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Max-product for the q-Bernstein-Chlodowsky operators

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Abstract. In this study, we introduce a new kind of nonlinear Bernstein-Chlodowsky operators based on q-integers. Firstly, we define the nonlinear $q-$Bernstein-Chlodowsky operators of max-product kind. Then, we give an error estimation for the $q-$Bernstein Chlodowsky operators of max-product kind by using a suitable generalizition of the Shisha-Mond Theorem. There follows an upper estimates of the approximation error for some subclasses of functions.

1. Introduction

In recent years, $q-$calculus plays a significant role in the approximation of functions by a linear positive operator so that the approximations are studied by suitable $q-$generalization of many operators known in the literature.

Lupaş [23] introduced $q-$Bernstein operators and studied approximation and shape-preserving properties for these operators. Phillips [24] presented another generalization of Bernstein operators based on the $q-$integers. In [21], the authors introduced the linear q-Bernstein-Chlodowsky operators and obtained approximation properties for these new operators. Many other interesting generalizations of linear positive operators based on the q-integers were defined and studied by several researchers [1], [14], [19]–[25].

In the Korovkin-type approximation theory, the main topic is the approximation of a continuous function by a sequence of linear positive operators (see [2, 22]). In recent years, nonlinear positive operators have been introduced instead of linear positive operators by Bede et al., [7] (see, also, [6]). Although the Korovkin theorem fails for these nonlinear operators, they obtained that the nonlinear operators have a similar approximation behavior to the linear operators.

In [4]–[13], the "max-product kind operators" were introduced and Jackson type error estimation was given in terms of the modulus of continuity. These studies have been given a significant improvement in the approximation theory owing to the non-linearity of operators.

Note that Bernstein operators have important applications in many fields. One of them is the creation of Bézier curves, which are very important in computer aided geometric designs, by using Bernstein basis functions.

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In [27], they studied a new construction of Bernstein operators with the help of Bézier bases with shape parameter λ and established various approximation results. Moreover, the Stancu-type modification of λ-Bernstein operators based on Bézier bases were introduced by Srivastava et al., [32]. Many other generalization of some operators based on the Bézier type were defined and studied by several researchers (see, [26], [28]–[31]). For future studies, the max-product type some operators based on Bézier bases can be examined in the light of these studies.

In this paper, we propose a further improvement in max-product type operators which is based on q-integers. Our main contribution is to give some novel results about the max-product type q-Bernstein-Chlodowsky operators in the approximation theory.

This paper is organized as follows: in Section 2 some preliminary remarks are given. Section 3 shows the construction of operators on q-integers. A fundamental theorem on error estimation by nonlinear q-operators and the approximation error for some family of functions are given in Section 4. Conclusion and a discussion on further developments are given in Section 5. In the appendix 6, the proofs of some theorem and lemma are included.

2. Preliminary remarks

This section deals with some preliminary definitions and fundamentals on the theory of nonlinear max-product operators.

2.1. Max-product operators

In the interval $[0,+\infty)$, it is defined a semi-ring structure with the operations "$\vee$" (maximum) and "$\cdot$" (product) . Then, $([0,+\infty), \vee, \cdot)$ is called "max-product algebra" (see [4, 6]). Let $I \subset [0,+\infty)$ be a finite or infinite interval and

$$CB_+(I) = \{f : I \to [0,+\infty);\ f \text{ continuous and bounded on } I\}.$$

The general form of discrete max-product type approximation operators $L_n : CB_+(I) \to CB_+(I)$ is defined by

$$L_n(f;x) = \bigvee_{i=0}^{n} K_n(x,x_i)f(x_i), \quad L_n(f;x) = \bigvee_{i=0}^{\infty} K_n(x,x_i)f(x_i)$$

where $f \in CB_+(I)$, $n \in \mathbb{N}$ and for all i, $K_n(.,x_i) \in CB_+(I)$, $x_i \in I$. These operators are nonlinear positive operators satisfying pseudo-linearity property, that is,

$$L_n(\alpha.f \vee \beta.g;x) = \alpha.L_n(f;x) \vee \beta.L_n(g;x)$$

for any $\alpha,\beta \in [0,+\infty)$ and for all $f,g : I \to [0,+\infty)$. Additionally, the max-product operators are positive homogeneous, i.e., $\forall\lambda \geq 0$, $L_n(\lambda.f;x) = \lambda.L_n(f;x)$. More details can be found in [7].

Güngör et al., [35] introduced the following nonlinear Bernstein-Chlodowsky operators of max-product kind by replacing the sum operator "Σ " with the max-operator "$\vee$".

$$C_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} h_{n,k}(x) f\left(\frac{b_n k}{n}\right)}{\bigvee_{k=0}^{n} h_{n,k}(x)},\ x \in [0,b_n] \tag{1}$$

with

$$h_{n,k}(x) = \binom{n}{k}\left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)^{n-k},\ f : [0,b_n] \to \mathbb{R}_+$$

where $x \in [0,b_n]$ and (b_n) is a sequence of positive real numbers such that $\lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} \dfrac{b_n}{\sqrt{n}} = 0$.

2.2. q-calculus

Let us first shortly summarize some elementary definitions about q-calculus.

Definition 2.1. *For any fixed real number $q > 0$ and non-negative integer n, the $q-$integer of the number n is defined by*

$$[n]_q := \begin{cases} \frac{1-q^n}{1-q} & , \quad q \neq 1 \\ n & , \quad q = 1 \end{cases} , \quad [0]_q := 0.$$

Definition 2.2. *The $q-$factorial is defined by*

$$[n]_q! := [n]_q ... [2]_q [1]_q \text{ with } [0]_q! := 1.$$

Definition 2.3. *For integers n and k, with $0 \leq k \leq n$, the q-binomial coefficients are defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

For more information about the $q-$calculus, see e.g. [3].

In the following we will define a class of nonlinear $q-$Bernstein-Chlodowsky operators of max-product kind and estimate the rate of pointwise convergence of these operators. Moreover, a better error estimate for some subclasses of functions will be also given.

3. Construction of the Operators

In this section, a nonlinear approximation operator by modifying the Bernstein-Chlodowsky operators (1) is defined. The construction is mainly based on the work of Bede et al. [6, 7].

Definition 3.1. *Let $f : [0, b_n] \to \mathbb{R}_+$ be a continuous function. The nonlinear $q-$Bernstein-Chlodowsky operators of max-product kind are defined as:*

$$C_{n,q}^{(M)}(f;x) = \frac{\bigvee_{i=0}^{n} s_{n,i,q}(x) f\left(\frac{[i]_q b_n}{[n]_q}\right)}{\bigvee_{i=0}^{n} s_{n,i,q}(x)} \tag{2}$$

with

$$s_{n,i,q}(x) = \begin{bmatrix} n \\ i \end{bmatrix}_q \left(\frac{x}{b_n}\right)^i \prod_{s=1}^{n-i} \left(1 - q^s \frac{x}{b_n}\right)$$

for all $n \in \mathbb{N}$, $q \in (0,1)$ and $x \in [0, b_n]$ where (b_n) be an increasing sequence of positive real numbers and satisfy the properties:

$$\lim_{n\to\infty} b_n = \infty \text{ and } \lim_{n\to\infty} \frac{b_n}{\sqrt{[n]_q}} = 0$$

for $n \in \mathbb{N}$.

We will show the convergence for the operators $C_{n,q}^{(M)}(f;x)$ defined by (2). However, in order to get such an approximation we have to replace the fixed single value $q \in (0,1)$ considered in Definition 3.1, with an appropriate sequence (q_n) whose terms still belong to the interval $(0,1)$. Otherwise, $[n]_q \to \frac{1}{1-q}$ as $n \to \infty$ for a fixed q.

Definition 3.2. *Let non-negative integer n and* (q_n) *is a sequence of real numbers such that* $q_n \in (0,1)$ *and* $\lim_{n\to\infty} q_n = 1$. *Then the q-sequence of q-integer of the number n is defined as*

$$[n]_{q_n} = \{[0]_{q_0}, [1]_{q_1}, [2]_{q_2}, ..., [n]_{q_n}\}.$$

There follows, according to Definition 2.1, that

$$[n]_{q_n} = \frac{1-q_n^n}{1-q_n}.$$

For example, let $q_n = \frac{n+1}{n+2}$. *Then*

$$\begin{aligned} [0]_{q_0} &= \frac{1-q_0^0}{1-q_0} = 0, \ \ q_0 \in (0,1) \\ [1]_{q_1} &= \frac{1-q_1^1}{1-q_1} = 1, \ \ q_1 \in (0,1) \\ [2]_{q_2} &= \frac{1-q_2^2}{1-q_2} = \frac{7}{4}, \ \ q_2 \in (0,1) \\ &\ldots. \\ [n]_{q_n} &= \frac{1-\left(\frac{n+1}{n+2}\right)^n}{1-\frac{n+1}{n+2}}, \ \ q_n \in (0,1) \end{aligned}$$

So that the q-sequence of the q-number is

$$[n]_{q_n} = \{[0]_{q_0}, [1]_{q_1}, [2]_{q_2}, ..., [n]_{q_n}\} = \{0, 1, \frac{7}{4}, ..., \frac{1-\left(\frac{n+1}{n+2}\right)^n}{1-\frac{n+1}{n+2}}\}.$$

In order to obtain convergence for the nonlinear $q-$*Bernstein-Chlodowsky operators of max-product kind (2), let* $q := (q_n)$ *(replacing q with* (q_n)*) is a sequence of real numbers such that* $q_n \in (0,1)$ *and* $\lim_{n\to\infty} q_n = 1$, *and* (b_n) *is an increasing sequence of positive real numbers and satisfy the properties:*

$$\lim_{n\to\infty} b_n = \infty \textit{ and } \lim_{n\to\infty} \frac{b_n}{\sqrt{[n]_{q_n}}} = 0,$$

for $n \in \mathbb{N}$.

$C^{(M)}_{n,q_n}(f;x)$ are positive and continuous on the interval $[0, b_n]$, and these operators satisfy the pseudo-linearity property and positive homogeneous. Moreover, $C^{(M)}_{n,q_n}(f;x)$ reduce to $C^{(M)}_n(f;x)$ given by (1) as $q_n \to 1^-$, and $C^{(M)}_{n,q_n}(f;0) - f(0) = 0$, for all n.

For all $x \in [0, b_n]$ and $q_n \in (0,1)$, since $\bigvee_{i=0}^{n} s_{n,i,q_n}(x) > 0$, $C^{(M)}_{n,q_n}(f;x)$ is well-defined.

Let us give the following simple examples for the maximum function.

Example 1. *Let* $f(x) = x^3 + x^2 + 1$, $b_n = n^{1/3}$ *and* $q_n = 1 - \dfrac{1}{n}$. *Assume that*

$$s_{n,i,q_n}(x) = \begin{bmatrix} n \\ i \end{bmatrix}_{q_n} \left(\frac{x}{b_n}\right)^i \prod_{s=1}^{n-i} (1 - q_n^s \frac{x}{b_n})$$

and

$$A_{n,q_n}(f;x) = s_{n,i,q_n}(x) f\left(\frac{[i]_{q_n} b_n}{[n]_{q_n}}\right).$$

In the first graph in Figure 1, we show the graph of s_{n,i,q_n} for values $i = 0, 1, 2$ (green,blue,red) and $n = 4$. Then we also show the graph of the maximum function of s_{n,i,q_n} (black) relative to these i points.

In the second graph in Figure 1, we show the graph of A_{n,q_n} for values $i = 0, 1, 2$ (green,blue,red) and $n = 4$. Then we also show the graph of the maximum function of A_{n,q_n} (black) relative to these i points.

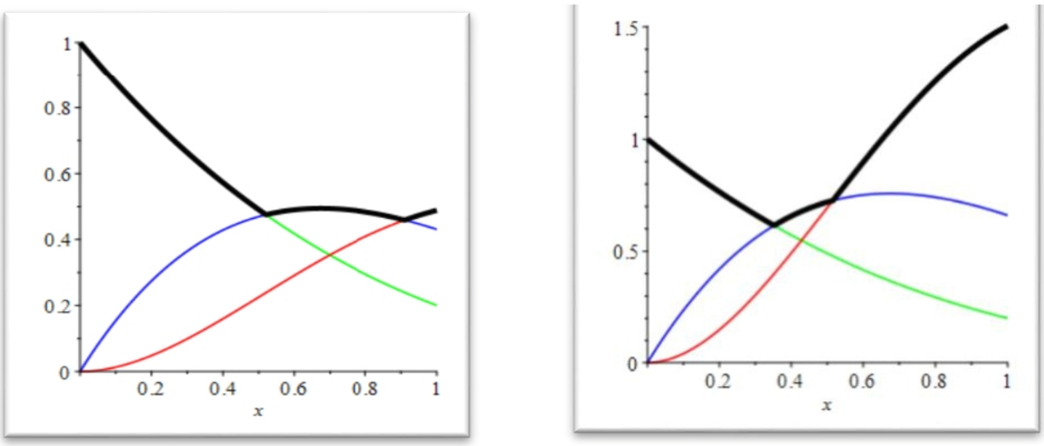

Figure 1: The graph of s_{n,i,q_n} and A_{n,q_n}, respectively, for $i = 0, 1, 2$ (green, blue, red) and its maximum function (black)

We need the following notations and lemmas for the proofs of the main results.

Definition 3.3. *For each $i, j \in \{0, 1, 2, ..., n\}$ and $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$, we define*

$$M_{i,n,j,q_n}(x) = \frac{s_{n,i,q_n}(x)\left|\frac{[i]_{q_n} b_n}{[n]_{q_n}} - x\right|}{s_{n,j,q_n}(x)} \tag{3}$$

and

$$m_{i,n,j,q_n}(x) = \frac{s_{n,i,q_n}(x)}{s_{n,j,q_n}(x)}. \tag{4}$$

If $i \geq j+1$, then

$$M_{i,n,j,q_n}(x) = \frac{s_{n,i,q_n}(x)\left(\frac{[i]_{q_n} b_n}{[n]_{q_n}} - x\right)}{s_{n,j,q_n}(x)} \tag{5}$$

and if $i \leq j-1$, then

$$M_{i,n,j,q_n}(x) = \frac{s_{n,i,q_n}(x)\left(x - \frac{[i]_{q_n} b_n}{[n]_{q_n}}\right)}{s_{n,j,q_n}(x)}. \tag{6}$$

Definition 3.4. *For each $i, j \in \{0, 1, 2, ..., n\}$, $i \geq j+2$ and $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$, we define*

$$\bar{M}_{i,n,j,q_n}(x) = \frac{s_{n,i,q_n}(x)\left(\frac{[i]_{q_n} b_n}{[n+1]_{q_n}} - x\right)}{s_{n,j,q_n}(x)} \tag{7}$$

and for each $i, j \in \{0,1,2,...,n\}$, $i \leq j-2$ *and* $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$,

$$\tilde{M}_{i,n,j,q_n}(x)(x) = \frac{s_{n,i,q_n}(x)\left(x - \dfrac{[i]_{q_n} b_n}{[n+1]_{q_n}}\right)}{s_{n,j,q_n}(x)}. \tag{8}$$

Lemma 3.5. *Let* $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$, *for all* $n \in \mathbb{N}$ *and* $j \in \{0,1,2,...,n\}$. *Then, we obtain the following inequalities:*
(a) *for all* $i \in \{0,1,2,...,n\}$, $i \geq j+2$, *we have*

$$\bar{M}_{i,n,j,q_n}(x) \leq M_{i,n,j,q_n}(x) \leq \bar{M}_{i,n,j,q_n}(x)\left(1 + \frac{2}{q_n^{n+1}}\right),$$

(b) *for all* $i \in \{0,1,2,...,n\}$, $i \leq j-1$, *we have*

$$M_{i,n,j,q_n}(x) \leq \tilde{M}_{i,n,j,q_n}(x) \leq M_{i,n,j,q_n}(x)\left(1 + \frac{2}{q_n^{n}}\right).$$

Proof. See the proof in the appendix. □

Lemma 3.6. *Let* $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$, *for all* $n \in \mathbb{N}$ *and* $i, j \in \{0,1,2,...,n\}$. *Then, we have*

$$m_{i,n,j,q_n}(x) \leq 1.$$

Proof. We have two cases: $a)$ $i \geq j$ and $b)$ $i \leq j$.

Case a. Let $i \geq j$. Since the function $g(x) = \frac{b_n - q_n^{n-i} x}{x}$ is nonincreasing on the interval $\left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$, it follows from (4) that

$$\begin{aligned} \frac{m_{i,n,j,q_n}(x)}{m_{i+1,n,j,q_n}(x)} &= \frac{[i+1]_{q_n}}{[n-i]_{q_n}} \frac{(b_n - q_n^{n-i} x)}{x} \\ &\geq \frac{[i+1]_{q_n}}{[n-i]_{q_n}} \frac{[n+1]_{q_n} b_n - q_n^{n-i}[j+1]_{q_n} b_n}{[j+1]_{q_n} b_n}, \end{aligned}$$

Since $[i+1]_{q_n} \geq [j+1]_{q_n}$, we get

$$\begin{aligned} \frac{m_{i,n,j,q_n}(x)}{m_{i+1,n,j,q_n}(x)} &\geq \frac{[n+1]_{q_n} - q_n^{n-i}[j+1]_{q_n}}{[n-i]_{q_n}} \\ &= \frac{1 - q_n^{n+1} - q_n^{n-i}(1 - q_n^{j+1})}{1 - q_n^{n-i}} \\ &= 1. \end{aligned}$$

Then, we conclude that

$$1 = m_{j,n,j,q_n}(x) \geq m_{j+1,n,j,q_n}(x) \geq m_{j+2,n,j,q_n}(x) \geq ... \geq m_{n,n,j,q_n}(x).$$

Therefore, the proof of the case (a) is complete.

Case b. Let $i \leq j$. The function $h(x) = \frac{x}{b_n - q_n^{n-i+1} x}$ is nondecreasing on the interval $\left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$. Also, since $[i]_{q_n} \leq [j]_{q_n}$, similarly to (a), we can write

$$\begin{aligned} \frac{m_{i,n,j,q_n}(x)}{m_{i-1,n,j,q_n}(x)} &\geq \frac{[n-i+1]_{q_n}}{[n+1]_{q_n} - q_n^{n-i+1}[i]_{q_n}} \\ &= \frac{1 - q_n^{n-i+1}}{1 - q_n^{n+1} - q_n^{n-i+1}(1 - q_n^{i})} \\ &= 1. \end{aligned}$$

Then, we clearly get

$$1 = m_{j,n,j,q_n}(x) \geq m_{j-1,n,j,q_n}(x) \geq m_{j-2,n,j,q_n}(x) \geq ... \geq m_{0,n,j,q_n}(x),$$

which completes the proof of the lemma. □

Lemma 3.7. *Let* $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$ *for all* $n \in \mathbb{N}$ *and* $j \in \{0, 1, 2, ..., n\}$.

(*a*) *if* $i \in \{j+2, j+3, ..., n-1\}$ *is such that* $[i+1]_{q_n} - \sqrt{q_n^i [i+1]_{q_n}} \geq [j+1]_{q_n}$, *then*

$$\bar{M}_{i,n,j,q_n}(x) \geq \bar{M}_{i+1,n,j,q_n}(x),$$

(*b*) *if* $i \in \{1, 2, ..., j-2\}$ *is such that* $[i]_{q_n} + \sqrt{q_n^{i-1} [i]_{q_n}} \geq [j]_{q_n}$, *then*

$$\tilde{M}_{i,n,j,q_n}(x) \geq \tilde{M}_{i-1,n,j,q_n}(x).$$

Proof. See the proof in the appendix. □

Lemma 3.8. *Let* $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$ *for all* $n \in \mathbb{N}$ *and* $j \in \{0, 1, 2, ..., n\}$. *Denoting*

$$s_{n,i,q_n}(x) = \begin{bmatrix} n \\ i \end{bmatrix}_{q_n} \left(\frac{x}{b_n}\right)^i \prod_{s=1}^{n-i} (1 - q_n^s \frac{x}{b_n}),$$

we have

$$\bigvee_{i=0}^{n} s_{n,i,q_n}(x) = s_{n,j,q_n}(x).$$

Proof. See the proof in the appendix. □

4. Degree of approximation by $C_{n,q_n}^{(M)}(f)$

In this section, we give an error estimate in terms of modulus of continuity for $C_{n,q_n}^{(M)}(f)$ by using the Shisha-Mond Theorem given for nonlinear max-product type operators in [6, 7].

Theorem 4.1. *Let* $q := (q_n)$ *is a sequence of real numbers such that* $q_n \in (0,1)$ *and* $\lim_{n\to\infty} q_n = 1$. *If* $f : [0, b_n] \to \mathbb{R}_+$ *is a continuous function and* $C_{n,q_n}^{(M)}(f; x)$ *are the nonlinear* $q-$*Bernstein-Chlodowsky operators of max-product kind defined in (2), then the following pointwise estimate holds*

$$\left|C_{n,q_n}^{(M)}(f; x) - f(x)\right| \leq 4\left(1 + \frac{2}{q_n^{n+1}}\right)\omega_1\left(f; \frac{b_n}{\sqrt{[n]_{q_n}}}\right),$$

where $\omega_1(f; \delta) = sup\{|f(x) - f(y)|; x, y \in [0, b_n], |x - y| \leq \delta\}$.

Proof. By using the Shisha-Mond Theorem, we get

$$\left|C_{n,q_n}^{(M)}(f; x) - f(x)\right| \leq \left(1 + \frac{1}{\delta_n} C_{n,q_n}^{(M)}(\varphi_x; x)\right)\omega_1(f; \delta_n),$$

where $\varphi_x(t) = |t-x|$. Therefore, it is enough to estimate only the following term

$$C_{n,q_n}^{(M)}(\varphi_x;x) = \frac{\bigvee_{i=0}^{n} s_{n,i,q_n}(x)\left|\frac{[i]_{q_n}b_n}{[n]_{q_n}} - x\right|}{\bigvee_{i=0}^{n} s_{n,i,q_n}(x)}.$$

Firstly, let $x \in \left[\frac{[j]_{q_n}b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n}b_n}{[n+1]_{q_n}}\right]$ for a fixed $j \in \{0,1,2,...,n\}$. Then, from Lemma 3.8, we have

$$C_{n,q_n}^{(M)}(\varphi_x;x) = \bigvee_{i=0}^{n} M_{i,n,j,q_n}(x), \tag{9}$$

where $M_{i,n,j,q_n}(x)$ is the same as (5). For $i \in \{0,1,2,...,n\}$ and $j=0$, we get $M_{i,n,0,q_n}(x) \leq \frac{b_n}{[n]_{q_n}}$ for all $x \in \left[0, \frac{b_n}{[n+1]_{q_n}}\right]$. Indeed, if $j=i=0$, then $M_{0,n,0,q_n}(x) = x \leq \frac{b_n}{[n]_{q_n}}$ for all $x \in \left[0, \frac{b_n}{[n+1]_{q_n}}\right]$. Also, if $i \in \{1,2,3,...,n\}$ and $j=0$, for all $x \in \left[0, \frac{b_n}{[n+1]_{q_n}}\right]$, we obtain that

$$\begin{aligned} M_{i,n,0,q_n}(x) &= \frac{s_{n,i,q_n}(x)\left(\frac{[i]_{q_n}b_n}{[n]_{q_n}} - x\right)}{s_{n,0,q_n}(x)} \\ &\leq \frac{[n-i+1]_{q_n}}{[n+1]_{q_n} - q_n^{n-i+1}} \cdots \frac{[n]_{q_n}}{[n+1]_{q_n} - q_n^{n}} \cdot \frac{b_n}{[i-1]_{q_n}![n]_{q_n}}, \end{aligned}$$

for all $x \in \left[0, \frac{b_n}{[n+1]_{q_n}}\right]$. Here, for each $k = 1,2,3,...,i$, we observe that $\frac{[n-i+k]_{q_n}}{[n+1]_{q_n} - q_n^{n-i+k}} \leq 1$. Consequently, the above inequality gives that $M_{i,n,0,q_n}(x) \leq \frac{b_n}{[n]_{q_n}}$.

Now, let $i \in \{0,1,2,...,n\}$ and $j \in \{1,2,...,n\}$. We can easily show (see the proof in the appendix) that the inequality

$$M_{i,n,j,q_n}(x) \leq \frac{\left(1 + \frac{2}{q_n^{n+1}}\right)b_n}{\sqrt{[n+1]_{q_n}}} \tag{10}$$

holds for $x \in \left[\frac{[j]_{q_n}b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n}b_n}{[n+1]_{q_n}}\right]$. As a result, from this claim and the above inequalities we conclude that

$$C_{n,q_n}^{(M)}(\varphi_x;x) \leq \frac{2b_n\left(1 + \frac{2}{q_n^{n+1}}\right)}{\sqrt{[n]_{q_n}}}$$

for all $i,j \in \{0,1,2,...,n\}$, $n \in \mathbb{N}$ and $x \in [0,b_n]$, and taking $\delta_n = \frac{2b_n\left(1+\frac{2}{q_n^{n+1}}\right)}{\sqrt{[n]_{q_n}}}$, we obtain the estimate

$$\left|C_{n,q_n}^{(M)}(f;x) - f(x)\right| \leq 4\left(1 + \frac{2}{q_n^{n+1}}\right)\omega_1\left(f; \frac{b_n}{\sqrt{[n]_{q_n}}}\right).$$

So that the proof is achieved. □

In the following, we illustrate the rate of convergence of the operators $C_{n,q_n}^{(M)}$ to certain functions by graphics. We also compare the convergence of the max-product (nonlinear) q-Bernstein-Chlodowsky operators and the classical linear q-Bernstein-Chlodowsky operators (see [21]) to functions.

Remark 4.2. *The classical linear q-Bernstein-Chlodowsky operators have the following form*

$$C_{n,q_n}(f;x) = \sum_{i=0}^{n} f\left(\frac{[i]_{q_n}}{[n]_{q_n}} b_n\right) \begin{bmatrix} n \\ i \end{bmatrix}_{q_n} \left(\frac{x}{b_n}\right)^i \prod_{s=0}^{n-i-1} \left(1 - q_n^s \frac{x}{b_n}\right),$$

where $x \in [0, b_n]$ *and* (b_n) *is a sequence of positive real numbers such that* $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$ *and* (q_n) *is a sequence of real numbers such that* $q_n \in (0,1)$ *and* $\lim_{n\to\infty} q_n = 1$ *(see [21]).*

Note that we prefer $\prod_{s=1}^{n-i}(1 - q_n^s \frac{x}{b_n})$, as opposed to the classical linear q-Bernstein-Chlodowsky operators due to some technical complexity from the q-calculus.

Example 2. *Let* $f(x) = |4sin(\pi x) - 2|$, $b_n = n^{1/5}$ *and* $q_n = 1 - \frac{1}{n}$.

In the first graph in Figure 2, we show the operators $C_{n,q_n}^{(M)}$ *approximation to* $f(x)$*(black) for the values* $n = 5, 15, 25$*(red, green, blue). Later, in the second graph in Figure 2, we illustrate the convergence of the operators* $C_{n,q_n}^{(M)}$*(blue) and* C_{n,q_n}*(red) to* $f(x)$*(black) for* $n = 25$.

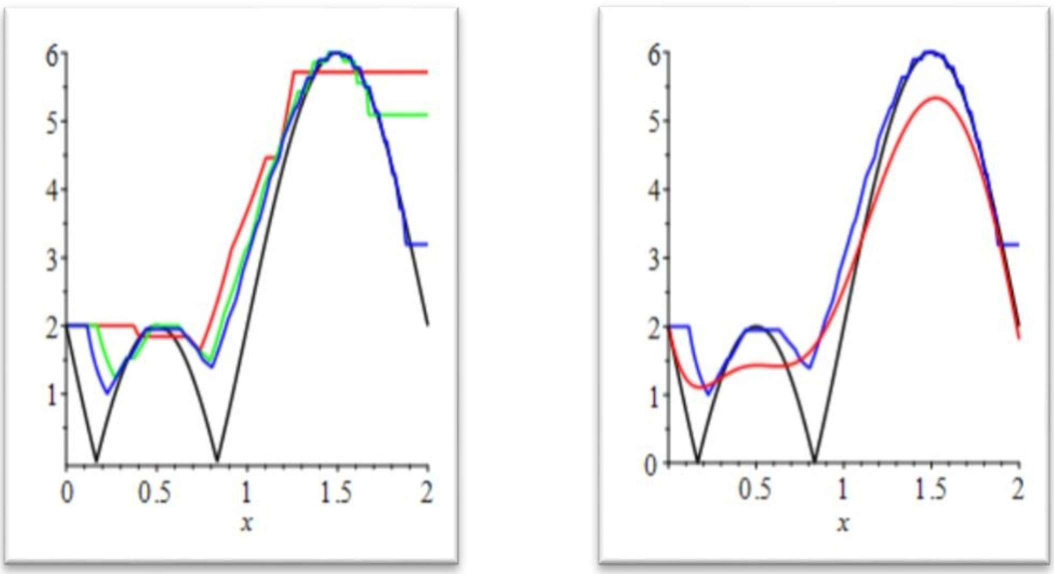

Figure 2: Approximation to the function $f(x) = |4sin(\pi x) - 2|$

Example 3. *Let* $f(x) = e^{-sin(\pi x)}$, $b_n = n^{1/5}$ *and* $q_n = 1 - \frac{1}{n}$.

In the first graph in Figure 3, we show the operators $C_{n,q_n}^{(M)}$ *approximation to* $f(x)$*(black) for the values* $n = 5, 15, 25$*(red, green, blue). Later, we illustrate the convergence of the operators* $C_{n,q_n}^{(M)}$*(blue) and* C_{n,q_n}*(red) to* $f(x)$*(black) for* $n = 25$ *in the second graph.*

As a result, from the second graph in Figure 2 and Figure 3, it is clearly seen that for the corresponding functions, the max- product (nonlinear) q-Bernstein-Chlodowsky operators approximate much better than the classical linear q-Bernstein-Chlodowsky operators.

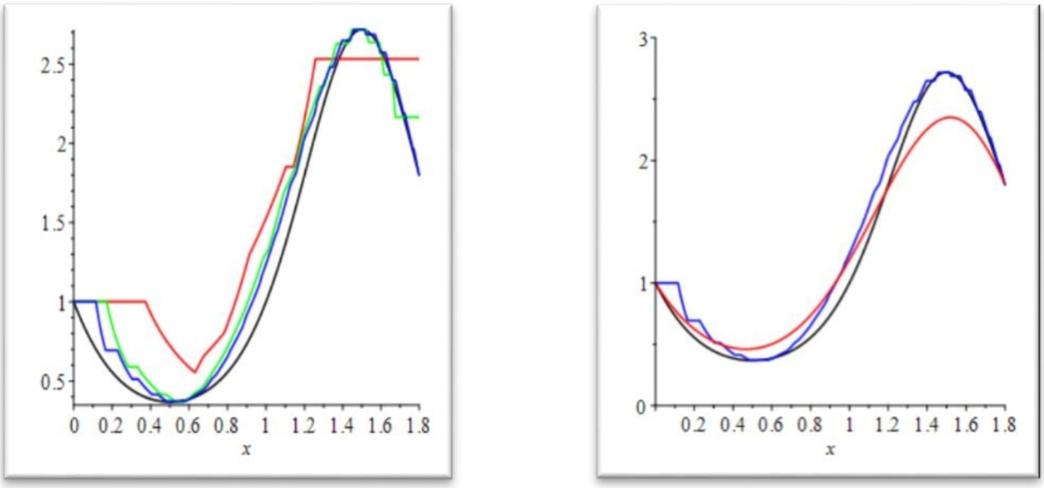

Figure 3: Approximation to the function $f(x) = e^{-sin(\pi x)}$

Example 4. *Let* $f(x) = x^4 e^{-4x}$, $b_n = n^{1/3}$ *and* $q_n = 1 - \dfrac{1}{n}$. *The approximation of* $C_{n,q_n}^{(M)}$ *to* $f(x)$ *is shown in Table 1 for different values* n.

Table 1: The error estimation of $f(x) = x^4 e^{-4x}$ by using modulus of continuity

n	Estimation for modulus of continuity of function
10	0.07257148244
10^2	0.06316331032
10^3	0.04847666096
10^4	0.03492674332
10^5	0.02442452536

Now, we make the following observation to get a better order of approximation for subclasses of functions f.

For any $i, j \in \{0, 1, 2, ..., n\}$, $n \in \mathbb{N}$, consider the function

$$f_{i,n,j,q_n}(x) : \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right] \to \mathbb{R}$$

defined by

$$f_{i,n,j,q_n}(x) = m_{i,n,j,q_n}(x) f\left(\frac{[i]_{q_n} b_n}{[n]_{q_n}}\right) = \frac{\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_{q_n} \left(\frac{x}{b_n}\right)^i \prod_{s=1}^{n-i} \left(1 - q_n^s \frac{x}{b_n}\right)}{\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]_{q_n} \left(\frac{x}{b_n}\right)^j \prod_{s=1}^{n-j} \left(1 - q_n^s \frac{x}{b_n}\right)} f\left(\frac{[i]_{q_n} b_n}{[n]_{q_n}}\right).$$

Thus, for any $j \in \{0, 1, 2, ..., n\}$ and $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$, we can write

$$C_{n,q_n}^{(M)}(f; x) = \bigvee_{i=0}^{n} f_{i,n,j,q_n}(x).$$

Lemma 4.3. *Let* $f : [0, b_n] \to [0, \infty)$. *If*

$$C_{n,q_n}^{(M)}(f; x) = \max\{f_{j,n,j,q_n}(x), f_{j+1,n,j,q_n}(x)\},$$

for all $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$, $n \in \mathbb{N}$. *Then*

$$\left|C_{n,q_n}^{(M)}(f; x) - f(x)\right| \le 3\omega_1\left(f; \frac{b_n}{[n]_{q_n}}\right).$$

Proof. See the proof in the appendix. □

Lemma 4.4. *Let* $f : [0, b_n] \to [0, \infty)$. *If*

$$C_{n,q_n}^{(M)}(f; x) = \max\{f_{j-1,n,j,q_n}(x), f_{j,n,j,q_n}(x)\},$$

for all $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$, $n \in \mathbb{N}$. *Then*

$$\left|C_{n,q_n}^{(M)}(f; x) - f(x)\right| \le 2\omega_1\left(f; \frac{b_n}{[n]_{q_n}}\right).$$

Proof. See the proof in the appendix. □

Lemma 4.5. *Let* $f : [0, b_n] \to [0, \infty)$, $n \in \mathbb{N}$. *If*

$$C_{n,q_n}^{(M)}(f; x) = \max\{f_{j-1,n,j,q_n}(x), f_{j,n,j,q_n}(x), f_{j+1,n,j,q_n}(x)\},$$

for all $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$. *Then*

$$\left|C_{n,q_n}^{(M)}(f; x) - f(x)\right| \le 3\omega_1\left(f; \frac{b_n}{[n]_{q_n}}\right).$$

Proof. Let $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$ be fixed such that

$$C_{n,q_n}^{(M)}(f; x) = f_{j,n,j,q_n}(x) \text{ or } C_{n,q_n}^{(M)}(f; x) = f_{j+1,n,j,q_n}(x).$$

Then $C_{n,q_n}^{(M)}(f; x) = \max\{f_{j,n,j,q_n}(x), f_{j+1,n,j,q_n}(x)\}$ and from Lemma 4.3, we get

$$\left|C_{n,q_n}^{(M)}(f; x) - f(x)\right| \le 3\omega_1\left(f; \frac{b_n}{[n]_{q_n}}\right).$$

If $C_{n,q_n}^{(M)}(f; x) = f_{j-1,n,j,q_n}(x)$, then $C_{n,q_n}^{(M)}(f; x) = \max\{f_{j-1,n,j,q_n}(x), f_{j,n,j,q_n}(x)\}$ and from Lemma 4.4, we get

$$\left|C_{n,q_n}^{(M)}(f; x) - f(x)\right| \le 2\omega_1\left(f; \frac{b_n}{[n]_{q_n}}\right)$$

which completes the proof. □

Lemma 4.6 (see [7]). *Let* $f : [0, b_n] \to [0, \infty)$ *be a concave function. Then the following properties hold:*

(a) the function $g : (0, b_n] \to [0, \infty)$, $g(x) = \frac{f(x)}{x}$ is nonincreasing,
(b) the function $h : [0, b_n) \to [0, \infty)$, $h(x) = \frac{f(x)}{b_n - x}$ is nondecreasing.

Corollary 4.7. *Let $f:[0,b_n]\to[0,\infty)$ be a concave function. Then*

$$\left|C_{n,q_n}^{(M)}(f;x)-f(x)\right|\le 3\omega_1\left(f;\frac{b_n}{[n]_{q_n}}\right)$$

for all $n\in\mathbb{N}$, $x\in[0,b_n]$.

Proof. Suppose that $x\in[0,b_n]$ and for $i,j\in\{0,1,2,...,n\}$, $x\in\left[\frac{[j]_{q_n}b_n}{[n+1]_{q_n}},\frac{[j+1]_{q_n}b_n}{[n+1]_{q_n}}\right]$. If $i\ge j$, then

$$\begin{aligned}
f_{i+1,n,j,q_n}(x) &= \frac{\begin{bmatrix} n \\ i+1 \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^{i+1}\prod\limits_{s=1}^{n-i-1}(1-q_n^s\frac{x}{b_n})}{\begin{bmatrix} n \\ j \end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^{j}\prod\limits_{s=1}^{n-j}(1-q_n^s\frac{x}{b_n})} f\left(\frac{[i+1]_{q_n}b_n}{[n]_{q_n}}\right)\\
&= \frac{\begin{bmatrix} n \\ i \end{bmatrix}_{q_n}}{\begin{bmatrix} n \\ j \end{bmatrix}_{q_n}}\frac{[n-i]_{q_n}}{[i+1]_{q_n}}\left(\frac{x}{b_n}\right)^{i-j}\left(\frac{x}{b_n-q_n^{n-i}x}\right) f\left(\frac{[i+1]_{q_n}b_n}{[n]_{q_n}}\right).
\end{aligned}$$

From Lemma 4.6(a), we shall write

$$f\left(\frac{[i+1]_{q_n}b_n}{[n]_{q_n}}\right)\Big/\frac{[i+1]_{q_n}b_n}{[n]_{q_n}}\le f\left(\frac{[i]_{q_n}b_n}{[n]_{q_n}}\right)\Big/\frac{[i]_{q_n}b_n}{[n]_{q_n}},$$

i.e.,

$$f\left(\frac{[i+1]_{q_n}b_n}{[n]_{q_n}}\right)\le\frac{[i+1]_{q_n}}{[i]_{q_n}}f\left(\frac{[i]_{q_n}b_n}{[n]_{q_n}}\right),$$

and since

$$\frac{x}{b_n-q_n^{n-i}x}\le\frac{[j+1]_{q_n}}{[n+1]_{q_n}-q_n^{n-i}[j+1]_{q_n}},$$

we get the following inequality:

$$\begin{aligned}
f_{i+1,n,j,q_n}(x) &\le \frac{\begin{bmatrix} n \\ i \end{bmatrix}_{q_n}}{\begin{bmatrix} n \\ j \end{bmatrix}_{q_n}}\frac{[n-i]_{q_n}}{[i+1]_{q_n}}\frac{[j+1]_{q_n}[i+1]_{q_n}}{([n+1]_{q_n}-q_n^{n-i}[j+1]_{q_n})[i]_{q_n}}\left(\frac{x}{b_n}\right)^{i-j} f\left(\frac{[i]_{q_n}b_n}{[n]_{q_n}}\right)\\
&= f_{i,n,j,q_n}(x)\frac{[j+1]_{q_n}}{[i]_{q_n}}.
\end{aligned}$$

For $i\ge j+1$ and $q_n\in(0,1)$, from the above inequality, we obtain $f_{i,n,j,q_n}(x)\ge f_{i+1,n,j,q_n}(x)$. Hence

$$f_{j+1,n,j,q_n}(x)\ge f_{j+2,n,j}(x)\ge ...\ge f_{n,n,j}(x). \tag{11}$$

We can use a similar method for $i\le j$. Then we find the inequality

$$f_{i-1,n,j,q_n}(x)\le f_{i,n,j,q_n}(x)\frac{[n]_{q_n}-[i-1]_{q_n}}{[n]_{q_n}-[i]_{q_n}}.$$

For $i\le j-1$ and $q_n\in(0,1)$, from the above inequality, we can write $f_{i,n,j,q_n}(x)\ge f_{i-1,n,j,q_n}(x)$. Hence

$$f_{j-1,n,j,q_n}(x)\ge f_{j-2,n,j}(x)\ge ...\ge f_{0,n,j}(x). \tag{12}$$

Consequently, by using (11) and (12), we get

$$C^{(M)}_{n,q_n}(f;x) = \max\{f_{j-1,n,j,q_n}(x), f_{j,n,j,q_n}(x), f_{j+1,n,j,q_n}(x)\},$$

and finally from Lemma 4.5

$$\left|C^{(M)}_{n,q_n}(f;x) - f(x)\right| \le 3\omega_1\left(f; \frac{b_n}{[n]_{q_n}}\right)$$

which proves the corollary. □

5. Conclusions

In recent years, the nonlinear max-product type operators have been studied by some authors (see, [4]–[13],[15]–[18]). In the present paper, nonlinear max-product type q-Bernstein-Chlodowsky operators are introduced. Moreover, the degree of approximation and the rate of convergence of the operators are investigated by using the modulus of continuity. Later, some upper estimates of approximation error for some subclasses of functions are obtained. As a result, the max-product type q-Bernstein-Chlodowsky operators approximate better than the classical linear q-Bernstein-Chlodowsky operators. In addition, until now, there is no such study in nonlinear max-product type operators based on q-integers. The purpose of this study is to fill this gap in the literature. As future work, the shape-preserving properties for these operators may be investigated and similar studies may be integrated into other convenient operators.

Finally, note that there is a clear connection between the classical q-calculus used in this study and the so-called (p,q)-calculus. The results of the q-analogues $(0 < q < 1)$ which we have discussed in this article can be easily translated into the corresponding results for the (p,q)-analogues $(0 < q < p \le 1)$ with some parametric and argument variations. Therefore, the additional p parameter is unnecessary. Moreover, in [33] (see, pp. 340), the authors clearly discourage some authors' tendency to trivially use the so-called (p,q)-calculus (see, also, pp. 1511-1512, in [34]).

6. Appendix

Proof of Lemma 3.5. (*a*) From (5) and (7), it is obvious that the inequality $\bar{M}_{i,n,j,q_n}(x) \le M_{i,n,j,q_n}(x)$. Also, we have

$$\frac{M_{i,n,j,q_n}(x)}{\bar{M}_{i,n,j,q_n}(x)} = \frac{\frac{[i]_{q_n} b_n}{[n]_{q_n}} - x}{\frac{[i]_{q_n} b_n}{[n+1]_{q_n}} - x} \le \frac{\frac{[i]_{q_n} b_n}{[n]_{q_n}} - \frac{[j]_{q_n} b_n}{[n+1]_{q_n}}}{\frac{[i]_{q_n} b_n}{[n+1]_{q_n}} - \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}}.$$

Now, using the fact that $[n+1]_{q_n} = [n]_{q_n} + q_n^n$ and $[i]_{q_n} \le [n]_{q_n}$ we can write

$$\begin{aligned}
\frac{M_{i,n,j,q_n}(x)}{\bar{M}_{i,n,j,q_n}(x)} &\le \frac{[i]_{q_n}[n]_{q_n} + [i]_{q_n} q_n^n - [j]_{q_n}[n]_{q_n}}{[n]_{q_n}\left([i]_{q_n} - [j+1]_{q_n}\right)} \\
&\le \frac{[i]_{q_n} + q_n^n - [j]_{q_n}}{[i]_{q_n} - [j+1]_{q_n}} \le \frac{[i]_{q_n} + q_n^n - [j]_{q_n}}{[i]_{q_n} - [j]_{q_n} - q_n^j} \\
&= 1 + \frac{q_n^j + q_n^n}{[i]_{q_n} - [j]_{q_n} - q_n^j} \le 1 + \frac{2}{[i]_{q_n} - [j]_{q_n} - q_n^j}
\end{aligned}$$

and since $i \ge j+2$, $j \le n$, we get

$$[i]_{q_n} - [j]_{q_n} - q_n^j \ge [j+2]_{q_n} - [j]_{q_n} - q_n^j \ge q_n^{j+1} \ge q_n^{n+1}.$$

Therefore, we obtain that

$$\frac{M_{i,n,j,q_n}(x)}{\bar{M}_{i,n,j,q_n}(x)} \leq 1 + \frac{2}{q_n^{n+1}},$$

which proves (a).

(b) By (6) and (8), we easily $M_{i,n,j,q_n}(x) \leq \tilde{M}_{i,n,j,q_n}(x)$. Additionally, similarly to (a), we observe that

$$\frac{M_{i,n,j,q_n}(x)}{\tilde{M}_{i,n,j,q_n}(x)} \leq 1 + \frac{2}{[i]_{q_n} - [j]_{q_n} - q_n^n}$$

and since $i \leq j-2$, $j \leq n$, we may write that

$$\begin{aligned} [i]_{q_n} - [j]_{q_n} - q_n^n &\geq [i]_{q_n} - [j-2]_{q_n} - q_n^n \\ &= q_n^{j-2} + q_n^{j-1} - q_n^n \\ &\geq q_n^n. \end{aligned}$$

Hence, we get

$$\frac{M_{i,n,j,q_n}(x)}{\tilde{M}_{i,n,j,q_n}(x)} \leq 1 + \frac{2}{q_n^n},$$

which completes the proof.

Proof of Lemma 3.7. (a) Let $i \in \{j+2, j+3, ..., n-1\}$ and $[i+1]_{q_n} - \sqrt{q_n^i [i+1]_{q_n}} \geq [j+1]_{q_n}$.

$$\frac{\bar{M}_{i,n,j,q_n}(x)}{\bar{M}_{i+1,n,j,q_n}(x)} = \frac{[i+1]_{q_n}}{[n-i]_{q_n}} \frac{b_n - q_n^{n-i}x}{x} \frac{\frac{[i]_{q_n} b_n}{[n+1]_{q_n}} - x}{\frac{[i+1]_{q_n} b_n}{[n+1]_{q_n}} - x}.$$

Since the function $\alpha(x) = \frac{b_n - q_n^{n-i}x}{x} \cdot \frac{[i]_{q_n} b_n - x[n+1]_{q_n}}{[i+1]_{q_n} b_n - x[n+1]_{q_n}}$ is nonincreasing on the interval $\left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$, we get

$$\begin{aligned} \frac{\bar{M}_{i,n,j,q_n}(x)}{\bar{M}_{i+1,n,j,q_n}(x)} &\geq \frac{[i+1]_{q_n}}{[n-i]_{q_n}} \frac{b_n - q_n^{n-i} \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}}{\frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}} \frac{[i]_{q_n} b_n - [j+1]_{q_n} b_n}{[i+1]_{q_n} b_n - [j+1]_{q_n} b_n} \\ &\geq \frac{[i+1]_{q_n}}{[n-i]_{q_n}} \frac{[n+1]_{q_n} b_n - q_n^{n-i}[i+1]_{q_n} b_n}{[j+1]_{q_n} b_n} \frac{[i]_{q_n} b_n - [j+1]_{q_n} b_n}{[i+1]_{q_n} b_n - [j+1]_{q_n} b_n} \\ &= \frac{[i+1]_{q_n}}{[j+1]_{q_n}} \frac{[i]_{q_n} - [j+1]_{q_n}}{[i+1]_{q_n} - [j+1]_{q_n}}. \end{aligned}$$

The hypothesis $[i+1]_{q_n} - \sqrt{q_n^i [i+1]_{q_n}} \geq [j+1]_{q_n}$ is equivalent to

$$[i+1]_{q_n} - \sqrt{[i+1]_{q_n}^2 - [i]_{q_n}[i+1]_{q_n}} \geq [j+1]_{q_n},$$

which implies that $[i+1]_{q_n}([i]_{q_n} - [j+1]_{q_n}) \geq [j+1]_{q_n}([i+1]_{q_n} - [j+1]_{q_n})$. Consequently,

$$\frac{\bar{M}_{i,n,j,q_n}(x)}{\bar{M}_{i+1,n,j,q_n}(x)} \geq 1.$$

(*b*) Let $i \in \{1, 2, ..., j-2\}$ and $[i]_{q_n} + \sqrt{q_n^{i-1}[i]_{q_n}} \geq [j]_{q_n}$. Then we may easily write

$$\frac{\tilde{M}_{i,n,j,q_n}(x)}{\tilde{M}_{i-1,n,j,q_n}(x)} = \frac{[n-i+1]_{q_n}}{[i]_{q_n}} \frac{x}{b_n - q_n^{n-i+1}x} \frac{x - \frac{[i]_{q_n} b_n}{[n+1]_{q_n}}}{x - \frac{[i-1]_{q_n} b_n}{[n+1]_{q_n}}}.$$

Since the function $\beta(x) = \frac{x}{b_n - q_n^{n-i+1}x} \cdot \frac{x[n+1]_{q_n} - [i]_{q_n} b_n}{x[n+1]_{q_n} - [i-1]_{q_n} b_n}$ is nondecreasing on the interval $\left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$, we get

$$\begin{aligned} \frac{\tilde{M}_{i,n,j,q_n}(x)}{\tilde{M}_{i-1,n,j,q_n}(x)} &\geq \frac{[n-i+1]_{q_n}}{[i]_{q_n}} \frac{[j]_{q_n} b_n}{[n+1]_{q_n} b_n - q_n^{n-i+1}[j]_{q_n} b_n} \frac{[j]_{q_n} b_n - [i]_{q_n} b_n}{[j]_{q_n} b_n - [i-1]_{q_n} b_n} \\ &= \frac{[j]_{q_n}}{[i]_{q_n}} \frac{[j]_{q_n} - [i]_{q_n}}{[j]_{q_n} - [i-1]_{q_n}}. \end{aligned}$$

The hypothesis $[i]_{q_n} + \sqrt{q_n^{i-1}[i]_{q_n}} \geq [j]_{q_n}$ is equivalent to $[i]_{q_n} + \sqrt{[i]_{q_n}^2 - [i]_{q_n}[i-1]_{q_n}} \geq [j]_{q_n}$, which implies that $[j]_{q_n}([j]_{q_n} - [i]_{q_n}) \geq [i]_{q_n}([j]_{q_n} - [i-1]_{q_n})$, we obtain

$$\frac{\tilde{M}_{i,n,j,q_n}(x)}{\tilde{M}_{i-1,n,j,q_n}(x)} \geq 1.$$

Proof of Lemma 3.8. We claim that for fixed $n \in \mathbb{N}$ and $0 \leq i < i+1 \leq n$, we have

$$0 \leq s_{n,i+1,q_n}(x) \leq s_{n,i,q_n}(x)$$

if and only if

$$0 \leq x \leq \frac{[i+1]_{q_n} b_n}{[n+1]_{q_n}}.$$

In this case, from the $\begin{bmatrix} n \\ i+1 \end{bmatrix}_{q_n} + q_n^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}_{q_n} = \begin{bmatrix} n+1 \\ i+1 \end{bmatrix}_{q_n}$ equation after some simplification, we can write

$$0 \leq \begin{bmatrix} n \\ i+1 \end{bmatrix}_{q_n} \left(\frac{x}{b_n}\right)^{i+1} \prod_{s=1}^{n-i-1} (1 - q_n^s \frac{x}{b_n}) \leq \begin{bmatrix} n \\ i \end{bmatrix}_{q_n} \left(\frac{x}{b_n}\right)^{i} \prod_{s=1}^{n-i} (1 - q_n^s \frac{x}{b_n})$$

and we can reduce the above inequality to

$$0 \leq x \leq \frac{[i+1]_{q_n} b_n}{[n+1]_{q_n}}.$$

Hence, by taking $i = 0, 1, 2, ..., n-1$ in the inequality above, we get

$$\begin{aligned} s_{n,1,q_n}(x) &\leq s_{n,0,q_n}(x) \text{ if and only if } x \in \left[0, b_n/[n+1]_{q_n}\right], \\ s_{n,2,q_n}(x) &\leq s_{n,1,q_n}(x) \text{ if and only if } x \in \left[0, b_n[2]_{q_n}/[n+1]_{q_n}\right], \\ &\ldots \\ s_{n,i+1,q_n}(x) &\leq s_{n,i,q_n}(x) \text{ if and only if } x \in \left[0, b_n[i+1]_{q_n}/[n+1]_{q_n}\right], \\ &\ldots \\ s_{n,n,q_n}(x) &\leq s_{n,n-1,q_n}(x) \text{ if and only if } x \in \left[0, b_n[n]_{q_n}/[n+1]_{q_n}\right]. \end{aligned}$$

As a result, for all $i = 0, 1, 2, ..., n$, we obtain

$$\begin{aligned} &\text{if} \quad 0 \leq x \leq \frac{b_n}{[n+1]_{q_n}} \text{ then } s_{n,i,q_n}(x) \leq s_{n,0,q_n}(x), \\ &\text{if} \quad \frac{b_n}{[n+1]_{q_n}} \leq x \leq \frac{[2]_{q_n} b_n}{[n+1]_{q_n}} \text{ then } s_{n,i,q_n}(x) \leq s_{n,1,q_n}(x), \\ &\ldots \\ &\text{if} \quad \frac{[n]_{q_n} b_n}{[n+1]_{q_n}} \leq x \leq b_n[n+1]_{q_n} \text{ then } s_{n,i,q_n}(x) \leq s_{n,n,q_n}(x). \end{aligned}$$

The proof is completed.

Proof of Theorem 4.1. Let $i \in \{0,1,2,...,n\}$ and $j \in \{1,2,...,n\}$. We claim that the inequality

$$M_{i,n,j,q_n}(x) \leq \frac{\left(1+\frac{2}{q_n^{n+1}}\right)b_n}{\sqrt{[n+1]_{q_n}}}$$

holds for $x \in \left[\frac{[j]_{q_n}b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n}b_n}{[n+1]_{q_n}}\right]$. The proof will be divided into the following five possible cases:

(a) $i \in \{j-1, j, j+1\}$,

(b) $i \geq j+2$ and $[i+1]_{q_n} - \sqrt{q_n^i[i+1]_{q_n}} < [j+1]_{q_n}$,

(c) $i \geq j+2$ and $[i+1]_{q_n} - \sqrt{q_n^i[i+1]_{q_n}} \geq [j+1]_{q_n}$,

(d) $i \leq j-2$ and $[i]_{q_n} + \sqrt{q_n^{i-1}[i]_{q_n}} \geq [j]_{q_n}$,

(e) $i \leq j-2$ and $[i]_{q_n} + \sqrt{q_n^{i-1}[i]_{q_n}} < [j]_{q_n}$.

Case a. If $i = j-1$, then it follows from Lemma 3.6 and (6)

$$M_{j-1,n,j,q_n}(x) \leq \frac{[j+1]_{q_n}b_n}{[n+1]_{q_n}} - \frac{[j-1]_{q_n}b_n}{[n]_{q_n}} \leq \frac{(q_n^{j-1}+q_n^j)b_n}{[n+1]_{q_n}} \leq \frac{2b_n}{[n+1]_{q_n}}.$$

If $i = j$, by Lemma 3.6 , we obtain that

$$M_{j,n,j,q_n}(x) = \left|\frac{[j]_{q_n}b_n}{[n]_{q_n}} - x\right| \leq \frac{b_n}{[n+1]_{q_n}}.$$

If $i = j+1$, then using again Lemma 3.6 and (5), we have

$$\begin{aligned} M_{j+1,n,j,q_n}(x) &\leq \frac{[n+1]_{q_n}[j+1]_{q_n}b_n - [n]_{q_n}[j]_{q_n}b_n}{[n]_{q_n}[n+1]_{q_n}} \\ &= \frac{\left(q_n^n[j]_{q_n} + q_n^j[n]_{q_n} + q_n^{n+j}\right)b_n}{[n]_{q_n}[n+1]_{q_n}} \\ &\leq \frac{\left(q_n^n + q_n^i + q_n^{n+j}\right)b_n}{[n+1]_{q_n}} \\ &\leq \frac{3b_n}{[n+1]_{q_n}}. \end{aligned}$$

Case b. Let $i \geq j+2$ and $[i+1]_{q_n} - \sqrt{q_n^i[i+1]_{q_n}} < [j+1]_{q_n}$, then

$$\bar{M}_{i,n,j,q_n}(x) = m_{i,n,j,q_n}(x)\left(\frac{[i]_{q_n}b_n}{[n+1]_{q_n}} - x\right) \leq \frac{[i]_{q_n}b_n}{[n+1]_{q_n}} - \frac{[j]_{q_n}b_n}{[n+1]_{q_n}}.$$

By hypothesis, since

$$q_n[i]_{q_n} - \sqrt{q_n^i[i+1]_{q_n}} < q_n[j]_{q_n},$$

we get

$$\bar{M}_{i,n,j,q_n}(x) \leq \frac{\sqrt{q_n^{i-2}[i+1]_{q_n}}b_n}{[n+1]_{q_n}} \leq \frac{\sqrt{q_n^{i-2}[n+1]_{q_n}}b_n}{[n+1]_{q_n}}.$$

since $i-2 \geq 0$, we can write

$$\bar{M}_{i,n,j,q_n}(x) \leq \frac{b_n}{\sqrt{[n+1]_{q_n}}}.$$

Also using Lemma 3.5(a), we obtain that

$$M_{i,n,j,q_n}(x) \leq \frac{\left(1+\frac{2}{q_n^{n+1}}\right)b_n}{\sqrt{[n+1]_{q_n}}}.$$

Case c. $i \geq j+2$ and $[i+1]_{q_n} - \sqrt{q_n^i[i+1]_{q_n}} \geq [j+1]_{q_n}$. In this case, we first show that the function $h(i) = [i+1]_{q_n} - \sqrt{q_n^i[i+1]_{q_n}}$ is increasing with respect to i. Indeed, we may write that

$$\begin{aligned} h(i+1)-h(i) &\geq [i+2]_{q_n} - [i+1]_{q_n} + \sqrt{q_n^i[i+1]_{q_n}} - \sqrt{q_n^i[i+2]_{q_n}} \\ &= q_n^{i+1} - \frac{q_n^{\frac{i}{2}}q_n^{i+1}}{\sqrt{[i+1]_{q_n}}+\sqrt{[i+2]_{q_n}}} \\ &\geq q_n^{i+1}\left(1-\frac{1}{\sqrt{[i+1]_{q_n}}+\sqrt{[i+2]_{q_n}}}\right) > 0. \end{aligned}$$

Hence, there exists $\tilde{i} \in \{0,1,2,...,n\}$ of maximum value such that

$$[\tilde{i}+1]_{q_n} - \sqrt{q_n^{\tilde{i}}[\tilde{i}+1]_{q_n}} < [j+1]_{q_n}.$$

Then for $\bar{i} = \tilde{i}+1$, we get $[\bar{i}+1]_{q_n} - \sqrt{q_n^{\bar{i}}[\bar{i}+1]_{q_n}} \geq [j+1]_{q_n}$ and

$$\begin{aligned} \bar{M}_{\bar{i},n,j,q_n}(x) &= m_{\bar{i},n,j,q_n}(x)\left(\frac{[\bar{i}]_{q_n}b_n}{[n+1]_{q_n}} - x\right) \\ &\leq \frac{[\tilde{i}+1]_{q_n}b_n}{[n+1]_{q_n}} - \frac{[j]_{q_n}b_n}{[n+1]_{q_n}}. \end{aligned}$$

Since $[\tilde{i}+1]_{q_n} - q_n^j - \sqrt{q_n^{\tilde{i}}[\tilde{i}+1]_{q_n}} < [j]_{q_n}$, we see that

$$\begin{aligned} \bar{M}_{\bar{i},n,j,q_n}(x) &\leq \frac{[\tilde{i}+1]_{q_n}b_n}{[n+1]_{q_n}} - \frac{\left([\tilde{i}+1]_{q_n} - q_n^j - \sqrt{q_n^{\tilde{i}}[\tilde{i}+1]_{q_n}}\right)b_n}{[n+1]_{q_n}} \\ &\leq \frac{\left(1+\sqrt{[n+1]_{q_n}}\right)b_n}{[n+1]_{q_n}} \\ &\leq \frac{2b_n}{\sqrt{[n+1]_{q_n}}}. \end{aligned}$$

Also, we have $\bar{i} \geq j+2$. Indeed, this is a consequence of the fact that $h(i)$ is increasing and it is easy to see that $h(j+1) < [j+1]_{q_n}$. By Lemma 3.7(a), it follows that $\bar{M}_{\bar{i},n,j,q_n}(x) \geq \bar{M}_{\bar{i}+1,n,j,q_n}(x) \geq ... \geq \bar{M}_{n,n,j,q_n}(x)$. We obtain

$\bar{M}_{i,n,j,q_n}(x) \le \frac{2b_n}{\sqrt{[n+1]_{q_n}}}$ for any $i \in \{\bar{i}, \bar{i}+1, ..., n\}$. Thus, for the same i's, it follows from Lemma 3.5(a) that

$$M_{i,n,j,q_n}(x) \le \frac{2\left(1+\frac{2}{q_n^{n+1}}\right)b_n}{\sqrt{[n+1]_{q_n}}}.$$

Case d. Let $i \le j-2$ and $[i]_{q_n} + \sqrt{q_n^{i-1}[i]_{q_n}} \ge [j]_{q_n}$. Then

$$\begin{aligned}\tilde{M}_{i,n,j,q_n}(x) &= m_{i,n,j,q_n}(x)\left(x - \frac{[i]_{q_n}b_n}{[n+1]_{q_n}}\right)\\ &\le \frac{[j+1]_{q_n}b_n}{[n+1]_{q_n}} - \frac{[i]_{q_n}b_n}{[n+1]_{q_n}}\\ &= \frac{\left([j]_{q_n}+q_n^j\right)b_n}{[n+1]_{q_n}} - \frac{[i]_{q_n}b_n}{[n+1]_{q_n}}.\end{aligned}$$

Using Lemma 3.5(b) and from hypothesis, we conclude that

$$\begin{aligned}M_{i,n,j,q_n}(x) &\le \frac{\left(\sqrt{q_n^{i-1}[i]_{q_n}}+q_n^j\right)b_n}{[n+1]_{q_n}}\\ &\le \frac{2b_n}{\sqrt{[n+1]_{q_n}}}.\end{aligned}$$

Case e. Let $i \le j-2$ and $[i]_{q_n} + \sqrt{q_n^{i-1}[i]_{q_n}} < [j]_{q_n}$. Let $\tilde{i} \in \{0,1,2,...,n\}$ be the minimum value such that $[\tilde{i}]_{q_n} + \sqrt{q_n^{\tilde{i}-1}[\tilde{i}]_{q_n}} \ge [j]_{q_n}$. Then $\bar{i} = \tilde{i}-1$ satisfies $[\tilde{i}-1]_{q_n} + \sqrt{q_n^{\tilde{i}-2}[\tilde{i}-1]_{q_n}} < [j]_{q_n}$ and

$$\begin{aligned}\tilde{M}_{\tilde{i}-1,n,j,q_n}(x) &= m_{\tilde{i}-1,n,j,q_n}(x)\left(x - \frac{[\tilde{i}-1]_{q_n}b_n}{[n+1]_{q_n}}\right)\\ &\le \frac{\left([j]_{q_n}+q_n^j\right)b_n}{[n+1]_{q_n}} - \frac{[\tilde{i}-1]_{q_n}b_n}{[n+1]_{q_n}}.\end{aligned}$$

Since $[\tilde{i}]_{q_n} + \sqrt{q_n^{\tilde{i}-1}[\tilde{i}]_{q_n}} \ge [j]_{q_n}$, we see that

$$\begin{aligned}\tilde{M}_{\tilde{i}-1,n,j,q_n}(x) &\le \frac{\left([\tilde{i}]_{q_n}+\sqrt{q_n^{\tilde{i}-1}[\tilde{i}]_{q_n}}+q_n^j\right)b_n}{[n+1]_{q_n}} - \frac{[\tilde{i}-1]_{q_n}b_n}{[n+1]_{q_n}}\\ &= \frac{\left(q_n^{\tilde{i}-1}+\sqrt{q_n^{\tilde{i}-1}[\tilde{i}]_{q_n}}+q_n^j\right)b_n}{[n+1]_{q_n}}\\ &\le \frac{\left(2+\sqrt{[n+1]_{q_n}}\right)b_n}{[n+1]_{q_n}}\\ &\le \frac{3b_n}{\sqrt{[n+1]_{q_n}}}.\end{aligned}$$

Moreover, in this case, we have $j \geq 2$, which implies $\bar{i} \leq j-2$. By Lemma 3.7(b), we get $\tilde{M}_{\bar{i}-1,n,j,q_n}(x) \geq \tilde{M}_{\bar{i}-2,n,j,q_n}(x) \geq ... \geq \tilde{M}_{0,n,j,q_n}(x)$ which implies, for the same i's, that

$$M_{i,n,j,q_n}(x) \leq \frac{3b_n}{\sqrt{[n+1]_{q_n}}}$$

due to Lemma 3.5(b).

As a result, if we combine all the results obtained above, the truth of the claim is shown.

Proof of Lemma 4.3. We have two cases:

Case a. Let $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$ be fixed such that $C^{(M)}_{n,q_n}(f;x) = f_{j,n,j,q_n}(x)$. Since

$$-\frac{b_n}{[n+1]_{q_n}} \leq x - \frac{[j]_{q_n} b_n}{[n]_{q_n}} \leq \frac{2b_n}{[n+1]_{q_n}} \text{ and } f_{j,n,j,q_n}(x) = f\left(\frac{[j]_{q_n} b_n}{[n]_{q_n}}\right),$$

we have

$$\left|C^{(M)}_{n,q_n}(f;x) - f(x)\right| = \left|f\left(\frac{[j]_{q_n} b_n}{[n]_{q_n}}\right) - f(x)\right| \leq 2\omega_1\left(f; \frac{b_n}{[n+1]_{q_n}}\right).$$

Case b. Let $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$ be fixed such that $C^{(M)}_{n,q_n}(f;x) = f_{j+1,n,j,q_n}(x)$. We have two subcase:

Subcase b1. If $C^{(M)}_{n,q_n}(f;x) \leq f(x)$, then $f_{j,n,j,q_n}(x) \leq f_{j+1,n,j,q_n}(x) \leq f(x)$ and we clearly get

$$\begin{aligned} \left|C^{(M)}_{n,q_n}(f;x) - f(x)\right| &= f(x) - f_{j+1,n,j,q_n}(x) \\ &\leq f(x) - f_{j,n,j,q_n}(x) \\ &\leq 2\omega_1\left(f; \frac{b_n}{[n+1]_{q_n}}\right). \end{aligned}$$

Subcase b2. If $C^{(M)}_{n,q_n}(f;x) > f(x)$, then

$$\begin{aligned} \left|C^{(M)}_{n,q_n}(f;x) - f(x)\right| &= f_{j+1,n,j,q_n}(x) - f(x) = m_{j+1,n,j}(x) f\left(\frac{[j+1]_{q_n} b_n}{[n]_{q_n}}\right) - f(x) \\ &\leq f\left(\frac{[j+1]_{q_n} b_n}{[n]_{q_n}}\right) - f(x). \end{aligned}$$

For $q_n \in (0,1)$, since

$$0 \leq \frac{[j+1]_{q_n} b_n}{[n]_{q_n}} - x \leq \frac{[j+1]_{q_n} b_n}{[n]_{q_n}} - \frac{[j]_{q_n} b_n}{[n+1]_{q_n}} = \frac{\left([n]_{q_n} q_n^j + [j]_{q_n} q_n^n + q_n^{n+j}\right) b_n}{[n]_{q_n} [n+1]_{q_n}} < \frac{3b_n}{[n]_{q_n}},$$

then

$$f\left(\frac{[j+1]_{q_n} b_n}{[n]_{q_n}}\right) - f(x) \leq 3\omega_1\left(f; \frac{b_n}{[n]_{q_n}}\right)$$

which completes the proof.

Proof of Lemma 4.4. We have two cases:

Case a. Let $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$ be fixed such that $C^{(M)}_{n,q_n}(f;x) = f_{j-1,n,j,q_n}(x)$. We have two subcase:

Subcase a1. If $C^{(M)}_{n,q_n}(f;x) \leq f(x)$, then following the proof of Lemma 4.3, we get

$$\left|C^{(M)}_{n,q_n}(f;x) - f(x)\right| \leq 2\omega_1\left(f; \frac{b_n}{[n+1]_{q_n}}\right).$$

Subcase a2. If $C^{(M)}_{n,q_n}(f;x) > f(x)$, then

$$\begin{aligned}\left|C^{(M)}_{n,q_n}(f;x) - f(x)\right| &= f_{j-1,n,j,q_n}(x) - f(x) = m_{j-1,n,j}(x) f\left(\frac{[j-1]_{q_n} b_n}{[n]_{q_n}}\right) - f(x)\\ &\leq f\left(\frac{[j-1]_{q_n} b_n}{[n]_{q_n}}\right) - f(x).\end{aligned}$$

Since
$0 \leq x - \frac{[j-1]_{q_n} b_n}{[n]_{q_n}} \leq \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}} - \frac{[j-1]_{q_n} b_n}{[n]_{q_n}} \leq \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}} - \frac{[j-1]_{q_n} b_n}{[n+1]_{q_n}} = \frac{q_n^{j-1}(1+q_n) b_n}{[n+1]_{q_n}} \leq \frac{2b_n}{[n]_{q_n}}$,
then

$$f\left(\frac{[j-1]_{q_n} b_n}{[n]_{q_n}}\right) - f(x) \leq 2\omega_1\left(f; \frac{b_n}{[n]_{q_n}}\right).$$

Case b. Let $x \in \left[\frac{[j]_{q_n} b_n}{[n+1]_{q_n}}, \frac{[j+1]_{q_n} b_n}{[n+1]_{q_n}}\right]$ be fixed such that $C^{(M)}_{n,q_n}(f;x) = f_{j,n,j,q_n}(x)$. As in the proof of Lemma 4.3, we get

$$\left|C^{(M)}_{n,q_n}(f;x) - f(x)\right| \leq 2\omega_1\left(f; \frac{b_n}{[n+1]_{q_n}}\right).$$

This completes the proof.

Remark 6.1. *The computations in this paper were performed by using Maple2021.*

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