Filomat 37:4 (2023), 1087–1095
https://doi.org/10.2298/FIL2304087G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

L^p-Inequalities and Parseval-type relations for the index ${}_2F_1$-transform

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Abstract. In this paper we consider a systematic study of several new L^p-boundedness properties for the index ${}_2F_1$-transform over the spaces $L_{\gamma,p}\big(\mathbb{R}_+\big)$, $1 \le p < \infty$, $\gamma \in \mathbb{R}$, and $L^\infty\big(\mathbb{R}_+\big)$. We also obtain Parseval-type relations over these spaces.

1. Introduction and preliminaries

This paper deals with the integral transform

$$F(y) = \int_0^\infty f(x)\mathbf{F}(\mu, \alpha, y, x)dx, \quad y > 0, \tag{1.1}$$

where

$$\mathbf{F}(\mu, \alpha, y, x) = {}_2F_1\left(\mu + \frac{1}{2} + iy, \mu + \frac{1}{2} - iy; \mu + 1; -x\right)x^\alpha$$

and ${}_2F_1(\mu + \frac{1}{2} + iy, \mu + \frac{1}{2} - iy; \mu + 1; -x)$ is the Gauss hypergeometric function. Here μ and α are complex numbers with $\Re\mu > -1/2$.

The Gauss hypergeometric function [3, p. 57] is defined for $|z| < 1$ as

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \quad (\lambda)_n := \lambda(\lambda + 1)\cdots(\lambda + n - 1),\ n = 1, 2\ldots(\lambda)_0 := 1.$$

see also [2]. For $|z| \ge 1$ is defined as its analytic continuation [18, p. 431] as

$${}_2F_1(a, b; c; z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)}\int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a}dt, \quad \Re c > \Re b > 0;\ |\arg(1 - z)| < \pi.$$

For more general definitions of the hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N} \cup \{0\})$ see [21]. Also for several important developments concerning the hypergeometric and other higher transcendental functions see [22].

2020 *Mathematics Subject Classification*. Primary 44A15; Secondary 33C05
Keywords. Index ${}_2F_1$-transform; Gauss hypergeometric function; L^p-inequalities; Parseval-type relations; Hölder inequality.
Received: 19 February 2022; Revised: 23 August 2022; Accepted: 05 October 2022
Communicated by Hari M. Srivastava
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The Gauss hypergeometric function satisfies the following differential equation [3, p. 56]

$$z(1-z)\frac{d^2w}{dz^2} + [c-(a+b+1)z]\frac{dw}{dz} - abw = 0,$$

where

$$w = w(z) = {}_2F_1(a,b;c;z).$$

The integral transform (1.1) was first mentioned in [28] as a particular case of a more general integral transform with the Meijer G-function as the kernel. Later in [1] it was also considered. In a series of papers Hayek, González and Negrín have considered several properties of the index ${}_2F_1$-transform both from a classical point of view and over spaces of generalized functions (cf. [8], [9], [10], [12], [13] and [14]).

First, we study L^p-boundedness properties for the index ${}_2F_1$-transform (1.1) over the space $L_{\gamma,p}(\mathbb{R}_+)$, $\gamma \in \mathbb{R}$, $1 \le p < \infty$ considered by Srivastava et al. in [20] and over the space $L^\infty(\mathbb{R}_+)$. In this sense we make use of the notation considered in [20] and therefore we denote by $L_{\gamma,p}(\mathbb{R}_+)$ the space of the complex-valued measurable functions defined on $\mathbb{R}_+$ such that

$$\|f\|_{\gamma,p} = \left(\int_0^\infty |f(x)|^p(1+x)^\gamma\, dx\right)^{1/p} < \infty \tag{1.2}$$

for $1 \le p < \infty$ and $\gamma \in \mathbb{R}$, and we denote by $L^\infty(\mathbb{R}_+)$ the space of the complex-valued measurable functions defined on $\mathbb{R}_+$ such that

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in(0,\infty)} \{|f(x)|\} < \infty.$$

We also consider the integral operator

$$(\mathfrak{G}g)(x) = \int_0^\infty g(y)\mathbf{F}(\mu,\alpha,y,x)dy, \quad x > 0, \tag{1.3}$$

which is related to the Olevskiĭ transform (see [16] and [27]).

According to the results and formulas in previous papers [4] and [6], we obtain L^p-boundedness properties for the index ${}_2F_1$-transform over the spaces $L_{\gamma,p}(\mathbb{R}_+)$, $1 \le p < \infty$, $\gamma \in \mathbb{R}$, and $L^\infty(\mathbb{R}_+)$.

Weighted norm inequalities for similar integral operators have been studied in several articles (see [4], [19] and [20], amongst others).

By using results of section 2 of [4] we prove that the operator $\mathfrak{G}$ is bounded from the space $L_{\gamma,p}(\mathbb{R}_+)$ into $L_{\gamma,p'}(\mathbb{R}_+)$, $1 < p < \infty$, $p + p' = pp'$, whenever $\gamma > p-1$ and $-1/p' < \Re\alpha < -1/p' + \Re\mu + 1/2 - \gamma/p'$. Also, for $\gamma \ge 0$ and $0 \le \Re\alpha < \Re\mu + 1/2$, the operator $\mathfrak{G}$ is bounded from $L_{\gamma,1}(\mathbb{R}_+)$ into $L^\infty(\mathbb{R}_+)$.

One has that under these conditions, if $f, g \in L_{\gamma,p}(\mathbb{R}_+)$, $1 \le p < \infty$, then one obtains the Parseval-type relation

$$\int_0^\infty (\mathfrak{F}f)(x)\, g(x)\, dx = \int_0^\infty f(x)\, (\mathfrak{G}g)(x)\, dx. \tag{1.4}$$

Let $\mathfrak{G}'$ be the adjoint of the operator $\mathfrak{L}$, i.e.,

$$\langle \mathfrak{G}' f, g\rangle = \langle f, \mathfrak{G}g\rangle. \tag{1.5}$$

The aforementioned Parseval-type relation (1.4) allows us to obtain an interesting connection between the operator $\mathfrak{G}'$ and the operator $\mathfrak{F}$.

We conclude that the operator $\mathfrak{G}'$ is the natural extension of the integral operator $\mathfrak{F}$, i.e.,

$$\mathfrak{G}'T_f = T_{\mathfrak{F}f}$$

where T_f is given by:

$$< T_f, g >= \int_0^\infty f(x)g(x)dx. \tag{1.6}$$

We also point out relevant connections of our work with various earlier related results (see [7], [15], [19], [20], [25] and [26]).

From [3, (7), p. 122 and (6), p. 155], we obtain

$$\mathbf{F}(\mu, \alpha, y, x) =$$

$$= \frac{\Gamma(\mu+1)x^\alpha}{\sqrt{\pi}\Gamma(\mu+\frac{1}{2})} \int_0^\pi \left(1 + 2x + 2\sqrt{x(x+1)}\cos\xi\right)^{-\mu-1/2-iy} (\sin\xi)^{2\mu} d\xi, \tag{1.7}$$

which is valid for

$$x > 0,\ y > 0,\ \Re\mu > -\frac{1}{2},\ \alpha \in \mathbb{C}.$$

Observe that one has

$$\begin{aligned} \sin\xi \geq 0, &\quad \xi \in [0, \pi], \\ 1 + 2\sqrt{x + 2x(x+1)}\cos\xi \geq 0, &\quad x > 0,\ \xi \in [0, \pi], \end{aligned}$$

and hence, it follows from (1.7) that

$$\left|\mathbf{F}(\mu, \alpha, y, x)\right|$$

$$\leq \frac{\left|\Gamma(\mu+1)\right| x^{\Re\alpha}}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right|} \int_0^\pi \left(1 + 2x + 2\sqrt{x(x+1)}\cos\xi\right)^{-\Re\mu-\frac{1}{2}} (\sin\xi)^{2\Re\mu} d\xi$$

$$= \frac{\left|\Gamma(\mu+1)\right| x^{\Re\alpha}}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right|} \int_0^\pi \left(1 + 2x + 2\sqrt{x(x+1)}\cos\xi\right)^{-\Re\mu-\frac{1}{2}} (\sin\xi)^{2\Re\mu} d\xi$$

$$= \frac{\left|\Gamma(\mu+1)\right| \Gamma(\Re\mu+\frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right| \Gamma(\Re\mu+1)} \mathbf{F}(\Re\mu, \Re\alpha, 0, x), \quad \Re\mu > -1/2. \tag{1.8}$$

Also, from [3, (7), p. 122] and [17, p.171, Entry (12.08) and p. 172, Entry (12.20)], for $\Re\mu > -1/2$ we have

$$\mathbf{F}(\Re\mu, \Re\alpha, 0, x) = O\left(x^{\Re\alpha}\right), \quad x \to 0^+, \tag{1.9}$$

$$\mathbf{F}(\Re\mu, \Re\alpha, 0, x) = O\left(x^{\Re\alpha-\Re\mu-\frac{1}{2}} \ln x\right), \quad x \to +\infty. \tag{1.10}$$

2. The operator $\mathfrak{F}$ over the space $L_{\gamma,p}(\mathbb{R}_+)$, $1 < p < \infty$

In this section we study the behaviour of the operator $\mathfrak{F}$ over the space $L_{\gamma,p}\big(\mathbb{R}_+\big)$, $1 < p < \infty$, $\gamma \in \mathbb{R}$, $\alpha, \mu \in \mathbb{C}$. Indeed, by following [5, Proposition 2.1], we derive Theorem 2.1 below

Theorem 2.1. *Let $1 < p < \infty$, $p + p' = pp'$. Then, for all $\gamma < -1$, $-1/p' < \Re\alpha < \Re\mu - 1/2 + (\gamma+1)/p$, and all q, $0 < q < \infty$, the operator $\mathfrak{F}$ given by (1.3) is bounded from $L_{\gamma,p}\big(\mathbb{R}_+\big)$ into $L_{\gamma,q}\big(\mathbb{R}_+\big)$. Furthermore, for all $\gamma \in \mathbb{R}$ and $-1/p' < \Re\alpha < \Re\mu - 1/2 + (\gamma+1)/p$, then the operator $\mathfrak{F}$ is bounded from $L_{\gamma,p}\big(\mathbb{R}_+\big)$ into $L^\infty\big(\mathbb{R}_+\big)$.*

Proof. By applying the Hölder inequality we get

$$\begin{aligned}
\left|(\mathfrak{F}f)(y)\right| &= \left|\int_0^\infty f(x)\mathbf{F}(\mu,\alpha,y,x)dx\right| \\
&\le \int_0^\infty \left|f(x)\right|\left|\mathbf{F}(\mu,\alpha,y,x)\right|dx \\
&= \int_0^\infty \left|f(x)\right|(1+x)^{\gamma/p}\left|\mathbf{F}(\mu,\alpha,y,x)\right|(1+x)^{-\gamma/p}\,dx \\
&\le \left(\int_0^\infty \left|f(x)\right|^p (1+x)^\gamma\,dx\right)^{1/p}\cdot\left(\int_0^\infty \left|\mathbf{F}(\mu,\alpha,y,x)\right|^{p'}(1+x)^{-\gamma p'/p}dx\right)^{1/p'} \\
&= \|f\|_{\gamma,p}\left(\int_0^\infty \left|\mathbf{F}(\mu,\alpha,y,x)\right|^{p'}(1+x)^{-\gamma p'/p}\,dx\right)^{1/p'}, \qquad (2.1)
\end{aligned}$$

which, from (1.8) and taking into account that $\Re\mu > -1/2$, leads us to the following inequality

$$\begin{aligned}
&\int_0^\infty \left|(\mathfrak{F}f)(y)\right|^q (1+y)^\gamma dy \\
&\le \|f\|_{\gamma,p}^q \int_0^\infty \left(\int_0^\infty |\mathbf{F}(\mu,\alpha,y,x)|^{p'}(1+x)^{-\gamma p'/p}\,dx\right)^{q/p'}(1+y)^\gamma dy \\
&\le \frac{\left|\Gamma(\mu+1)\right|\Gamma(\Re\mu+\frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right|\Gamma(\Re\mu+1)}\|f\|_{\gamma,p}^q\left(\int_0^\infty \mathbf{F}(\Re\mu,\Re\alpha,0,x)^{p'}(1+x)^{-\gamma p'/p}\,dx\right)^{q/p'}\cdot\int_0^\infty (1+y)^\gamma\,dy \qquad (2.2) \\
&\le \frac{\left|\Gamma(\mu+1)\right|\Gamma(\Re\mu+\frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right|\Gamma(\Re\mu+1)}\|f\|_{\gamma,p}^q\left(\int_0^\infty \mathbf{F}(\Re\mu,\Re\alpha,0,x)^{p'}(1+x)^{-\gamma p'/p}\,dx\right)^{q/p'}\cdot(-1-\gamma)^{-1}. \qquad (2.3)
\end{aligned}$$

Now from (1.9) and (1.10), the integral in (2.3) converges under the conditions for this Theorem. So, we have that the operator $\mathfrak{F}$ is bounded from $L_{\gamma,p}\big(\mathbb{R}_+\big)$ into $L_{\gamma,q}\big(\mathbb{R}_+\big)$.

Analogously, one has

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x\in(0,\infty)}\{|\mathfrak{F}f(x)|\} \qquad (2.4) \\
&\le \|f\|_{\gamma,p}\operatorname*{ess\,sup}_{x\in(0,\infty)}\left\{\left(\int_0^\infty \left|\mathbf{F}(\mu,\alpha,y,x)\right|^{p'}(1+x)^{-\gamma p'/p}dx\right)^{1/p'}\right\} \qquad (2.5) \\
&\le \frac{\left|\Gamma(\mu+1)\right|\Gamma(\Re\mu+\frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right|\Gamma(\Re\mu+1)}\|f\|_{\gamma,p}^q\left(\int_0^\infty \left|\mathbf{F}(\Re\mu,\Re\alpha,0,x)\right|^{p'}(1+x)^{-\gamma p'/p}dx\right)^{1/p'}. \qquad (2.6)
\end{aligned}$$

We next observe that, under the conditions of this Theorem and by virtue of (1.9) and (1.10), the integral in (2.6) converges. So, clearly, we have

$$||\mathfrak{F}f||_{\infty} \leq C||f||_{\gamma,p},$$

where C is a real constant depending on p and γ. Consequently, the operator $\mathfrak{F}$ is bounded from $L_{\gamma,p}\big(\mathbb{R}_+\big)$ into $L^{\infty}\big(\mathbb{R}_+\big)$. □

3. The operator $\mathfrak{F}$ on the space $L_{\gamma,1}(\mathbb{R}_+)$

In this section we study the behaviour of the operator $\mathfrak{F}$ over the space $L_{\gamma,1}\big(\mathbb{R}_+\big)$, $\gamma \in \mathbb{R}$, $\alpha, \mu \in \mathbb{C}$. Indeed, by following [5, Proposition 3.1], we derive Theorem 3.1 below

Theorem 3.1. *For all $\gamma < -1$ and $0 \leq \Re\alpha < \Re\mu + 1/2 + \gamma$, and any q, $0 < q < \infty$, the operator $\mathfrak{F}$ given by* (1.3) *is bounded from $L_{\gamma,1}\big(\mathbb{R}_+\big)$ into $L_{\gamma,q}\big(\mathbb{R}_+\big)$. Also, for all $\gamma \in \mathbb{R}$ and $0 \leq \Re\alpha < \Re\mu + 1/2 + \gamma$, then the operator $\mathfrak{F}$ is bounded from $L_{\gamma,1}\big(\mathbb{R}_+\big)$ into $L^{\infty}\big(\mathbb{R}_+\big)$.*

Proof. Note that

$$\begin{aligned}
\left|(\mathfrak{F}f)(y)\right| &= \left|\int_0^{\infty} f(x)\mathbf{F}(\mu,\alpha,y,x)dx\right| \\
&\leq \int_0^{\infty} \left|f(x)\right| \left|\mathbf{F}(\mu,\alpha,y,x)\right| dx \\
&= \int_0^{\infty} \left|f(x)\right| (1+x)^{\gamma} \left|\mathbf{F}(\mu,\alpha,y,x)\right| (1+x)^{-\gamma} dx \\
&\leq \int_0^{\infty} \left|f(x)\right| (1+x)^{\gamma} dx \cdot \sup_{x\in(0,\infty)} \left\{ \frac{|\mathbf{F}(\mu,\alpha,y,x)|}{(1+x)^{\gamma}} \right\} \\
&= ||f||_{\gamma,1} \cdot \sup_{x\in(0,\infty)} \left\{ \frac{|\mathbf{F}(\mu,\alpha,y,x)|}{(1+x)^{\gamma}} \right\},
\end{aligned} \tag{3.1}$$

which, from (1.8) and taking into account that $\Re\mu > -1/2$, leads us to the following inequality

$$\begin{aligned}
&\int_0^{\infty} \left|(\mathfrak{F}f)(y)\right|^{q} (1+y)^{\gamma} dy \\
&\leq ||f||_{\gamma,1}^{q} \int_0^{\infty} \left(\sup_{x\in(0,\infty)} \left\{ \frac{|\mathbf{F}(\mu,\alpha,y,x)|}{(1+x)^{\gamma}} \right\} \right)^{q} (1+y)^{\gamma} dy \\
&\leq ||f||_{\gamma,1}^{q} \frac{\left|\Gamma(\mu+1)\right| \Gamma(\Re\mu+\frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right| \Gamma(\Re\mu+1)} \left(\sup_{x\in(0,\infty)} \left\{ \frac{\mathbf{F}(\Re\mu,\Re\alpha,0,x)}{(1+x)^{\gamma}} \right\} \right)^{q} \int_0^{\infty} (1+y)^{\gamma} dy \\
&= ||f||_{\gamma,1}^{q} \frac{\left|\Gamma(\mu+1)\right| \Gamma(\Re\mu+\frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right| \Gamma(\Re\mu+1)} \left(\sup_{x\in(0,\infty)} \left\{ \frac{\mathbf{F}(\Re\mu,\Re\alpha,0,x)}{(1+x)^{\gamma}} \right\} \right)^{q} \cdot (-1-\gamma)^{-1}.
\end{aligned}$$

Therefore, in view of (1.9) and (1.10), we obtain

$$||\mathfrak{F}f||_{\gamma,q} \leq C|f||_{\gamma,1},$$

where C is a real constant depending on q and γ. Consequently, the operator $\mathfrak{F}$ is bounded from $L_{\gamma,1}\big(\mathbb{R}_+\big)$ into $L_{\gamma,q}\big(\mathbb{R}_+\big)$.

Similarly, by using (1.8), we get

$$\|\mathfrak{F}f\|_\infty \leq \|f\|_{\gamma,1} \cdot \operatorname*{ess\,sup}_{y\in(0,\infty)} \sup_{x\in(0,\infty)} \left\{ \frac{\left|\mathbf{F}(\mu,\alpha,y,x)\right|}{(1+x)^\gamma} \right\}$$

$$\leq \|f\|_{\gamma,1} \cdot \frac{\left|\Gamma(\mu+1)\right| \Gamma(\Re\mu + \frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right| \Gamma(\Re\mu+1)} \sup_{x\in(0,\infty)} \left\{ \frac{\mathbf{F}(\Re\mu, \Re\alpha, 0, x)}{(1+x)^\gamma} \right\},$$

which, in light of (1.9) and (1.10), yields to

$$\|\mathfrak{F}f\|_\infty \leq C\|f\|_{\gamma,1},$$

for a certain real constant C depending on γ. Thus, clearly, the operator $\mathfrak{F}$ is bounded from $L_{\gamma,1}\big(\mathbb{R}_+\big)$ into $L^\infty\big(\mathbb{R}_+\big)$, which evidently completes the proof of Theorem 3.1. □

4. The operator $\mathfrak{F}$ on the space $L^\infty(\mathbb{R}_+)$

In this section we study the behaviour of the operator $\mathfrak{F}$ over the space $L^\infty\big(\mathbb{R}_+\big)$, $\alpha, \mu \in \mathbb{C}$. Indeed, by following [5, Proposition 4.1], we derive Theorem 4.1 below.

Theorem 4.1. *For $\gamma < -1$ and $-1 < \Re\alpha < \Re\mu - 1/2$, and any q, $0 < q < \infty$, the operator $\mathfrak{F}$ given by (1.1) is bounded from $L^\infty\big(\mathbb{R}_+\big)$ into $L_{\gamma,q}\big(\mathbb{R}_+\big)$. Moreover, for $\gamma \in \mathbb{R}$ and $-1 < \Re\alpha < \Re\mu - 1/2$, the operator $\mathfrak{F}$ is bounded from $L^\infty\big(\mathbb{R}_+\big)$ into $L^\infty\big(\mathbb{R}_+\big)$.*

Proof. One has

$$|(\mathfrak{F}f)(y)| \leq \int_0^\infty |f(x)||\mathbf{F}(\mu,\alpha,y,x)|dx \leq \|f\|_\infty \cdot \int_0^\infty |\mathbf{F}(\mu,\alpha,y,x)|dx,$$

so that, for any q, $0 < q < \infty$, we get

$$|(\mathfrak{F}f)(y)|^q \leq \|f\|_\infty^q \cdot \left(\int_0^\infty |\mathbf{F}(\mu,\alpha,y,x)|dx \right)^q.$$

We thus find that

$$\int_0^\infty |(\mathfrak{F}f)(y)|^q (1+y)^\gamma dy \leq \|f\|_\infty^q \cdot \int_0^\infty \left(\int_0^\infty |\mathbf{F}(\mu,\alpha,y,x)|dx \right)^q (1+y)^\gamma dy,$$

and, therefore, that

$$\|\mathfrak{F}f\|_{\gamma,q} \leq \|f\|_\infty \cdot \left(\int_0^\infty \left(\int_0^\infty |\mathbf{F}(\mu,\alpha,y,x)|dx \right)^q (1+y)^\gamma dy \right)^{1/q}.$$

In view of (1.8), we have

$$\|\mathfrak{F}f\|_{\gamma,q} \leq \|f\|_\infty \cdot \frac{\left|\Gamma(\mu+1)\right| \Gamma(\Re\mu + \frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right| \Gamma(\Re\mu+1)} \left(\int_0^\infty \mathbf{F}(\Re\mu, \Re\alpha, 0, x)dx \right) \left(\int_0^\infty (1+y)^\gamma dy \right)^{1/q}$$

$$= ||f||_\infty \cdot \frac{\left|\Gamma(\mu+1)\right| \Gamma(\Re\mu + \frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right| \Gamma(\Re\mu+1)} \left(\int_0^\infty \mathbf{F}(\Re\mu, \Re\alpha, 0, x) dx\right) (-1-\gamma)^{-1/q}. \tag{4.1}$$

Thus, by applying (1.9) and (1.10), we see that the integral in (4.1) converges. So, we have

$$||\mathfrak{F}f||_{\gamma,q} \leq C||f||_\infty,$$

for certain real constant C depending on γ and q. Therefore, the operator $\mathfrak{F}$ is bounded from $L^\infty\big(\mathbb{R}_+\big)$ into $L_{\gamma,q}\big(\mathbb{R}_+\big)$.

Also, in view of (1.8), we get

$$||\mathfrak{F}f||_{\gamma,q} \leq ||f||_\infty \cdot \operatorname*{ess\,sup}_{y\in(0,\infty)} \left\{\int_0^\infty \left|\mathbf{F}(\mu,\alpha,y,x)\right| dx\right\}$$

$$\leq \frac{\left|\Gamma(\mu+1)\right| \Gamma(\Re\mu + \frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right| \Gamma(\Re\mu+1)} ||f||_\infty \cdot \int_0^\infty \mathbf{F}(\Re\mu, \Re\alpha, 0, x) dx. \tag{4.2}$$

Thus, by applying (1.9) and (1.10), we see that the integral in (4.1) converges. Hence we have

$$||\mathfrak{F}f||_\infty \leq C||f||_\infty,$$

for certain real constant C. Consequently, the operator $\mathfrak{F}$ is bounded from $L^\infty\big(\mathbb{R}_+\big)$ into $L^\infty\big(\mathbb{R}_+\big)$. □

5. The operator $\mathfrak{G}$ over the space $L_{\gamma,p}(\mathbb{R}_+)$, $1 < p < \infty$

In this section we deal with the behaviour of the operator $\mathfrak{G}$ on the spaces $L_{\gamma,p}\big(\mathbb{R}_+\big)$, $\gamma \in \mathbb{R}$ and $1 < p < \infty$.

Theorem 5.1. *Set $1 < p < \infty$ and $p + p' = pp'$. Then for all $\gamma > p - 1$ and $-1/p' < \Re\alpha < -1/p' + \Re\mu + 1/2 - \gamma/p'$, the operator $\mathfrak{G}$ given by* (1.3) *is bounded from $L_{\gamma,p}\big(\mathbb{R}_+\big)$ into $L_{\gamma,p'}\big(\mathbb{R}_+\big)$.*

Proof. Taking into account the hypothesis of this Theorem, using (1.8), (1.9) and (1.10), one has that $\mathbf{F}(\Re\mu, \Re\alpha, 0, x) \in L_{\gamma,p}\big(\mathbb{R}_+\big)$ and moreover, since

$$\int_0^\infty (1+y)^{-\gamma p'/p} dy = \frac{p}{\gamma p'},$$

from Proposition 2.1 in [4] the result holds. □

As a consequence of Proposition 2.2 in [4] one has

Theorem 5.2. *Assume $1 < p < \infty$ and $p + p' = pp'$, then for $\gamma > p - 1$ and $-1/p' < \Re\alpha < -1/p' + \Re\mu + 1/2 - \gamma/p'$, the following mixed Parseval-type relation holds*

$$\int_0^\infty (\mathfrak{F}f)(x) g(x) dx = \int_0^\infty f(x) (\mathfrak{G}g)(x) dx,$$

for $f, g \in L_{\gamma,p}\big(\mathbb{R}_+\big)$.

Also, as a consequence of Corollary 2.1 in [4] one has

Corollary 5.3. *Assume $1 < p < \infty$ and $p + p' = pp'$, then for $\gamma > p - 1$ and $-1/p' < \Re\alpha < -1/p' + \Re\mu + 1/2 - \gamma/p'$, one has for $f \in L_{\gamma,p}\big(\mathbb{R}_+\big)$*

$$\mathfrak{G}' T_f = T_{\mathfrak{F}f}$$

on $\left(L_{\gamma,p}\big(\mathbb{R}_+\big)\right)'$.

6. The operator $\mathfrak{G}$ over the spaces $L_{\gamma,1}(\mathbb{R}_+)$

This section is devoted to the study of the behaviour of the operator $\mathfrak{G}$ on the spaces $L_{\gamma,1}\left(\mathbb{R}_+\right)$.

Theorem 6.1. *If $\gamma \geq 0$ and $0 \leq \Re\alpha < \Re\mu + 1/2$, then the operator $\mathfrak{G}$ given by (1.3) is bounded from $L_{\gamma,1}\left(\mathbb{R}_+\right)$ into $L^{\infty}\left(\mathbb{R}_+\right)$.*

Proof. Observe that for $\gamma \geq 0$ and $0 \leq \Re\alpha < \Re\mu + 1/2$, and using (1.8), (1.9) and (1.10), we get that $\mathbf{F}(\Re\mu, \Re\alpha, 0, x)$ is essentially bounded of $(0, \infty)$. Then from Proposition 2.1 in [4] the result holds. □

As a consequence of Proposition 2.1 in [4] we get the following mixed Parseval relation

Theorem 6.2. *The following mixed Parseval relation holds*

$$\int_0^{\infty} (\mathfrak{F}f)(x)g(x)dx = \int_0^{\infty} f(x)(\mathfrak{G}g)(x)dx, \tag{6.1}$$

for $f, g \in L_{\gamma,1}\left(\mathbb{R}_+\right)$ with $\gamma \geq 0$ and $0 \leq \Re\alpha < \Re\mu + 1/2$.

Also, as a consequence of Corollary 3.1 in [4] we have the following

Corollary 6.3. *For $f \in L_{\gamma,1}\left(\mathbb{R}_+\right)$ with $\gamma \geq 0$ and $0 \leq \Re\alpha < \Re\mu + 1/2$ is holds that*

$$\mathfrak{G}' T_f = T_{\mathfrak{F}f}$$

on $\left(L_{\gamma,1}\left(\mathbb{R}_+\right)\right)'$.

7. Conclusions

Starting from the properties of the Gaussian hypergeometric function and considering suitable conditions on the parameters μ and α, we have deduced some boundedness properties between L^p spaces with different weights for the operator $\mathfrak{F}$ defined by the index ${}_2F_1$-transform (1.1). Analogous boundedness have been obtained for the operator $\mathfrak{G}$ given by (1.3) and related to the Olevskiĭ integral transform.

In addition, Parseval-type relations have been derived for the operators $\mathfrak{F}$ and $\mathfrak{G}$ over the L^p spaces considered.

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