Filomat 37:4 (2023), 1053–1063
https://doi.org/10.2298/FIL2304053Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence and UH-stability of integral boundary problem for a class of nonlinear higher-order Hadamard fractional Langevin equation via Mittag-Leffler functions

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Abstract. The Langevin equation is a very important mathematical model in describing the random motion of particles. The fractional Langevin equation is a powerful tool in complex viscoelasticity. Therefore, this paper focuses on a class of nonlinear higher-order Hadamard fractional Langevin equation with integral boundary value conditions. Firstly, we employ successive approximation and Mittag-Leffler function to transform the differential equation into an equivalent integral equation. Then the existence and uniqueness of the solution are obtained by using the fixed point theory. Meanwhile, the Ulam-Hyers (UH) stability is proved by inequality technique and direct analysis.

1. Introduction

In order to describe the random motion of particles annihilated in the fluid due to the collision between particles and fluid molecules, the French physicist Paul Langevin proposed the famous Langevin equation in 1908. Many random phenomena and processes can be described by Langevin equation [2, 3]. However, for some very complex stochastic systems, the description of Langevin equation is not so accurate. Therefore, the classical Langevin equation has been extended and modified. For example, Kubo [13, 14] put forward a general Langevin equation to simulate the complex viscoelastic anomalous diffusion process. It is worth noting that the derivatives in these generalized equations are of integer order. Because fractional differential has advantages in describing the process of memory and viscoelasticity, another generalization of the Langevin equation is the fractional Langevin equation. For example, Eab and Lim [4] applied fractional Langevin equation to describe the single-file diffusion. Sandev and Tomovski [16] established a fractional Langevin equation model to study the motion of free particles driven by power-law noise. In addition, some new achievements have been made in the study of fractional calculus and fractional Langevin equation in the recently published papers (see [1, 9, 17–20, 39]).

As we all know, for a system with practical application background, its stability is very important. According to practical needs, scientists have put forward many concepts of system stability. Ulam-Hyers stability is one of the most important stability, which is raised by Ulam and Hyers [10, 21] in 1940s. In recent ten years, the research on Ulam-Hyers stability of fractional differential systems has been highly praised by

2020 *Mathematics Subject Classification*. Primary 34B15, 34D20; Secondary 34K37
Keywords. UH-stability, Hadamard fractional Langevin equation, higher-order integral boundary problem, Mittag-Leffler function
Received:04 February 2022; Revised: 22 August 2022; Accepted: 28 November 2022
Communicated by Hari M. Srivastava
Research supported by the start-up funds for high-level talents of Taizhou University
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many scholars. There have many works dealing with Ulam-Hyers stability [8, 15, 22, 25–27, 29, 30, 33–35, 37] and generalized Ulam-Hyers stability [11, 23, 28, 31, 32, 38] of fractional systems. However, only a few previous papers [5, 6, 24, 31, 32, 36] are involved in Ulam-Hyers stability of fractional Langevin systems. We singled out several enlightening research results for special presentation. For example, in [5], the authors considered the Ulam-Hyers stability of the following fractional Langevin equations

$$({}^{\mathrm{RL}}D^{\alpha}_{0+}y)(x) - \lambda y(x) = f(x),$$

and

$$({}^{\mathrm{RL}}D^{\alpha}_{0+}y)(x) - \lambda({}^{\mathrm{RL}}D^{\beta}_{0+}y)(t) = g(x),$$

where $\lambda \in \mathbb{R}$, $x > 0$, $n-1 < \alpha \le n$, $m-1 < \beta \le m$, $m \le n$, $m, n \in \mathbb{N}$, $f, g :\in C(\mathbb{R}^+, \mathbb{R})$, and ${}^{\mathrm{RL}}D^{*}_{0+}$ is the standard Riemann-Liouville fractional derivative of order $*$.

In [24], the authors discussed the Ulam-Hyers stability of the following fractional Langevin equations with impulses

$$\begin{cases} {}^{\mathrm{LC}}D^{\beta}_{t}({}^{\mathrm{LC}}D^{\alpha}_{t} - \lambda)x(t) = f(t, x(t)),\ t \in J \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta x(t_k) := x(t_k^+) - x(t_k^-) = I_k,\ I_k \in \mathbb{R} \end{cases}$$

where $J = (0, 1)$, α, β and $\alpha + \beta$ are belongs to $(0, 1)$ and $\lambda > 0$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, ${}^{\mathrm{LC}}D^{*}_{t}$ is the standard Liouville-Caputo fractional derivative of order $*$, the impulsive points satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, the symbols $x(t_k^+) = \lim\limits_{x\to 0^+} x(t_k + \varepsilon)$ and $x(t_k^-) = \lim\limits_{x\to 0^-} x(t_k - \varepsilon)$ are the right and left limits of $x(t)$ at the point $t = t_k$.

However, among the published research results on fractional order systems, the research papers involving Hadamard fractional derivatives are relatively rare than Riemann Liouville or Liouville-Caputo fractional derivatives. Therefore, there are few papers on UH-stability of Langevin system with Hadamard fractional derivative. Inspired by aforementioned, this paper mainly considers a class of nonlinear higher-order Hadamard fractional Langevin equation with integral boundary condition of the form

$$\begin{cases} {}^{\mathrm{LH}}D^{\beta}_{a^+}\Big[{}^{\mathrm{LH}}D^{\alpha-\beta}_{a^+} - \lambda\Big]u(t) = f(t, u),\ a < t \le T, \\ {}^{\mathrm{LH}}J^{\alpha}_{a^+}u(T) = \kappa \cdot {}^{\mathrm{LH}}J^{\beta}_{a^+}u(T),\ u'(a) = u''(a) = \ldots = u^{(n-1)}(a) = 0, \end{cases} \tag{1}$$

where $0 < a < T$, $0 < \beta < \alpha$, $n-1 < \alpha \le n$, $n \ge 2$, $\lambda > 0$, $\kappa \in \mathbb{R}$. ${}^{\mathrm{LH}}D^{\alpha}_{a^+}$ and ${}^{\mathrm{LH}}D^{\alpha-\beta}_{a^+}$ represent Hadamard fractional derivative of type Riemann-Liouville, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.

We focus on the existence and Ulam-Hyers stability of solutions of (1) by using fixed point theory and Mittag-Leffler functions. The remaining structure of the paper is as follows. Section 2 introduces some definitions and lemmas about Hadamard fractional calculus, Mittag-Leffler functions and Ulam-Hyers stability. In Section 3, the existence and Ulam-Hyers stability of (1) are proved. Section 4 makes a brief summary.

2. Preliminaries

This section mainly introduces some concepts and lemmas of Hadamard fractional calculus of type Riemann-Liouville, the Mittag-Leffler function and the concept of Ulam-Hyers stability for (1).

Definition 2.1. *[12] For $a > 0$, the left-sided Hadamard fractional integral of order $\alpha > 0$ for a function $u : (a, +\infty) \to \mathbb{R}$ is defined by*

$${}^{\mathrm{LH}}J^{\alpha}_{a^+}u(t) = \frac{1}{\Gamma(\alpha)}\int_{a^+}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{u(s)}{s}ds,$$

provided the integral exists, here $\log(\cdot) = \log_e(\cdot)$, $\Gamma(\cdot)$ *is the gamma function.*

Definition 2.2. *([12]) For $a > 0$, $\alpha > 0$, the α-order Hadamard fractional derivative of type Riemann-Liouville of a function $u \in C^n[a,\infty)$ is defined by*

$$ {}^{\mathrm{LH}}D^{\alpha}_{a^+}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_{a^+}^{t}\left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{u(s)}{s}ds, \ n-1<\alpha\le n, \ n=[\alpha]+1, $$

where $[\alpha]$ denotes the integer part of the real number $\alpha > 0$.

Lemma 2.3. *[12] Assume that $u \in C(a,T)\cap L^1(a,T)$ with α-order Hadamard fractional derivative of type Riemann-Liouville. Then*

$$ {}^{\mathrm{LH}}J^{\alpha}_{a^+}\,{}^{\mathrm{LH}}D^{\alpha}_{a^+}u(t) = u(t) + c_1\left(\log\frac{t}{a}\right)^{\alpha-1} + c_2\left(\log\frac{t}{a}\right)^{\alpha-2} + \ldots + c_n\left(\log\frac{t}{a}\right)^{\alpha-n}, $$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n-1,n$ and $n = [\alpha]+1$.

Lemma 2.4. *[12] If $\alpha,\beta > 0$, then the following properties hold:*

$$ {}^{\mathrm{LH}}D^{\alpha}_{a^+}\,{}^{\mathrm{LH}}J^{\alpha}_{a^+}u(t) = u(t), \ {}^{\mathrm{LH}}J^{\alpha}_{a^+}\,{}^{\mathrm{LH}}J^{\beta}_{a^+}u(t) = {}^{\mathrm{LH}}J^{\alpha+\beta}_{a^+}u(t), \ {}^{\mathrm{LH}}D^{\alpha}_{a^+}\,{}^{\mathrm{LH}}D^{\beta}_{a^+}u(t) = {}^{\mathrm{LH}}D^{\alpha+\beta}_{a^+}u(t), $$

$$ {}^{\mathrm{LH}}D^{\alpha}_{a^+}\left(\log\frac{t}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{t}{a}\right)^{\beta-\alpha-1}, \ {}^{\mathrm{LH}}J^{\alpha}_{a^+}\left(\log\frac{t}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{t}{a}\right)^{\beta+\alpha-1}, $$

especially, ${}^{LH}D^{\alpha}_{a^+}\left(\log\frac{t}{a}\right)^{\alpha-j} = 0$, $j = 1,2,\ldots,[\alpha]+1$.

Definition 2.5. *[12] The one-parameter Mittag-Leffler function $E_\alpha(z)$ and the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ are defined by the series expansion*

$$ E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \ \alpha,\ z \in \mathbb{C},\ \Re(\alpha) > 0, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \ \alpha,\ \beta,\ z \in \mathbb{C},\ \Re(\alpha) > 0, $$

where $\mathbb{C}$ is the complex field, $\Re(\alpha)$ is the real part of complex number α.

Let $J = [a,T]$, $\mathbb{X} = C(J,\mathbb{R})$. Then $\mathbb{X}$ is a Banach space with the norm $\|u\| = \sup_{t\in J}|u(t)|$. We shall study the existence and Ulam-Hyers stability of solution of (1) in $(\mathbb{X},\|\cdot\|)$. Now we introduce the concept and property of Ulam-Hyers stability. Let $z \in \mathbb{X}$, $\epsilon > 0$, consider the following inequality

$$ \left| {}^{\mathrm{LH}}D^{\beta}_{a^+}\left[{}^{\mathrm{LH}}D^{\alpha-\beta}_{a^+} - \lambda\right]z(t) - f(t,z)\right| \le \epsilon, \ a < t \le T. \tag{2} $$

Definition 2.6. *Assume that for all $\epsilon > 0$ and each solution $z \in \mathbb{X}$ of inequality (2), there have a constant $M > 0$ and a solution $u \in \mathbb{X}$ of system (1) such that*

$$ \|z(t) - u(t)\| \le M\epsilon, $$

then the equation (1) is called Ulam-Hyers stable.

Remark 2.7. *A function $z \in \mathbb{X}$ is a solution of inequality (2) if and only if there exists a function $\phi \in \mathbb{X}$ such that*

(a) $|\phi(t)| \le \epsilon$, $a < t \le T$.

(b) ${}^{\mathrm{LH}}D^{\beta}_{a^+}\left[{}^{\mathrm{LH}}D^{\alpha-\beta}_{a^+} - \lambda\right]z(t) = f(t,z) + \phi(t), \ a < t \le T.$

Lemma 2.8. *[7] Let $\mathbb{E}$ be a non-empty closed subset of a Banach space $\mathbb{X}$. If $\mathscr{M} : \mathbb{E} \to \mathbb{E}$ is a contraction mapping, then $\mathscr{M}$ has a unique fixed point $x^* \in \mathbb{E}$.*

3. Main results

In this section, we focus on the existence and stability of solutions for system (1). To this end, we need to prove the following important lemmas.

Lemma 3.1. *For any $0 < a < t$ and $\gamma, \delta > 0$, the following integral holds.*

$$\int_a^t \left(\log \frac{t}{s}\right)^{\gamma-1} \cdot \left(\log \frac{s}{a}\right)^{\delta-1} \frac{ds}{s} = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} \left(log \frac{t}{a}\right)^{\gamma+\delta-1}. \tag{3}$$

Proof. Let $\log \frac{s}{a} = \xi$ and denote the integral on the left-hand side of (3) as I, then

$$\begin{aligned} I = &\int_0^{log \frac{t}{a}} \left(log \frac{t}{a} - \xi\right)^{\gamma-1} \xi^{\delta-1} d\xi \overset{\xi=\eta \log \frac{t}{a}}{=\!=\!=\!=} \int_0^1 \left(\log \frac{t}{a} - \eta \log \frac{t}{a}\right)^{\gamma-1} \left(\eta \log \frac{t}{a}\right)^{\delta-1} \log \frac{t}{a} d\eta \\ =&\left(\log \frac{t}{a}\right)^{\gamma+\delta-1} \int_0^1 (1-\eta)^{\gamma-1} \eta^{\delta-1} d\eta = \left(\log \frac{t}{a}\right)^{\gamma+\delta-1} \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}. \end{aligned}$$

The proof is completed. □

Lemma 3.2. *Consider the following μ-order Hadamard fractional differential equation of type Riemann-Liouville*

$${}^{\mathrm{LH}}D^{\mu}_{a^+} u(t) + \lambda u(t) = h(t), \ t > a, \tag{4}$$

where $a, \mu > 0$, $l - 1 < \mu \leq l$, $l \geq 2$, $\lambda > 0$. ${}^{\mathrm{LH}}D^{\mu}_{a^+}$ represents Hadamard fractional derivative of type Riemann-Liouville. If $h(t) \in C((a, +\infty), \mathbb{R})$ is a given function, then the general solution of (4) is read by

$$u(t) = \sum_{j=1}^{l} d_j \left(\log \frac{t}{a}\right)^{\mu-j} E_{\mu,\mu-j+1}\left[-\lambda \left(\log \frac{t}{a}\right)^{\mu}\right] + \int_{a^+}^t \left(\log \frac{t}{s}\right)^{\mu-1} E_{\mu,\mu}\left[-\lambda \left(\log \frac{t}{s}\right)^{\mu}\right] h(s) \frac{ds}{s},$$

where $d_1, d_2, \dots, d_l$ are some unknown constants.

Proof. From Definitions 2.1-2.2 and Lemmas 2.3-2.4 together with (4), we have

$$\begin{aligned} u(t) = &- \lambda \, {}^{\mathrm{LH}}J^{\mu}_{a^+} u(t) + {}^{\mathrm{LH}}J^{\mu}_{a^+} h(t) + c_1 \left(\log \frac{t}{a}\right)^{\mu-1} + c_2 \left(\log \frac{t}{a}\right)^{\mu-2} + \dots + c_l \left(\log \frac{t}{a}\right)^{\mu-l} \\ =&\frac{-\lambda}{\Gamma(\mu)} \int_{a^+}^t \left(\log \frac{t}{s}\right)^{\mu-1} \frac{u(s)}{s} ds + \frac{1}{\Gamma(\mu)} \int_{a^+}^t \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} ds \\ &+ c_1 \left(\log \frac{t}{a}\right)^{\mu-1} + c_2 \left(\log \frac{t}{a}\right)^{\mu-2} + \dots + c_l \left(\log \frac{t}{a}\right)^{\mu-l}, \end{aligned} \tag{5}$$

where $c_1, c_2, \dots, c_l$ are some unknown constants. Next, we apply the successive approximation method to solve the equation (4). Let $c_j = \frac{d_j}{\Gamma(\alpha-j+1)}$,

$$u_0(t) = \sum_{j=1}^{l} \frac{d_j}{\Gamma(\mu-j+1)} \left(\log \frac{t}{a}\right)^{\mu-j}, \tag{6}$$

then the following recursive formula is obtained

$$u_m(t) = u_0(t) + \frac{-\lambda}{\Gamma(\mu)} \int_{a^+}^t \left(\log \frac{t}{s}\right)^{\mu-1} \frac{u_{m-1}(s)}{s} ds + \frac{1}{\Gamma(\mu)} \int_{a^+}^t \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} ds. \tag{7}$$

(7) can be rewritten into the following fractional integral form

$$u_m(t) = u_0(t) - \lambda \cdot {}^{\mathrm{LH}}J^{\mu}_{a^+} u_{m-1}(t) + {}^{\mathrm{LH}}J^{\mu}_{a^+} h(t). \tag{8}$$

By employing (7) or (8) and (3), one has

$$\begin{aligned} u_1(t) =& u_0(t) + \frac{-\lambda}{\Gamma(\mu)} \sum_{j=1}^{l} \frac{d_j}{\Gamma(\mu - j + 1)} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \left(\log \frac{s}{a}\right)^{\mu-j} \frac{ds}{s} + \frac{1}{\Gamma(\mu)} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} ds \\ =& u_0(t) + \frac{-\lambda}{\Gamma(\mu)} \sum_{j=1}^{l} \frac{d_j}{\Gamma(\mu - j + 1)} \frac{\Gamma(\mu)\Gamma(\mu - j + 1)}{\Gamma(2\mu - j + 1)} \left(\log \frac{t}{a}\right)^{2\mu-j} + \frac{1}{\Gamma(\mu)} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} ds \\ =& \sum_{j=1}^{l} d_j \sum_{k=1}^{2} \frac{(-\lambda)^{k-1} \left(\log \frac{t}{a}\right)^{k\mu-j}}{\Gamma(k\mu - j + 1)} + \frac{1}{\Gamma(\mu)} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} ds, \end{aligned} \tag{9}$$

and

$$\begin{aligned} u_2(t) =& u_0(t) + \frac{-\lambda}{\Gamma(\mu)} \sum_{j=1}^{l} d_j \sum_{k=1}^{2} \frac{(-\lambda)^{k-1}}{\Gamma(k\mu - j + 1)} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \left(\log \frac{s}{a}\right)^{k\mu-j} \frac{ds}{s} \\ &+ \frac{-\lambda}{(\Gamma(\mu))^2} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} \Big[\int_{a^+}^{s} \left(\log \frac{s}{\tau}\right)^{\mu-1} h(\tau) \frac{d\tau}{\tau} \Big] ds + \frac{1}{\Gamma(\mu)} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} ds \\ =& u_0(t) + \frac{-\lambda}{\Gamma(\mu)} \sum_{j=1}^{l} d_j \sum_{k=1}^{2} \frac{(-\lambda)^{k-1}}{\Gamma(k\mu - j + 1)} \frac{\Gamma(\mu)\Gamma(k\mu - j + 1)}{\Gamma((k+1)\mu - j + 1)} \left(\log \frac{t}{a}\right)^{(k+1)\mu-j} \\ &+ \frac{-\lambda}{(\Gamma(\mu))^2} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} \Big[\int_{a^+}^{s} \left(\log \frac{s}{\tau}\right)^{\mu-1} h(\tau) \frac{d\tau}{\tau} \Big] ds + \frac{1}{\Gamma(\mu)} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} ds \\ =& \sum_{j=1}^{l} d_j \sum_{k=1}^{3} \frac{(-\lambda)^{k-1} \left(\log \frac{t}{a}\right)^{k\mu-j}}{\Gamma(k\mu - j + 1)} + \frac{1}{\Gamma(\mu)} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} ds \\ &+ \frac{-\lambda}{(\Gamma(\mu))^2} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} \Big[\int_{a^+}^{s} \left(\log \frac{s}{\tau}\right)^{\mu-1} h(\tau) \frac{d\tau}{\tau} \Big] ds. \end{aligned} \tag{10}$$

Noting that

$$\begin{aligned} & \frac{-\lambda}{(\Gamma(\mu))^2} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \frac{h(s)}{s} \Big[\int_{a^+}^{s} \left(\log \frac{s}{\tau}\right)^{\mu-1} h(\tau) \frac{d\tau}{\tau} \Big] ds \\ =& \frac{-\lambda}{(\Gamma(\mu))^2} \int_{a^+}^{t} h(\tau) \Big[\int_{\tau^+}^{t} \left(\log \frac{t}{s}\right)^{\mu-1} \left(\log \frac{s}{\tau}\right)^{\mu-1} \frac{ds}{s} \Big] \frac{d\tau}{\tau} \\ =& \frac{-\lambda}{(\Gamma(\mu))^2} \int_{a^+}^{t} h(\tau) \Big[\frac{\Gamma(\mu)\Gamma(\mu)}{\Gamma(2\mu)} \left(\log \frac{t}{\tau}\right)^{2\mu-1} \Big] \frac{d\tau}{\tau} \\ =& \frac{-\lambda}{\Gamma(2\mu)} \int_{a^+}^{t} \left(\log \frac{t}{\tau}\right)^{2\mu-1} h(\tau) \frac{d\tau}{\tau}. \end{aligned} \tag{11}$$

Bring (11) into (10), we have

$$u_2(t) = \sum_{j=1}^{l} d_j \sum_{k=1}^{3} \frac{(-\lambda)^{k-1} \left(\log \frac{t}{a}\right)^{k\mu-j}}{\Gamma(k\mu - j + 1)} + \int_{a^+}^{t} \sum_{k=1}^{2} \frac{(-\lambda)^{k-1}}{\Gamma(k\mu)} \left(\log \frac{t}{s}\right)^{k\mu-1} h(s) \frac{ds}{s}. \tag{12}$$

Repeating the above process, we derive

$$
\begin{aligned}
u_m(t) &= \sum_{j=1}^{l} d_j \sum_{k=1}^{m+1} \frac{(-\lambda)^{k-1}\left(\log\frac{t}{a}\right)^{k\mu-j}}{\Gamma(k\mu-j+1)} + \int_{a^+}^{t} \sum_{k=1}^{m} \frac{(-\lambda)^{k-1}}{\Gamma(k\mu)}\left(\log\frac{t}{s}\right)^{k\mu-1} h(s)\frac{ds}{s} \\
&= \sum_{j=1}^{l} d_j \sum_{k=0}^{m} \frac{(-\lambda)^{k}\left(\log\frac{t}{a}\right)^{k\mu+\mu-j}}{\Gamma(k\mu+\mu-j+1)} + \int_{a^+}^{t} \sum_{k=0}^{m-1} \frac{(-\lambda)^{k}}{\Gamma(k\mu+\mu)}\left(\log\frac{t}{s}\right)^{k\mu+\mu-1} h(s)\frac{ds}{s}. \qquad (13)
\end{aligned}
$$

Letting $m \to +\infty$ on both sides of (13) together with Definition 2.5, we get

$$
\begin{aligned}
u(t) &= \sum_{j=1}^{l} d_j \sum_{k=0}^{\infty} \frac{(-\lambda)^{k}\left(\log\frac{t}{a}\right)^{k\mu+\mu-j}}{\Gamma(k\mu+\mu-j+1)} + \int_{a^+}^{t} \sum_{k=0}^{\infty} \frac{(-\lambda)^{k}}{\Gamma(k\mu+\mu)}\left(\log\frac{t}{s}\right)^{k\mu+\mu-1} h(s)\frac{ds}{s} \\
&= \sum_{j=1}^{l} d_j \left(\log\frac{t}{a}\right)^{\mu-j} \sum_{k=0}^{\infty} \frac{\left[-\lambda\left(\log\frac{t}{a}\right)^{\mu}\right]^k}{\Gamma(k\mu+\mu-j+1)} + \int_{a^+}^{t} \sum_{k=0}^{\infty} \frac{\left[-\lambda\left(\log\frac{t}{s}\right)^{\mu}\right]^k}{\Gamma(k\mu+\mu)}\left(\log\frac{t}{s}\right)^{\mu-1} h(s)\frac{ds}{s} \\
&= \sum_{j=1}^{l} d_j \left(\log\frac{t}{a}\right)^{\mu-j} E_{\mu,\mu-j+1}\left[-\lambda\left(\log\frac{t}{a}\right)^{\mu}\right] + \int_{a^+}^{t} \left(\log\frac{t}{s}\right)^{\mu-1} E_{\mu,\mu}\left[-\lambda\left(\log\frac{t}{s}\right)^{\mu}\right] h(s)\frac{ds}{s}. \qquad (14)
\end{aligned}
$$

The proof is completed. □

Lemma 3.3. *Consider the following BVP of Hadamard fractional differential equation of type Riemann-Liouville*

$$
\begin{cases} {}^{\mathrm{LH}}D^{\beta}_{a^+}\left[{}^{\mathrm{LH}}D^{\alpha-\beta}_{a^+}+\lambda\right]u(t) = h(t), \ a < t \le T, \\ {}^{\mathrm{LH}}J^{\alpha}_{a^+}u(T) = \kappa \cdot {}^{\mathrm{LH}}J^{\beta}_{a^+}u(T), \ u'(a) = u''(a) = \ldots = u^{(n-1)}(a) = 0. \end{cases} \qquad (15)
$$

where $0 < a < T$, $0 < \beta < \alpha$, $n-1 < \alpha \le n$, $n \ge 2$, $\lambda > 0$, $\kappa \in \mathbb{R}$. ${}^{\mathrm{LH}}D^{\alpha}_{a^+}$ *and* ${}^{\mathrm{LH}}D^{\alpha-\beta}_{a^+}$ *represent Hadamard fractional derivative of type Riemann-Liouville. If* $h(t) \in C((a,+\infty),\mathbb{R})$ *is a given function and* $H(T,a) \neq 0$*, then the unique solution of* (15) *is given by*

$$
u(t) = \int_{a^+}^{t} G_\alpha(t,s)h(s)\frac{ds}{s} - \frac{G_\alpha(t,a)}{H(T,a)}\int_{a^+}^{T} H(T,s)h(s)\frac{ds}{s}, \qquad (16)
$$

where

$$
G_\eta(t,s) = \left(\log\frac{t}{s}\right)^{\eta-1} E_{\alpha-\beta,\eta}\left[-\lambda\left(\log\frac{t}{a}\right)^{\alpha-\beta}\right], \qquad (17)
$$

$$
H(t,s) = G_{2\alpha}(t,s) - \kappa G_{\alpha+\beta}(t,s). \qquad (18)
$$

Proof. Let $v(t) = {}^{\mathrm{LH}}D^{\beta}_{a^+}u(t)$ and denote $[\alpha-\beta] = p$, then the first equation of (15) becomes

$$
{}^{\mathrm{LH}}D^{\alpha-\beta}_{a^+}v(t) + \lambda v(t) = h(t). \qquad (19)
$$

It follows from Lemma 3.2 that the general solution of equation (19) is

$$
\begin{aligned}
v(t) = &\sum_{j=1}^{p} d_j\left(\log\frac{t}{a}\right)^{\alpha-\beta-j} E_{\alpha-\beta,\alpha-\beta-j+1}\left[-\lambda\left(\log\frac{t}{a}\right)^{\alpha-\beta}\right] \\
&+ \int_{a^+}^{t}\left(\log\frac{t}{s}\right)^{\alpha-\beta-1} E_{\alpha-\beta,\alpha-\beta}\left[-\lambda\left(\log\frac{t}{s}\right)^{\alpha-\beta}\right] h(s)\frac{ds}{s}. \qquad (20)
\end{aligned}
$$

In view of (3), (20) and Lemma 2.3, we have

$$\begin{aligned} u(t) = {}^{\mathrm{LH}}J_{a^+}^{\beta} v(t) = {} & c_1 \left(\log \frac{t}{a}\right)^{\beta-1} + c_2 \left(\log \frac{t}{a}\right)^{\beta-2} + \ldots + c_{n-p} \left(\log \frac{t}{a}\right)^{\beta-n+p} \\ & + \sum_{j=1}^{p} d_j \left(\log \frac{t}{a}\right)^{\alpha-j} E_{\alpha-\beta,\alpha-j+1}\left[-\lambda \left(\log \frac{t}{a}\right)^{\alpha-\beta}\right] \\ & + \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha-\beta,\alpha}\left[-\lambda \left(\log \frac{t}{s}\right)^{\alpha-\beta}\right] h(s) \frac{ds}{s}, \end{aligned} \tag{21}$$

where $d_1, d_2, \ldots, d_p, c_1, c_2, \ldots, c_{n-p}$ are some unknown constants. Noticing that $0 < \beta < \alpha$ and $n-1 < \alpha \le n$, according to boundary value conditions $u^{(j)}(a) = 0$, $j = 1, 2, \ldots, n-1$, we obtain $d_2 = d_3 = \ldots = d_p = c_1 = c_2 = \ldots = c_{n-p} = 0$. So (21) is simplified as

$$u(t) = d_1 \left(\log \frac{t}{a}\right)^{\alpha-1} E_{\alpha-\beta,\alpha}\left[-\lambda \left(\log \frac{t}{a}\right)^{\alpha-\beta}\right] + \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha-\beta,\alpha}\left[-\lambda \left(\log \frac{t}{s}\right)^{\alpha-\beta}\right] h(s) \frac{ds}{s}. \tag{22}$$

From (22), we have

$$\begin{aligned} {}^{\mathrm{LH}}J_{a^+}^{\beta} u(T) = {} & d_1 \left(\log \frac{T}{a}\right)^{\alpha+\beta-1} E_{\alpha-\beta,\alpha+\beta}\left[-\lambda \left(\log \frac{T}{a}\right)^{\alpha-\beta}\right] \\ & + \int_{a^+}^{T} \left(\log \frac{T}{s}\right)^{\alpha+\beta-1} E_{\alpha-\beta,\alpha+\beta}\left[-\lambda \left(\log \frac{T}{s}\right)^{\alpha-\beta}\right] h(s) \frac{ds}{s}, \end{aligned} \tag{23}$$

$${}^{\mathrm{LH}}J_{a^+}^{\alpha} u(T) = d_1 \left(\log \frac{T}{a}\right)^{2\alpha-1} E_{\alpha-\beta,2\alpha}\left[-\lambda \left(\log \frac{T}{a}\right)^{\alpha-\beta}\right] + \int_{a^+}^{T} \left(\log \frac{T}{s}\right)^{2\alpha-1} E_{\alpha-\beta,2\alpha}\left[-\lambda \left(\log \frac{T}{s}\right)^{\alpha-\beta}\right] h(s) \frac{ds}{s}. \tag{24}$$

By (23) and (24) associated with boundary value conditions ${}^{\mathrm{LH}}J_{a^+}^{\alpha} u(T) = \kappa \cdot {}^{\mathrm{LH}}J_{a^+}^{\beta} u(T)$, we obtain

$$d_1 = -\frac{1}{H(T,a)} \int_{a^+}^{T} H(T,s) h(s) \frac{ds}{s}. \tag{25}$$

Substituting (25) into (22), we get (16). The proof is completed. □

From Lemma 3.3, we have the following important conclusion.

Lemma 3.4. *Let $0 < \beta < \alpha$, $n-1 < \alpha \le n$, $n \ge 2$, $\lambda > 0$, $\kappa \in \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$. If $H(T,a) \ne 0$, then $u(t) \in \mathbb{X}$ is a solution of equation* (1) *if and only if $u(t) \in \mathbb{X}$ is a solution of the following fractional integral equation*

$$u(t) = \int_{a^+}^{t} G_\alpha(t,s) f(s,u(s)) \frac{ds}{s} - \frac{G_\alpha(t,a)}{H(T,a)} \int_{a^+}^{T} H(T,s) f(s,u(s)) \frac{ds}{s}, \ a < t \le T, \tag{26}$$

where $G_\eta(t,s)$ and $H(t,s)$ are defined as (17) *and* (18)*, respectively.*

According to Lemma 3.4, we define a mapping $\mathscr{M} : \mathbb{X} \to \mathbb{X}$ as follows:

$$(\mathscr{M}u)(t) = \int_{a^+}^{t} G_\alpha(t,s) f(s,u(s)) \frac{ds}{s} - \frac{G_\alpha(t,a)}{H(T,a)} \int_{a^+}^{T} H(T,s) f(s,u(s)) \frac{ds}{s}, \ a < t \le T. \tag{27}$$

Thus the solution of the equation (1) is equivalent to the fixed point of the mapping $\mathscr{M}$ defined as (27). For convenience and unification, we make a priori estimation of Mittag-Leffler function under certain conditions.

Lemma 3.5. *Let* $0 < a < s < t \le T$, $0 < \beta < \alpha$, $\alpha \ge 2$, $\eta \ge 1$. *If* $0 < \lambda < \frac{1}{\left(\log \frac{T}{a}\right)^{\alpha-\beta}}$, *then*

(1) $\left|E_{\alpha-\beta,\eta}\left[-\lambda\left(\log \frac{t}{s}\right)^{\alpha-\beta}\right]\right| \le \frac{1}{1-\lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}}$.

(2) $|G_\eta(t,s)| \le \frac{\left(\log \frac{t}{s}\right)^{\eta-1}}{1-\lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}}$, $|H(t,s)| \le \frac{\left(\log \frac{t}{s}\right)^{2\alpha-1}}{1-\lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}} + \frac{\kappa\left(\log \frac{t}{s}\right)^{\alpha+\beta-1}}{1-\lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}}$.

Proof. When $\eta \ge 1$, $0 < \beta < \alpha$ and $m \in \mathbb{N}$, $1 = \Gamma(1) \le \Gamma(m(\alpha-\beta)+\eta)$. Thus we have

$$\left|E_{\alpha-\beta,\eta}\left[-\lambda\left(\log \frac{t}{s}\right)^{\alpha-\beta}\right]\right| = \left|\sum_{m=0}^{\infty} \frac{\left[-\lambda\left(\log \frac{t}{s}\right)^{\alpha-\beta}\right]^m}{\Gamma(m(\alpha-\beta)+\eta)}\right| \le \sum_{m=0}^{\infty}\left|\left[-\lambda\left(\log \frac{t}{s}\right)^{\alpha-\beta}\right]^m\right|$$

$$\le \sum_{m=0}^{\infty}\left[\lambda\left(\log \frac{t}{s}\right)^{\alpha-\beta}\right]^m \le \sum_{m=0}^{\infty}\left[\lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}\right]^m = \frac{1}{1-\lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}}.$$

Thus the conclusion (1) holds, and $E_{\alpha-\beta,\eta}\left[-\lambda\left(\log \frac{t}{s}\right)^{\alpha-\beta}\right]$ is absolutely uniformly convergent. By (1) associated with (17) and (18), one easily gets (2). The proof is completed. $\square$

In addition, our main results need the following assumptions.

(C_1) Assume that $f \in C(J \times \mathbb{R}, \mathbb{R})$, and for all $t \in J$, $\overline{u}, \overline{v} \in \mathbb{R}$, there has a constant $Q > 0$ such that

$$|f(t,\overline{u}) - f(t,\overline{v})| \le Q|\overline{u} - \overline{v}|.$$

(C_2) Let a, T, α, β, λ and κ are some real constants satisfy $0 < a < T$, $0 < \beta < \alpha$, $n-1 < \alpha \le n$, $n \ge 2$ and $\lambda > 0$. Assume that $0 < \lambda < \frac{1}{\left(\log \frac{T}{a}\right)^{\alpha-\beta}}$, $H(T,a) \ne 0$ and $0 < \rho < 1$, where

$$H(T,a) = \left(\log \frac{T}{a}\right)^{2\alpha-1} E_{\alpha-\beta,2\alpha}\left[-\lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}\right] - \kappa\left(\log \frac{T}{a}\right)^{\alpha+\beta-1} E_{\alpha-\beta,\alpha+\beta}\left[-\lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}\right]$$

$$\rho = \frac{TQ\left(\log \frac{T}{a}\right)^{\alpha}}{|H(T,a)|\left(1-\lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}\right)^2}\left[\frac{1}{\alpha}|H(T,a)|\left(1-\lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}\right) + \frac{1}{2\alpha}\left(\log \frac{T}{a}\right)^{2\alpha-1} + \frac{\kappa}{\alpha+\beta}\left(\log \frac{T}{a}\right)^{\alpha+\beta-1}\right].$$

Theorem 3.6. *If* (C_1) *and* (C_2) *are true, then we have the following two claims*

(i) *the eqution* (1) *has a unique solution* $u^*(t) \in \mathbb{X}$;

(ii) *the equation* (1) *is Ulam-Hyers stable, namely, assume that* $z(t) \in \mathbb{X}$ *is a solution of inequality* (2), *and* $u^*(t) \in \mathbb{X}$ *is a unique solution of equation* (1), *then*

$$\|z(t) - u^*(t)\| \le \rho\epsilon.$$

Proof. Now, we apply Lemma 2.8 to prove that the claim (i) in Theorem 3.6 holds. Define a mapping $\mathscr{M} : \mathbb{X} \to \mathbb{X}$ as (27), it suffices to verify that $\mathscr{M}$ is compressive. Indeed, $u \in \mathbb{X}$ and $f \in C(J \times \mathbb{R}, \mathbb{R})$ indicate

$(\mathscr{M}u)(t) \in \mathbb{X}$. For $\forall\, t \in J$, $u(t), v(t) \in \mathbb{X}$, by employing (C_1) and (C_2) together with (17), (18), (27) and Lemma 3.5, one has

$$
\begin{aligned}
&|(\mathscr{M}u)(t) - (\mathscr{M}v)(t)| \leq \int_{a^+}^{t} \left|G_\alpha(t,s)\right| \cdot \left|f(s,u(s)) - f(s,v(s))\right| \frac{ds}{s} \\
&+ \left|\frac{G_\alpha(t,a)}{H(T,a)}\right| \int_{a^+}^{T} |H(T,s)| \cdot \left|f(s,u(s)) - f(s,v(s))\right| \frac{ds}{s} \\
\leq& \int_{a^+}^{t} \left|G_\alpha(t,s)\right| \cdot Q|u(s) - v(s)| \frac{ds}{s} + \frac{|G_\alpha(t,a)|}{|H(T,a)|} \int_{a^+}^{T} |H(T,s)| \cdot Q|u(s) - v(s)| \frac{ds}{s} \\
\leq& \int_{a^+}^{t} \frac{\left(\log \frac{t}{s}\right)^{\alpha-1}}{1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}} Q|u(s) - v(s)| \frac{ds}{s} + \frac{\left(\log \frac{t}{a}\right)^{\alpha-1}}{|H(T,a)|\left[1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}\right]} \\
&\times \int_{a^+}^{T} \left[\frac{\left(\log \frac{T}{s}\right)^{2\alpha-1}}{1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}} + \frac{\kappa\left(\log \frac{T}{s}\right)^{\alpha+\beta-1}}{1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}}\right] \cdot Q|u(s) - v(s)| \frac{ds}{s} \\
\leq& \frac{1}{1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}} \int_{a^+}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \cdot Q \cdot \|u - v\| + \frac{\left(\log \frac{t}{a}\right)^{\alpha-1}}{|H(T,a)|\left[1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}\right]} \\
&\times \frac{1}{1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}} \int_{a^+}^{T} \left[\left(\log \frac{T}{s}\right)^{2\alpha-1} + \kappa\left(\log \frac{T}{s}\right)^{\alpha+\beta-1}\right] \frac{ds}{s} \cdot Q \cdot \|u - v\| \\
=& \frac{t\left(\log \frac{t}{a}\right)^{\alpha}}{\alpha\left[1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}\right]} \cdot Q \cdot \|u - v\| + \frac{\left(\log \frac{t}{a}\right)^{\alpha-1}}{|H(T,a)|\left[1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}\right]} \\
&\times \frac{1}{1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}} \left[\frac{T}{2\alpha}\left(\log \frac{T}{a}\right)^{2\alpha} + \frac{T\kappa}{\alpha+\beta}\left(\log \frac{T}{a}\right)^{\alpha+\beta}\right] \cdot Q \cdot \|u - v\| \\
\leq& \frac{TQ\left(\log \frac{T}{a}\right)^{\alpha}}{|H(T,a)|\left(1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}\right)^2} \left[\frac{1}{\alpha}|H(T,a)|\left(1 - \lambda\left(\log \frac{T}{a}\right)^{\alpha-\beta}\right)\right. \\
&\left. + \frac{1}{2\alpha}\left(\log \frac{T}{a}\right)^{2\alpha-1} + \frac{\kappa}{\alpha+\beta}\left(\log \frac{T}{a}\right)^{\alpha+\beta-1}\right] \|u - v\| = \rho\|u - v\|.
\end{aligned} \tag{28}
$$

(28) gives

$$\|(\mathscr{M}u)(t) - (\mathscr{M}v)(t)\| \leq \rho\|u - v\|. \tag{29}$$

It follows from (29) and (C_2) that $\mathscr{M} : \mathbb{X} \to \mathbb{X}$ is a contractive mapping. From Lemma 2.8, we conclude that $\mathscr{M}$ exists a unique fixed point $u^*(t) \in \mathbb{X}$, which is a unique solution of equation (1). Thus, the claim (i) holds.

Next, we will prove that (ii) in Theorem 3.6 holds. Let $z(t), u^*(t) \in \mathbb{X}$ be a solution of inequality (2) and a unique solution of equation (1), respectively. By Remark 2.7, The derivation of similar Lemma 3.4 is obtained

$$z(t) = \int_{a^+}^{t} G_\alpha(t,s)[f(s,z(s)) + \phi(s)] \frac{ds}{s} - \frac{G_\alpha(t,a)}{H(T,a)} \int_{a^+}^{T} H(T,s)[f(s,z(s)) + \phi(s)] \frac{ds}{s}. \tag{30}$$

Since $u^*(t)$ satisfies the equation (26), we derive from (26) and (30) that

$$z(t) - u^*(t) = \int_{a^+}^{t} G_\alpha(t,s)\phi(s)\frac{ds}{s} - \frac{G_\alpha(t,a)}{H(T,a)}\int_{a^+}^{T} H(T,s)\phi(s)\frac{ds}{s}. \tag{31}$$

Similar to (28), we have

$$\begin{aligned}|z(t) - u^*(t)| \leq & \int_{a^+}^{t} G_\alpha(t,s)|\phi(s)|\frac{ds}{s} + \frac{G_\alpha(t,a)}{|H(T,a)|}\int_{a^+}^{T} |H(T,s)||\phi(s)|\frac{ds}{s} \\ \leq & \epsilon\Big[\int_{a^+}^{t} G_\alpha(t,s)\frac{ds}{s} + \frac{G_\alpha(t,a)}{|H(T,a)|}\int_{a^+}^{T} |H(T,s)|\frac{ds}{s}\Big] \leq \rho\epsilon. \end{aligned} \tag{32}$$

(32) leads to

$$\|z(t) - u^*(t)\| \leq \rho\epsilon. \tag{33}$$

So we conclude from (33) and Definition 2.6 that equation (1) is Ulam-Hyers stable. The proof is completed. □

4. Summaries

It is well known that the Langevin equation a powerful tool in describing the random motion of particles in fluid. In a particularly complex viscous liquid, the integer-order Langevin equation describing the motion of particles is no longer accurate. Some scholars began to use fractional Langevin equation as model to study this problem, and achieved good results. Therefore, the study of the dynamic properties of fractional Langevin equation can provide mathematical theoretical support for such physical problems. In this paper, we study the existence, uniqueness and Ulam-Hyers stability of solutions to the integral boundary value problem for a class of nonlinear higher-order fractional Langevin equation. The mathematical theories and methods of this paper can be used as a reference for the study of other fractional differential equations.

Acknowledgments

The author would like to thank the anonymous referees for their useful and valuable suggestions.

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