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Approximation properties of Bernstein-Stancu operators preserving e^{-2x}

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Abstract. Bernstein-Stancu operators are one of the most powerful tool that can be used in approximation theory. In this manuscript, we propose a new construction of Bernstein-Stancu operators which preserve the constant and e^{-2x}, $x > 0$. In this direction, the approximation properties of this newly defined operators have been examined in the sense of different function spaces. In addition to these, we present the Voronovskaya type theorem for this operators. At the end, we provide two computational examples to demonstrate that the new operator is an approximation procedure.

1. Introduction

In recent years, approximation theory has attracted the attention of a number of mathematicians, especially in the field of mathematical analysis. In this context, a lot of new positive and linear operators have been introduced and presented their approximation properties. In this direction, Bernstein's major work on Bernstein polynomials keep the conductor status in approximation theory for many a long day. More specifically, Bernstein polynomials are defined as

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \tag{1}$$

for each bounded map on $[0,1]$, $n \geq 1$ and $x \in [0,1]$.

Immediately after, its expansion to unbounded interval has became common in literature. However, uniform approximation by polynomial on infinite intervals cannot be anticipated, therefore it is natural to strive an alteration where the interval of the operator arises from rational functions.

In 1969, Stancu would like to pick the nodes in another different way, to get more flexibility. Stancu took into account the nodes such as when $n \to \infty$, the distance between two successive nodes and the distance between 0 and first node and also between the last node and 1 tending all to zero. In this way, he defined and studied the following linear and positive operators, which are called Bernstein-Stancu polynomials in the literature:

$$B_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k+\alpha}{n+\beta}\right), \tag{2}$$

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for every bounded function on $[0,1]$, $n \geq 1$, $x \in [0,1]$ and $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha \leq \beta$. Note that, if we choose $\alpha = \beta = 0$, we get the standard Bernstein polynomials given in (1).

On the other hand, King's inspiration [15] make magnifical influence on the approximation theory and has been finely practised to a number of well-known sequences of operators [4], [11], [12], [9], [16], [17], [18], [20], [21]. The principal motivation of King is preserving quadratic function x^2 instead of the unit function x for the standard Bernstein operators that approximate better in proportion to the past. Regarding the King's idea, the innovative paper is due to Acar et. al. [1], who introduced the modified Szász-Mirakyan operators preserving constants and e^{2ax}, $a > 0$. This idea has been the source of inspiration a number of qualified papers in approximation theory and successfully applied to several well known sequences of operators too. In more detail, 1 and e^{ax} for $a > 0$ in [22], e^{ax} and e^{2ax} for $a > 0$ in [6], [7] have been preserved with the help of modified some well known positive linear operators, such as Bernstein, Szász-Mirakyan and Baskakov operators. Soon after, in [13], constant and e^{-x}, and in [2], constant and e^{-2x} have been preserved in a similar manner. Regarding the similar motivation, the most recent paper is due to Usta [19], who obtained a general class of linear and positive approximation procedure defined on unbounded and bounded intervals designed using an suitable function and Voronovskaya-type theorems.

This paper aims to introduce a modified version of Bernstein-Stancu operators which fix constant and e^{-2x}. In the meantime, we present the approximation properties of this newly defined operators for both in some weighted functions spaces and in spaces of continuous functions. Additionally, we provide theoretical background in an attempt to show that this operators have better error estimation than the original operators on certain intervals.

The entire composition of this study is composed of seven sections including this one. The rest of this work is organized as follows: In Section 2, we summarize the fundamental facts which we use our main theorems. Then, in Section 3, the new type Bernstein-Stancu operators which fix the constant and e^{-2x} are introduced. In section 4, approximation properties of newly constructed operators have been introduced while asymptotic formula given in Section 5. Finally, in Section 6, we provide some computational examples while some conclusions discussed in Section 7.

2. Fundamental Facts

Throughout this and following sections, we shall represent by I_n the set of $\left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$. We will use the notation $C(I_n)$ for the space of all continuous real valued functions on I_n. In this manner, we shall use $C_b(I_n)$ for the space consisting of all bounded functions in $C(I_n)$. Here, $C_b(I_n)$ endowed with the natural order and the uniform norm $\|\cdot\|_\infty$, is Banach lattice. Additionally, let $C_*(I_n)$ and $C_0(I_n)$ be the Banach sublattices of all space of real-valued bounded continuous functions on I_n described as,

$$C_*(I_n) = \{f \in C(I_n) : \exists \lim_{x \to \frac{\alpha}{n+\beta}} f(x) \in \mathbb{R} \quad \wedge \lim_{x \to \frac{n+\alpha}{n+\beta}} f(x) \in \mathbb{R}\},$$

and

$$C_0(I_n) = \{f \in C_*(I_n) : \lim_{x \to \frac{\alpha}{n+\beta}} f(x) = 0 \quad \wedge \lim_{x \to \frac{n+\alpha}{n+\beta}} f(x) = 0\},$$

respectively. Now let us consider the weighted space

$$F_m := \{f \in C(I_n) : \sup_{x \in I_n} \rho_m(x)|f(x)| \in \mathbb{R}\},$$

where $\rho_m(x) = \left(x - \frac{\alpha}{n+\beta}\right)^{\xi} \left(\frac{n+\alpha}{n+\beta} - x\right)^{\eta}$ is the weight function with $\xi, \eta \geq 0$ for $m \geq 1$ and $x \in I_n$. It is quite apparent that this weighted space endowed with the norm

$$\|f\|_{\xi,\eta} = \sup_{x \in I_n} \rho_m(x)|f(x)|,$$

where $f \in F_m$ and its natural subspaces

$$F_m^* = \{f \in F_m : \exists \lim_{x \to \frac{\alpha}{n+\beta}} \rho_m(x) f(x) \in \mathbb{R} \quad \wedge \lim_{x \to \frac{n+\alpha}{n+\beta}} \rho_m(x) f(x) \in \mathbb{R}\},$$

and

$$F_m^0 = \{f \in F_m : \lim_{x \to \frac{\alpha}{n+\beta}} \rho_m(x) f(x) = 0 \quad \wedge \lim_{x \to \frac{n+\alpha}{n+\beta}} \rho_m(x) f(x) = 0\}.$$

It must be noted that, $C_0(I_n)$ is dense in F_m^0 as a consequence of the Stone-Weierstrass theorem.

In addition to this, throughout this and the next sections, we take a fixed real parameter $\mu > 0$ and take the exponential function f_μ, defined by

$$f_\mu(x) = e^{-\mu x}, \tag{3}$$

for $x \in [0, 1]$. Moreover, as usual, we denote by e_j the polynomials functions defined by $e_j(t) = t^j$ ($j \in \mathbb{N}$).

It might be functional to recall the following identities for every $n \geq 1$

1. $B_n^{\alpha,\beta}(e_0) = e_0$,
2. $B_n^{\alpha,\beta}(e_1) = \dfrac{ne_1 + \alpha}{n + \beta}$,
3. $B_n^{\alpha,\beta}(e_2) = \dfrac{(n^2 - n)e_2 + (2\alpha n + n)e_1 + \alpha^2}{(n + \beta)^2}$.

In particular, if one take the function described for each $x \geq 0$, as

$$\phi_t^m = (e_1 - te_0)^m,$$

then for every $x \in [0, 1]$,

1. $B_n^{\alpha,\beta}(\phi_t^0) = 0$,
2. $B_n^{\alpha,\beta}(\phi_t^1) = \dfrac{ne_1 + \alpha}{n + \beta} - e_1$,
3. $B_n^{\alpha,\beta}(\phi_t^2) = \dfrac{(n^2 - n)e_2 + (2\alpha n + n)e_1 + \alpha^2}{(n + \beta)^2} - \dfrac{2(ne_2 + \alpha e_1)}{n + \beta} + e_2$.

In conclusion, one can easily deduce that the following equality for the exponential function given in (3),

$$B_n^{\alpha,\beta}(f_\mu)(x) = e^{-\mu\alpha/(n+\beta)} \left(1 - x\left(1 - e^{-\mu/(n+\beta)}\right)\right)^n. \tag{4}$$

As a result, $(B_n^{\alpha,\beta})_{n\geq 1}$ is an approximation procedure in $C[0, 1]$; i.e., for every $f \in C[0, 1]$,

$$\lim_{n\to\infty} B_n^{\alpha,\beta}(f) = f,$$

uniformly on $[0, 1]$.

3. Bernstein-Stancu operators preserving e^{-2x}

After taking into consideration the above-stated, we can introduce a general version of Bernstein-Stancu operators which fix the function f_2. For this purpose, first of all, we need to introduce a sequence $(s_n)_{n\geq 1}$ of a real functions such that the operators,

$$\mathfrak{B}_n^{\alpha,\beta} := B_n^{\alpha,\beta} \circ s_n, \tag{5}$$

have the function f_2 as a constant point for every $n \geq 1$. To do this, with the help of (4), we have

$$e^{-2\alpha/(n+\beta)} \left[1 - s_n(x)\left(1 - e^{-2/(n+\beta)}\right)\right]^n = e^{-2x},$$

such that

$$s_n(x) = \frac{1 - e^{2\alpha/[n(n+\beta)]-2x/n}}{1 - e^{-2/(n+\beta)}}.$$

where $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha \leq \beta$. Here is a point that

$$\lim_{n\to\infty} s_n(x) = x.$$

In addition to this, thanks to the fact that $1 - e^{-x} \leq x$ for $x \geq \frac{\alpha}{n+\beta}$, we can smoothly deduce that,

$$\begin{aligned} 1 - e^{2\alpha/[n(n+\beta)]-2x/n} &\leq \frac{2x}{n} e^{2\alpha/[n(n+\beta)]} + 1 - e^{2\alpha/[n(n+\beta)]}, \\ &\leq \frac{2x}{n} e^{2\alpha/[n(n+\beta)]}, \end{aligned}$$

since $1 - e^{2\alpha/[n(\beta+n)]} < 0$. So it yields,

$$0 < s_n(x) \leq K_n x, \tag{6}$$

as

$$s_n\left(\frac{\alpha}{\beta+n}\right) = 0,$$

where

$$K_n := \frac{2e^{2\alpha/[n(n+\beta)]}}{n(1 - e^{-2/(n+\beta)})},$$

for every $n \geq 1$ and $x > \left(\frac{\alpha}{\beta+n}\right)$. Here is a point that

$$\lim_{n\to\infty} K_n = 1,$$

and additionally $K_n \geq 1$ for $n \geq 1$. Taking into account all of these, the new sequence $(\mathfrak{B}_n^{\alpha,\beta})_{n\geq 1}$ can be expressed as

$$\mathfrak{B}_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right) \varphi_{n,k}^{\alpha,\beta}(x), \tag{7}$$

where

$$\varphi_{n,k}^{\alpha,\beta}(x) = \binom{n}{k} \left(\frac{1 - e^{2\alpha/[n(n+\beta)]-2x/n}}{1 - e^{-2/(n+\beta)}}\right)^k \left(1 - \frac{1 - e^{2\alpha/[n(n+\beta)]-2x/n}}{1 - e^{-2/(n+\beta)}}\right)^{n-k},$$

for every $f \in C_b(I_n)$, $n \geq 1$ and $x \in I_n$ such that $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha \leq \beta$.

Now, we can provide the following lemma without detailed proof since it can be obtained with elementary calculus.

Lemma 3.1. *For each $x \in I_n$ and $n \in \mathbb{N}$, then the following identities hold:*

1. $\mathfrak{B}_n^{\alpha,\beta}(e_0;x) = 1$,
2. $\mathfrak{B}_n^{\alpha,\beta}(e_1;x) = \dfrac{n}{n+\beta}s_n(x) + \dfrac{\alpha}{n+\beta}$,
3. $\mathfrak{B}_n^{\alpha,\beta}(e_2;x) = \dfrac{n^2-n}{(n+\beta)^2}s_n^2(x) + \dfrac{n^2+n\beta+2\alpha n}{(n+\beta)^3}s_n(x) + \dfrac{\alpha^2 n+\alpha^2\beta+2\alpha^2}{(n+\beta)^3}$.

where $\alpha,\beta \in \mathbb{R}$ and $0 \le \alpha \le \beta$.

In addition to this, we can present the following lemma in a similar way.

Lemma 3.2. *For each $x \in I_n$ and $n \in \mathbb{N}$, then the following identities hold:*

1. $\mathfrak{B}_n^{\alpha,\beta}(\phi_t^0;x) = 1$,
2. $\mathfrak{B}_n^{\alpha,\beta}(\phi_t^1;x) = \dfrac{n}{n+\beta}s_n(x) + \dfrac{\alpha}{n+\beta} - x$,
3. $\mathfrak{B}_n^{\alpha,\beta}(\phi_t^2;x) = \dfrac{n^2-n}{(n+\beta)^2}s_n^2(x) + \dfrac{n^2+n\beta+2\alpha n}{(n+\beta)^3}s_n(x) + \dfrac{\alpha^2 n+\alpha^2\beta+2\alpha^2}{(n+\beta)^3} - \dfrac{2nx}{n+\beta}s_n(x) - \dfrac{2x\alpha}{n+\beta} + x^2$,

where $\alpha,\beta \in \mathbb{R}$ and $0 \le \alpha \le \beta$.

In addition to this, with the help of (4), for each $\mu > 0$, we deduce that

$$\begin{aligned} \mathfrak{B}_n^{\alpha,\beta}(f_\mu;x) &= e^{-\mu\alpha/(n+\beta)}\left(1 - s_n(x)\left(1-e^{-\mu/(n+\beta)}\right)\right)^n, \\ &= e^{-\mu\alpha/(n+\beta)}\left(1 - \frac{(1-e^{2\alpha/[n(n+\beta)]-2x/n})\left(1-e^{-\mu/(n+\beta)}\right)}{1-e^{-2/(n+\beta)}}\right)^n. \end{aligned}$$

In particular, if one take the function described for each $x \ge 0$, as

$$\Psi_t^m = (e^{-t} - e^{-x})^m,$$

then we can smoothly obtain the following lemma.

Lemma 3.3. *For each $x \in I_n$ and $n \in \mathbb{N}$, then the following identities hold:*

1. $\mathfrak{B}_n^{\alpha,\beta}(\Psi_t^0;x) = 1$,
2. $\mathfrak{B}_n^{\alpha,\beta}(\Psi_t^1;x) = e^{-\alpha/(n+\beta)}\left(1 - s_n(x)\left(1-e^{-1/(n+\beta)}\right)\right)^n - e^{-x}$,
3. $\mathfrak{B}_n^{\alpha,\beta}(\Psi_t^2;x) = 2e^{-2x} - 2e^{-x}e^{-\alpha/(n+\beta)}\left(1 - s_n(x)\left(1-e^{-1/(n+\beta)}\right)\right)^n$,

where $\alpha,\beta \in \mathbb{R}$ and $0 \le \alpha \le \beta$.

Now, let focus on the properties of the function $s_n(x)$.

Proposition 3.4. *For each $n \ge 1$ and any $x \in I_n$, we have*

$$s_n(x) \ge \left(1 + \frac{\beta}{n}\right)x - \frac{\alpha}{n}, \tag{8}$$

where $\alpha,\beta \in \mathbb{R}$ and $0 \le \alpha \le \beta$.

Proof. First of all, for $n \geq 1$ we know that s_n is convex down increasing function in I_n. Additionally, because of $s_n\left(\frac{\alpha}{n+\beta}\right) = 0$ and $s_n\left(\frac{\alpha+n}{\beta+n}\right) = 1$, we can easily deduce that $s_n(x) \geq \left(1 + \frac{\beta}{n}\right) - \frac{\alpha}{n}$ for $x \in I_n$, thus the proof is completed. □

Proposition 3.5. *For $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha \leq \beta$, $\lim_{n\to\infty} s_n = e_1$ uniformly on compact subintervals of I_n.*

Proof. It is clear that, $\lim_{n\to\infty} s_n = e_1$ pointwise on I_n. In addition to this, every $s_n(x)$ is concave the convergence is indeed uniform on every compact interval of I_n. □

4. Approximation Properties of the Sequence $(\mathfrak{B}_n^{\alpha,\beta})_{n\geq 1}$

In this section, we now present the approximation properties of introduced operator which preserve exponential function in several spaces.

Theorem 4.1. *Let $x > 0$ be fixed and $\mathfrak{B}_n^{\alpha,\beta}$, $n \geq 1$, be the operator defined in 7. Then,*

1. *$\mathfrak{B}_n^{\alpha,\beta}$ is a linear and positive operator from $C_*(I_n)$ into itself; in addition to this, $\|\mathfrak{B}_n^{\alpha,\beta}\|_{C_*(I_n)} = 1$,*
2. *$\mathfrak{B}_n^{\alpha,\beta}(C_0(I_n)) \subset C_0(I_n)$.*

for $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha \leq \beta$.

Proof.
1. It can be easily shown that for each $n \in \mathbb{N}$, $s_n(x)$ is positive function which given in (6). As a explicit consequence of that, one can say deduce that $\mathfrak{B}_n^{\alpha,\beta}$ is a positive operator. Additionally, if $f \in C_*(I_n)$, one can say that $B_n^{\alpha,\beta}(f) \in C_*(I_n)$ resulting from (2) which yields $B_n^{\alpha,\beta}(f) \in C(I_n)$. Then, it can be easily seen that $B_n^{\alpha,\beta}(f) \in C(I_n)$ since $s_n(x)$ satisfy the continuity and the relation (5). Moreover, it is obvious that $\lim_{x\to\left\{\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right\}} \mathfrak{B}_n^{\alpha,\beta}(f)(x) = \lim_{x\to\left\{\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right\}}(f)(x) \in \mathbb{R}$. As a consequences, $\|\mathfrak{B}_n^{\alpha,\beta}\|_{C_*(I_n)} = \|\mathfrak{B}_n^{\alpha,\beta}(e_0)\|_\infty = 1$ due to the positivity of each $\mathfrak{B}_n^{\alpha,\beta}$.
2. From the direct consequence of (i) and
$$\lim_{x\to\left\{\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right\}} \mathfrak{B}_n^{\alpha,\beta}(f)(x) = 0 \quad \text{and} \quad \lim_{x\to\left\{\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right\}} (f)(x) = 0,$$
whenever $f \in C_0(I_n)$, one can easily show the proof of (ii).
□

Theorem 4.2. *For $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha \leq \beta$ and the fixed $n \geq 1$, consider the operators L_n^* defined by (7). Then*

1. $\lim_{n\to\infty} \mathfrak{B}_n^{\alpha,\beta}(f) = f$ *uniformly on I_n if $f \in C_*(I_n)$,*
2. $\lim_{n\to\infty} \mathfrak{B}_n^{\alpha,\beta}(f) = f$ *uniformly on compacts subsets of I_n if $f \in C_b(I_n)$.*

Proof.
1. In an effort to prove the first part of theorem, firstly, we need to show that
$$\lim_{n\to\infty} \mathfrak{B}_n^{\alpha,\beta}(f_\mu) = f_\mu \quad \text{uniformly on} \quad I_n, \tag{9}$$
for every $\mu > 0$. In accordance with this purpose, for every $k > 0$, we use the following useful inequality given in [14, Lemma 3.1]
$$e^{-k\sigma_n} - e^{-k} < \frac{k_n}{2e}, \quad n \geq 1, \tag{10}$$

where $\sigma_n = \frac{1-e^{-k_n}}{k_n}$ and $(k_n)_{n\geq 1}$ is a sequence of strictly positive real numbers. Following the similar method of the proof of [14, Corollary 3.4], we can deduce that

$$\begin{aligned} |\mathfrak{B}_n^{\alpha,\beta}(f_\mu)(x) - (f_\mu)(x)| &\leq e^{-\mu\frac{\alpha}{n+\beta}}\left(1 - s_n(x)\left(1 - e^{-\mu/(n+\beta)}\right)\right)^n - e^{-\mu x}, \\ &= e^{-\mu\frac{\alpha}{n+\beta}} e^{n\log\left(1-s_n(x)\left(1-e^{-\mu/(n+\beta)}\right)\right)} - e^{-\mu x}, \\ &\leq e^{-\mu\frac{\alpha}{n+\beta}} e^{-\frac{n}{n+\beta}\mu s_n(x)\frac{\left(1-e^{-\mu/(n+\beta)}\right)}{\mu/(n+\beta)}} - e^{-\mu x}, \\ &\leq e^{-\mu\frac{\alpha}{n+\beta}} e^{-\frac{n}{n+\beta}\mu s_n(x)\frac{\left(1-e^{-\mu/(n+\beta)}\right)}{\mu/(n+\beta)}} - e^{-\mu s_n(x)\frac{n}{n+\beta}-\mu\frac{\alpha}{n+\beta}}, \\ &= e^{-\mu\frac{\alpha}{n+\beta}}\left[e^{-\frac{n}{n+\beta}\mu s_n(x)\frac{\left(1-e^{-\mu/(n+\beta)}\right)}{\mu/(n+\beta)}} - e^{-\mu s_n(x)\frac{n}{n+\beta}}\right], \end{aligned}$$

because $\ln x \leq x - 1$ and the inequality (8) holds. Then using the (10) for $k = \mu s_n(x)\frac{n}{n+\beta}$ and $k_n = \frac{\mu}{n+\beta}$, we deduce that

$$|\mathfrak{B}_n^{\alpha,\beta}(f_\mu)(x) - f_\mu(x)| \leq e^{-\mu\frac{\alpha}{n+\beta}}\frac{\mu}{2e(n+\beta)},$$

and

$$\|\mathfrak{B}_n^{\alpha,\beta}(f_\mu) - f_\mu\|_\infty \leq e^{-\mu\frac{\alpha}{n+\beta}}\frac{\mu}{2e(n+\beta)}, \tag{11}$$

for $x \in I_n$ and the proof of (9) is completed. Then, relying the direct result of (9) and [8], we can prove the first part of theorem.

2. For the second part of theorem, we notice that

$$|\mathfrak{B}_n^{\alpha,\beta}(e_1)(x) - e_1(x)| \leq x\left(K_n\frac{n}{n+\beta} - 1\right) + \frac{\alpha}{n+\beta},$$

and

$$|\mathfrak{B}_n^{\alpha,\beta}(e_2)(x) - e_2(x)| \leq \left(\frac{n^2-n}{(n+\beta)^2}K_n^2 - 1\right)x^2 + \frac{n^2+n\beta+2\alpha n}{(n+\beta)^3}K_n + \frac{\alpha^2 n + \alpha^2\beta + 2\alpha^2}{(n+\beta)^3},$$

thereby, $\lim_{n\to\infty}\mathfrak{B}_n^{\alpha,\beta}(\{e_0,e_1,e_2\}) = \{e_0,e_1,e_2\}$ uniformly on compact subsets of I_n, owing to the fact that $\lim_{n\to\infty} K_n = 1$. As a result, as $\{e_0,e_1,e_2\} \subset F_2^*$, the consequence follows from [5, Theorem 3.5].
□

In order to estimate the rate of convergence of $(\mathfrak{B}_n^{\alpha,\beta}(f))$ for $n \geq 1$ to f in Theorem 4.2, we need to brush up our knowledge about the modulus of continuity. In this estimation, we will take advantage of the following definition of modulus of continuity introduced in [14]:

Definition 4.3. *Let $f \in C_*(I_n)$. Then the modulus of continuity of a function, $\omega^*(f,\delta)$, is defined for $\delta \geq 0$ by*

$$\omega^*(f,\delta) = \sup_{\substack{x,t\in I_n \\ |e^{-x}-e^{-t}|\leq\delta}} |f(x) - f(t)|. \tag{12}$$

In other words, this modulus of continuity can be stated concerning the standard modulus of continuity by

$$\omega^*(f,\delta) = \omega(\mathbf{f},\delta),$$

where $\mathbf{f} : C_*(I_n) \to C(I_n)$ is the continuous function defined by

$$\mathbf{f}(\theta) = \begin{cases} f(-\ln\theta) & \text{if } \theta \in (0,1], \\ 1 & \text{if } \theta = 0. \end{cases}$$

Then the following theorem would be helpful in order to express next theorems.

Theorem 4.4. *[14] If $P_n : C_*(I_n) \to C_*(I_n)$ is a sequence of positive and linear operators for $n \geq 1$ with*

$$\begin{aligned} A_n &= \|P_n(e_0) - e_0\|_\infty, \\ B_n &= \|P_n(f_1) - f_1\|_\infty, \\ C_n &= \|P_n(f_2) - f_2\|_\infty, \end{aligned}$$

where $A_n, B_n, C_n \to 0$ as $n \to \infty$, then,

$$\|P_n(f) - f\|_\infty \leq \|f\|_\infty A_n + (2 + A_n)\omega^*\left(f, \sqrt{A_n + 2B_n + C_n}\right),$$

for $f \in C_*(I_n)$.

In this regard, it is clear that there is a close relation between $\omega^*(f, \delta)$ and the particular Korovkin subset chosen for the space $C_*(I_n)$, (see [14]). At this moment we can state the following theorem with the help of above.

Theorem 4.5. *For every $f \in C_*(I_n)$ and $n \geq 1$,*

$$\|\mathfrak{B}_n^{\alpha,\beta}(f) - f\|_\infty \leq 2\omega^*\left(f, \sqrt{e^{\frac{\alpha}{n+\beta}} \frac{1}{e(n+\beta)}}\right),$$

under the same assumptions of Theorem 4.2.

Proof. It is obvious that, A_n and C_n equal to zero due to the their definitions. On the other hand, it is clear that $B_n = e^{\frac{\alpha}{n+\beta}} \frac{1}{2e(n+\beta)}$ from the (11). with $\mu = 1$ for every $n \geq 1$. So the proof is completed. □

5. Pointwise Convergence of $(\mathfrak{B}_n^{\alpha,\beta})_{n\geq 1}$

This part of paper is devoted pointwise convergence of the sequence of $\mathfrak{B}_n^{\alpha,\beta}$. To present the convergence we present Voronovskaya-type theorem in quantitative mean which shall allows us both degree of aimed convergence and upper bound for the error of approximation.

The quantitative Voronovskaya-type theorem for the operators acting on bounded intervals and unbounded intervals can be found in the papers [3], [10], respectively. Here we take the modulus of continuity given in (12). The main theorem of this section is:

Theorem 5.1. *Let $f, f'' \in C_*(I_n)$. Then the inequality*

$$\begin{aligned} &\left| n\left[\mathfrak{B}_n^{\alpha,\beta}(f, x) - f(x)\right] - x(1-x)f'(x) - \frac{1}{2}x(1-2\alpha-x)f''(x) \right| \\ \leq\ & \left|f'(x)\right| |a_n(x)| + \left|f''(x)\right| |b_n(x)| + 2\,|2b_n(x) + x(1-2\alpha-x)| + 2c_n(x)\omega^*\left(f''; \sqrt{1/n}\right), \end{aligned}$$

holds for any $x \in I_n$, $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha \leq \beta$ where

$$\begin{aligned} a_n(x) &= n\mathfrak{B}_n^{\alpha,\beta}(\phi_t^1; x) - x(1-x), \\ b_n(x) &= \frac{1}{2}n\mathfrak{B}_n^{\alpha,\beta}(\phi_t^2; x) - x(1-2\alpha-x), \\ c_n(x) &= n^2\sqrt{\mathfrak{B}_n^{\alpha,\beta}(\phi_t^4; x)}\sqrt{\mathfrak{B}_n^{\alpha,\beta}(\Psi_t^4; x)}. \end{aligned}$$

where ϕ_t^m and Ψ_t^m defined in Section 3.

Proof. Thanks to the Taylor expansion of f at the point $x \in I_n$, we can write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \tau(t,x)(t-x)^2, \tag{13}$$

where

$$\tau(t,x) := \frac{f''(\upsilon) - f''(x)}{2},$$

and υ is a number between x and t. If we apply the introduced operators $\mathfrak{B}_n^{\alpha,\beta}$ to both sides of equality (13) and using the fact $\mathfrak{B}_n^{\alpha,\beta}(e_0) = e_0$, we immediately obtain

$$\left|\mathfrak{B}_n^{\alpha,\beta}(f,x) - f(x) - f'(x)\mathfrak{B}_n^{\alpha,\beta}(\phi_t^1;x) - \frac{1}{2}f''(x)\mathfrak{B}_n^{\alpha,\beta}(\phi_t^2;x)\right| \leq \left|\mathfrak{B}_n^{\alpha,\beta}(\tau\phi_t^2;x)\right|.$$

Then we deduce that by manipulating the above inequality,

$$\begin{aligned} &\left|n\left[\mathfrak{B}_n^{\alpha,\beta}(f,x) - f(x)\right] - x(1-x)f'(x) - \frac{1}{2}x(1-2\alpha-x)f''(x)\right| \\ \leq\ &\left|f'(x)\right|\left|n\mathfrak{B}_n^{\alpha,\beta}(\phi_t^1;x) - x(1-x)\right| + \frac{1}{2}\left|f''(x)\right|\left|n\mathfrak{B}_n^{\alpha,\beta}(\phi_t^2;x) - x(1-2\alpha-x)\right| + \left|n\mathfrak{B}_n^{\alpha,\beta}(\tau\phi_t^2;x)\right|. \end{aligned}$$

In order to get simpler demonstration, we shall denote by

$$\begin{aligned} a_n(x) &= n\mathfrak{B}_n^{\alpha,\beta}(\phi_t^1;x) - x(1-x), \\ b_n(x) &= \frac{1}{2}n\mathfrak{B}_n^{\alpha,\beta}(\phi_t^2;x) - x(1-2\alpha-x). \end{aligned}$$

As a direct consequence of Lemma 3.2, it is obvious that $a_n \to 0$ and $b_n(x) \to 0$ as $n \to \infty$ at any point $x \in I_n$. So we obtain,

$$\left|n\left[\mathfrak{B}_n^{\alpha,\beta}(f,x) - f(x)\right] - x(1-x)f'(x) - \frac{1}{2}x(1-2\alpha-x)f''(x)\right| \leq \left|f'(x)\right||a_n(x)| + \left|f''(x)\right||b_n(x)| + \left|n\mathfrak{B}_n^{\alpha,\beta}(\tau\phi_t^2;x)\right|.$$

To complete the proof successfully, we must estimate the last term $\left|n\mathfrak{B}_n^{\alpha,\beta}(\tau\phi_t^2;x)\right|$. Taking into consideration an inequality in Holhoş's paper [14], we deduce that

$$|h(t,x)| \leq \left(1 + \frac{\left(e^{-x} - e^{-t}\right)^2}{\delta^2}\right)\omega^*(f'';\delta),$$

and

$$\begin{cases} |h(t,x)| \leq 2\omega^*(f'';\delta) & \text{if } \left|e^{-x} - e^{-t}\right| \leq \delta, \\ |h(t,x)| \leq 2\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\omega^*(f'';\delta) & \text{if } \left|e^{-x} - e^{-t}\right| > \delta. \end{cases}$$

Accordingly, we have $|h(t,x)| \leq 2\left(1 + \frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\omega^*(f'';\delta)$. With the help of this fact, we obtain

$$\left|n\mathfrak{B}_n^{\alpha,\beta}(\tau\phi_t^2;x)\right| \leq 2n\omega^*(f'';\delta)\mathfrak{B}_n^{\alpha,\beta}(\phi_t^2;x) + \frac{2n}{\delta^2}\omega^*(f'';\delta)\mathfrak{B}_n^{\alpha,\beta}(\phi_t^2\Psi_t^2;x),$$

and applying well-known Cauchy-Schwarz inequality, we deduce that

$$n\mathfrak{B}_n^{\alpha,\beta}(|\tau\phi_t^2|;x) \leq 2n\omega^*(f'';\delta)\,\mathfrak{B}_n^{\alpha,\beta}(\phi_t^2;x) + \frac{2n}{\delta^2}\omega^*(f'';\delta)\sqrt{\mathfrak{B}_n^{\alpha,\beta}(\phi_t^4;x)}\sqrt{\mathfrak{B}_n^{\alpha,\beta}(\Psi_t^4;x)}.$$

Choosing $\delta = \sqrt{\frac{1}{n}}$ and using the notations $c_n(x) = n^2\sqrt{\mathfrak{B}_n^{\alpha,\beta}(\phi_t^4;x)}\sqrt{\mathfrak{B}_n^{\alpha,\beta}(\Psi_t^4;x)}$, we obtain

$$\begin{aligned} &\left| n\left[\mathfrak{B}_n^{\alpha,\beta}(f,x) - f(x)\right] - x(1-x)f'(x) - \frac{1}{2}x(1-2\alpha-x)f''(x)\right| \\ \leq\ & \left|f'(x)\right||a_n(x)| + \left|f''(x)\right||b_n(x)| + 2\left|2b_n(x) + x(1-2\alpha-x)\right| + 2c_n(x)\omega^*\left(f''; \sqrt{1/n}\right), \end{aligned}$$

thus the proof is completed. □

Corollary 5.2. *Let $f, f'' \in C_*(I_n)$. Then the inequality*

$$\lim_{n\to\infty} n\left[\mathfrak{B}_n^{\alpha,\beta}(f,x) - f(x)\right] = x(1-x)f'(x) + \frac{1}{2}x(1-2\alpha-x)f''(x),$$

holds for any $x \in I_n$.

6. Numerical Examples

In this part, two computational illustrations for the newly constructed Bernstein-Stancu type operators are given due to display their approximation properties.

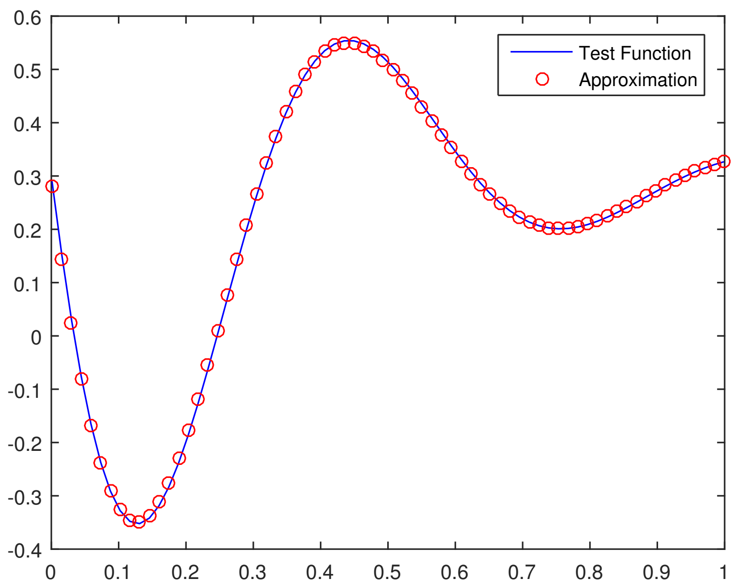

Figure 1: $\mathfrak{B}_n^{\alpha,\beta}(f,x)$ approximation of test function $f(x) = -\sin(10x)e^{-3x} + 0.3$ on a equally placed evaluation grid of I_n where $\alpha = 1$, $\beta = 2$ and $n = 1000$.

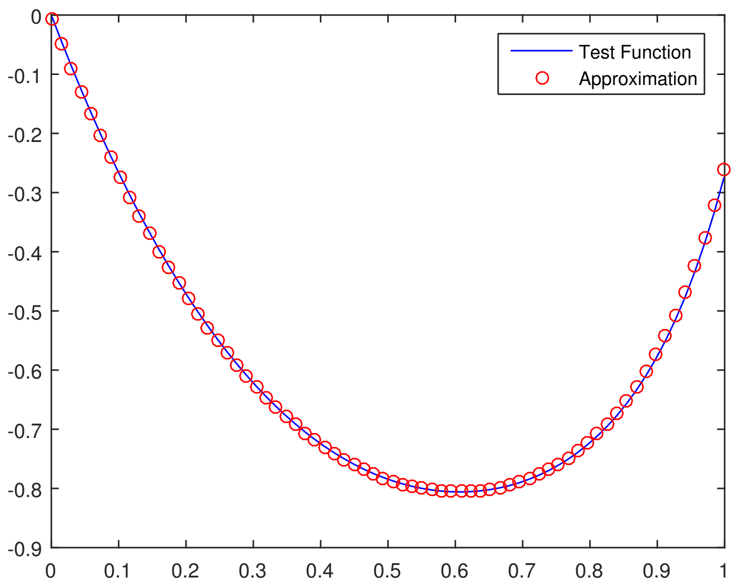

Figure 2: $\mathfrak{B}_n^{\alpha,\beta}(f,x)$ approximation of test function $f(x) = \frac{x^{10}}{3} + \frac{x^2}{2} - 3xe^{-x}$ on a equally placed evaluation grid of I_n where $\alpha = 1, \beta = 2$ and $n = 1000$.

For these numerical experiments, we consider the next two test functions such that,

$$f(x) = -\sin(10x)e^{-3x} + 0.3,$$

and

$$f(x) = \frac{x^{10}}{3} + \frac{x^2}{2} - 3xe^{-x},$$

for $n = 1000$. The results of the approximation can be seen in Figures 1 and Figure 2.

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