0)$ (see (3.2)) and $A_{\lambda,\mu} \in (m, m)$. So we complete the proof using Lemmas 2.4 and 2.5.

Remark 3.3. Condition (XIII) is redundant in Theorem 3.5 for $\mu_n \neq O(1)$, since in this case the validity of (XIII) follows from (XI).

Theorem 3.6. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(m_0^\lambda, m^\mu\right)$ if and only if conditions* (VIII) (*or* (X)) *and* (XI) *are satisfied, and*

(XX) $Ae^k \in m^\mu$.

Besides, if $\mu_n = O(1)$, then it is necessary to replace $O(1)$ by $o(1)$ *in* (XI).

Outline of the proof. Since in this case $A_\lambda \in (m, c)$ (see (3.2) and $A_{\lambda,\mu} \in (m, m)$, we complete the proof using Lemmas 2.3, 2.5 and Remark 3.2.

Theorem 3.7. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(m_0^\lambda, m_0^\mu\right)$ if and only if conditions* (XI), (XIII) *and* (XVIII) *are satisfied. Besides, for $\mu_n \neq O(1)$ condition* (XIII) *is redundant.*

Outline of the proof. Since in this case $A_\lambda \in (m, c)$ (see (3.2)) and $A_{\lambda,\mu} \in (m, m)$, we complete the proof using Lemmas 2.4 and 2.5.

Theorem 3.8. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(m_0^\lambda, c^\mu\right)$ if and only if conditions* (VIII), (IX) (*or conditions* (X), (XI))) *and* (VI) *are satisfied. Besides, if $\mu_n = O(1)$, then it is necessary to replace* $O(1)$ by $o(1)$ *in* (XI).

Outline of the proof. Since in this case $A_\lambda \in (m, c)$ and $A_{\lambda,\mu} \in (m, c)$, we complete the proof using Lemma

2.3.

Theorem 3.9. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(m_0^\lambda, c_0^\mu\right)$ if and only if conditions* (XIV) *and* (XVII) *are satisfied.*
Outline of the proof. Since in this case $A_\lambda \in (m, c)$ and $A_{\lambda,\mu} \in (m, c_0)$, we complete the proof using Lemmas 2.3 and 2.4.

Theorem 3.10. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(c_0^\lambda, m^\mu\right)$ if and only if conditions* (X), (XI) *and* (XX) *are satisfied.*
Outline of the proof. Since in this case $b(x) = 0$ for each $x \in c_0^\lambda$, $A_\lambda \in (c_0, c)$ and $A_{\lambda,\mu} \in (c_0, m)$, we complete the proof using Lemmas 2.3 and 2.5.

Theorem 3.11. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(c_0^\lambda, m_0^\mu\right)$ if and only if condition* (XI) *with $a_k = 0$ and condition* (XIV) *are satisfied.*
Outline of the proof. Since in this case $b(x) = 0$ for each $x \in c_0^\lambda$, $A_\lambda \in (c_0, c_0)$ and $A_{\lambda,\mu} \in (c_0, m)$, we complete the proof using Lemmas 2.2 and 2.5.

Theorem 3.12. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(c_0^\lambda, c^\mu\right)$ if and only if conditions* (VI), (VII) (X) *and* (XI) *are satisfied.*
Outline of the proof. Since in this case $b(x) = 0$ for each $x \in c_0^\lambda$, $A_\lambda \in (c_0, c)$ and $A_{\lambda,\mu} \in (c_0, c)$, we complete the proof using Lemma 2.1.

Theorem 3.13. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(c_0^\lambda, c_0^\mu\right)$ if and only if conditions* (XIV), (XV) *and* (XI) *with $a_k = 0$ are satisfied.*
Outline of the proof. Since in this case $b(x) = 0$ for each $x \in c_0^\lambda$, $A_\lambda \in (c_0, c)$ and $A_{\lambda,\mu} \in (c_0, c_0)$, we complete the proof using Lemmas 2.1 and 2.2.

Theorem 3.14. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(c^\lambda, m_0^\mu\right)$ if and only if conditions* (I), (XI) *with $a_k = 0$ and* (XIX) *are satisfied, and $a^\lambda = 0$.*
Outline of the proof. In this case $\phi(x) = 0$ for each $x \in c^\lambda$, $A_\lambda \in (c, c_0)$ and $A_{\lambda,\mu} \in (c, m)$. Since condition (X) follows from (XI) for $a_k = 0$, we complete the proof using Lemmas 2.1 and 2.3.

Theorem 3.15. *Let $\lambda_n \neq O(1)$. A matrix $A = (a_{nk}) \in \left(c^\lambda, m_0^\mu\right)$ if and only if conditions* (I), (XI) *with $a_k = 0$, and conditions* (XIX) and (XX) *are satisfied.*
Outline of the proof. In this case $\phi(x) = 0$ for each $x \in c^\lambda$, $A_\lambda \in (c, c_0)$ and $A_{\lambda,\mu} \in (c, m)$. Since condition (X) follows from (XI) for $a_k = 0$, we complete the proof using Lemmas 2.1 and 2.3.

References

[1] A. Aasma, H. Dutta, P.N. Natarajan, An Introductory Course in Summability Theory, John Wiley and Sons, Hoboken, USA, 2017.
[2] J. Boos, Classical and Modern Methods in Summability, University Press, Oxford, 2000.
[3] G. Kangro, On the summability factors of the Bohr-Hardy type for a given speed I., Proc. Estonian Acad. Sci. Phys. Math. 18 (1969) 137-146 (in Russian).
[4] G. Kangro, On the summability factors of the Bohr-Hardy type for a given speed II)., Eesti NSV Tead. Akad. Toimetised 18(4) (1969) 387-395.
[5] G. Kangro, Summability factors for the series λ-bounded by the methods of Riesz and Cesàro, Acta Comment. Univ. Tartuensis 277 (1971) 136-154 (in Russian).
[6] T. Leiger, Methods of functional analysis in summability theory, Tartu University, Tartu, 1992 in Estonian).
[7] T. Leiger, M. Maazik, The λ-inclusion of summation matrices, Tartu Riikl. *Ü*l. Toimetised 770 (1987), 61-68 (in Russian).
[8] M. Stieglitz, H. Tietz, Matrixtransformationen von Folgenr*ä*umen. Eine Ergebnis*ü*bersicht, Math. Z. 154 (1977) 1-14.