



Harvesting effect on Prey–predator system with strong Allee effect in prey and Herd Behaviour in both

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Abstract. In this paper, we introduce a prey-predator system with Allee effect in prey where both the prey and predator species are subject to harvesting and exhibit herd behaviour. The new idea of this paper is to consider the Allee effect, herd behaviour and harvesting together. Herd behaviour may be proved as a buffer against environmental obstacles. On the contrary, Allee effect and harvesting together may wash out the population from the system. So, the optimal harvesting policy is significant for the ecosystem concerned with our proposed model. Moreover, taking some hypothetical data, a rigorous numerical illustration and sensitivity analysis of the main parameters are offered here to validate the mathematical findings. To summarise, we can say that our model is an endeavor aiming at the ecological balance in nature.

1. Introduction

Modelling of the prey-predator system has been a fertile field for theoretical ecology. The exploitation of biological resources ([1],[2],[3]) is a significant threat to the ecological balance in nature. Furthermore, the human being is directly related to ecology and is too needed to take proper initiatives to maintain the ecological balance. Near about two million species exist in this world and they are always fighting with each other for existence. A noticeable fact is that, different species are related in some instances. The predator-prey ([4]-[7]) situation is defined as the situation when the growth rate of one species decreases while that of other increases. The Lotka-Volterra model [8] is the prime source of population models of previous days. Researchers also considered different models in this field like the Holling tanner model, ratio-dependent model, etc. ([9]-[11]). Harvesting [12] is a matter of interest for both ecologists and economists in recent times. So, harvesting models ([13]-[18]) are getting so much importance in the dynamics of the prey-predator system. Allee effect ([19]-[23]) is another recent trend to work with these types of models. The positive density reliance of populace enlargement at squat densities is known as Allee effect ([24]-[31]). In this study, strong Allee effect has considered ([32]-[35]), whereas different types of Allee effect ([36]-[44]) can be used for this purpose. Jana and Elsayed [45] presented a single species model with strong Allee effect and harvesting. Min et al. [40] discussed a diffusive prey-predator system with strong Allee effect. Yu et al. [43] recently analysed an integrated pest management predator-prey model with weak Allee effect.

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Considering the recent trend, we introduce our present model using suitable assumptions regarding Allee effect, herd behaviour and harvesting. For the modelling of Allee effect, the growth equation is of the form,

$$\frac{dX_1}{dT} = rX_1 \left(1 - \frac{X_1}{K}\right) \left(\frac{X_1}{K_0} - 1\right) \quad (1)$$

where, $X_1(T)$ is the population density at the time T , K is the carrying capacity of the environment, r is the intrinsic per capita growth rate and $0 < K_0 \ll K$. When the population size goes below the threshold level K_0 , then the population growth rate decreases and goes to extinction.

The functional response or trophic function ([46]-[47]) is defined as the function which describes the number of prey consumed per predator per unit time for given quantities of prey and predator. Till now, many functional responses like Beddington-DeAngelis type, Holling type-I, II, III, etc. ([48]-[50]) have been developed by different researchers. Among them, the Holling type-II functional response [51] can be mathematically expressed as:

$$G(X_1) = \frac{aX_1}{1 + aT_h X_1} \quad (2)$$

where, $X_1(T)$ is the prey density at time T , T_h denotes the average handling time of predator for each prey and a means the search efficiency of the predator for prey.

It is a usual assumption that both prey and predator species live independently, for which any predator can interact with any prey. But the populations who live in a group, always enjoy the benefits like high success in mate finding, defense against predators, protection from foul weather etc. The dictionary meaning of herd is a large group of animals, especially hoofed mammals, that live together as livestock. To describe this social behaviour, Cosner et al. [52], Chattopadhyay et al. [53], Ajraldi et al. [54] gave different concepts. Among them, Cosner et al. [52] and Ajraldi et al. [54] considered the functional response as the square root of the predator and prey variable, respectively. In the real world, we find the prey-predator situation where predator species are very large regarding some of their prey species. For these predators, there is no need for time to handle the prey species. We consider this fact here and take the average handling time $T_h = 0$ for the functional response. The recent study of Anacleto et al. [51] is about the delayed predator-prey model with Allee effect and Holling type-II functional response. We have modified their considerations by taking herd behaviour and harvesting together. With all these above concerns and reviewing some related literature ([55]-[61]), we consider the predation functional response as

$$G_1(X_1) = a\sqrt{X_1} \quad (3)$$

where a , X_1 are same as in (2). Lots of research work have already been done in this field. But as per the author's concern, no one considered the Allee effect, herd behaviour and harvesting together. For our proposed model's biological relevance, we can consider the example of some predatory fish species like perch, muskie, pike, walleye, sharks, billfish, etc. People always harvest these fish to be consumed and made oil, medicine etc. out of it. Any ecosystem containing big predatory fish usually shows a low density of small fish. Generally, the little fish are preyed upon by the big fish almost in no time. For example, Shark species prey on seals, sea lions, and other marine mammals.

Moreover, harvesting of all these species is always going on by human beings. Therefore, to protect such an ecosystem, we must need an acceptable harvesting policy. So, the real-world situation demands an optimal harvesting strategy for the sustainable development of the ecosystem. Hence, this study ultimately aims to discuss the threats and some possible ways out for balanced ecology.

2. Mathematical Formulation of Proposed System

We are considering the following notations and assumptions to formulate our proposed model:

Notations:

$X_1(T)$: Density of prey species at the time T

$X_2(T)$: Density of predator species at the time T

r : Intrinsic per capita growth rate of the prey

K : Carrying capacity of the prey

K_0 : Prey population Allee threshold in the absence of predator

a : Search efficiency of the predator

b : Biomass conversion rate

d : Death rate of the predator in the absence of prey

q_1 : Catchability coefficient of prey species

q_2 : Catchability coefficient of predator species

Assumptions:

Here we will derive the mathematical modelling in differential form based on the discussion in the introduction section regarding the growth of species, their relations, functional responses (1,2,3) and the following assumptions:

- (i) Both the populations live in herds.
- (ii) A multiplicative Allee effect in prey population growth.
- (iii) The average handling time of the predator for each prey is zero.
- (iv) Both the prey and predator are subject to harvesting effort F .

Therefore, the proposed model can be represented by the following set of non-linear differential equations:

$$\begin{aligned}\frac{dX_1}{dT} &= rX_1 \left(1 - \frac{X_1}{K}\right) \left(\frac{X_1}{K_0} - 1\right) - a\sqrt{X_1}\sqrt{X_2} - q_1FX_1, \\ \frac{dX_2}{dT} &= ab\sqrt{X_1}\sqrt{X_2} - (d + q_2F)X_2\end{aligned}\quad (4)$$

with initial condition $X_1(0) > 0$ and $X_2(0) > 0$.

Here $0 < K_0 \ll K$, and q_1FX_1, q_2FX_2 are the catch rate function (based on the catch per unit effort) for the prey and predator, respectively. The modified functional response of (3) has been considered here because of herd behaviour. Prey species always want to avoid their predators. But in an ecosystem, it is not possible every time. So, the herd behaviour acts as a buffer against predatory attacks. Harvesting is also unexpected for the populations of any ecosystem. But they cannot neglect the harvesting process. As a result, both prey and predator species form herds as a fighting strategy. So, we must have a biological significance to discuss our proposed model system. It is obvious that the parameters we used here are non-negative.

Now let $x_1 = \frac{X_1}{K}$, $x_2 = \frac{X_2}{K}$, $t = r\frac{K}{K_0}T$ and also put $m = \frac{K_0}{K}$, $\alpha = \frac{aK_0}{rK}$, $\beta = \frac{abK_0}{rK}$, $\delta = \frac{dK_0}{rK}$, $E = \frac{FK_0}{rK}$. Then after some simplifications, the system (4) reduces to our working equations of the form,

$$\begin{aligned}\frac{dx_1}{dt} &= x_1(1 - x_1)(x_1 - m) - \alpha\sqrt{x_1}\sqrt{x_2} - q_1Ex_1, \\ \frac{dx_2}{dt} &= \beta\sqrt{x_1}\sqrt{x_2} - (\delta + q_2E)x_2,\end{aligned}\quad (5)$$

with initial condition

$$x_1(0) > 0 \text{ and } x_2(0) > 0$$

3. Positivity and Boundedness of the System (5)

Let us consider the following theorems for the positivity of the system (5) :

Theorem 3.1. *Beginning in \mathbb{R}_+^2 , all the solutions $(x_1(t), x_2(t))$ of (5) remain positive for all $t \geq 0$.*

Proof. The first equation of (5) gives,

$$x_1(t) = x_1(0) \exp\left[\int_0^t \left\{ (1 - x_1(s))(x_1(s) - m) - \frac{\alpha \sqrt{x_2(s)}}{\sqrt{x_1(s)}} - q_1 E \right\} ds\right] > 0$$

Again the second equation of (5) gives,

$$x_2(t) = x_2(0) \exp\left[\int_0^t \left\{ \frac{\beta \sqrt{x_1(s)}}{\sqrt{x_2(s)}} - (\delta + q_2 E) \right\} ds\right] > 0$$

So we can say that $x_1(t) > 0$ and $x_2(t) > 0$ for all $t \geq 0$ which completes the proof. \square

Now the following theorem ensures the boundedness of the system (5).

Theorem 3.2. *All the solutions of (5) which start in \mathbb{R}_+^2 , are uniformly bounded.*

Proof. Let $\{x_1(t), x_2(t)\}$ be a solution of (5). Then we have the possible two cases.

Case 1: $x_1(0) \leq 1$.

Let us consider for all values of $t \geq 0$, $x_1(t) \leq 1$. Now, if this supposing is false then, we must find two positive real numbers t_1 and t_2 for which $x_1(t_1) = 1$ and $x_1(t) > 1$ for all $t \in (t_1, t_2)$. So, for all $t \in (t_1, t_2)$ the system (5) gives,

$$x_1(t) = x_1(0) \exp\left[\int_0^t f(x_1(s), x_2(s)) ds\right], \text{ where } f(x_1(s), x_2(s)) = \left\{ (1 - x_1(s))(x_1(s) - m) - \frac{\alpha \sqrt{x_2(s)}}{\sqrt{x_1(s)}} - q_1 E \right\}.$$

This implies,

$$\begin{aligned} x_1(t) &= x_1(0) \left[\exp\int_0^{t_1} f(x_1(s), x_2(s)) ds \right] \left[\exp\int_{t_1}^t f(x_1(s), x_2(s)) ds \right] \\ &= x_1(t_1) \exp\left[\int_{t_1}^t f(x_1(s), x_2(s)) ds\right] \text{ for all } t \in (t_1, t_2) \end{aligned}$$

So, for all $t \in (t_1, t_2)$, value of $f(x_1(s), x_2(s))$ must be negative as $m < 1$ and hence $x_1(t) < x_1(t_1) = 1$. This is a contradiction as we assumed that $x_1(t) > 1$ for all $t \in (t_1, t_2)$. Consequently, our claim is valid.

Case 2: $x_1(0) > 1$.

Here we claim $\limsup_{t \rightarrow \infty} x_1(t) \leq 1$ and suppose that our assert is false. Then $x_1(t) > 1$ for all $t > 0$.

Therefore, $f(x_1(t), x_2(t)) < 0$ where $f(x_1(s), x_2(s)) = \left\{ (1 - x_1(s))(x_1(s) - m) - \frac{\alpha \sqrt{x_2(s)}}{\sqrt{x_1(s)}} - q_1 E \right\}$. Now, the first equation of (5) gives,

$$x_1(t) = x_1(0) \exp\left[\int_0^t f(x_1(s), x_2(s)) ds\right] < x_1(0)$$

Again we can obtain that, $\frac{dx_1}{dt} < (x_1(0) - m)x_1(1 - x_1)$ where $x_1(0) - m > 0$. This ensures that $\limsup_{t \rightarrow \infty} x_1(t) \leq 1$ which contradicts our assumption and hence our assert is correct. So, we conclude that $\limsup_{t \rightarrow \infty} x_1(t) \leq 1$.

Now let $W = \beta x_1 + \alpha x_2$. Then for large value of t we must have,

$$\begin{aligned} \frac{dW}{dt} &= \beta \frac{dx_1}{dt} + \alpha \frac{dx_2}{dt} \\ &= \beta x_1(1 - x_1)(x_1 - m) - \alpha \delta x_2 - \beta q_1 E x_1 - \alpha q_2 E x_2 \\ &\leq \beta x_1 \{(1 + m)x_1 - m - x_1^2\} - \alpha \delta x_2 \\ &\leq \beta(1 + m)x_1 - \alpha \delta x_2 \\ &\leq 2\beta(1 + m) - \mu W, \text{ where } \mu = \min\{(1 + m), \delta\} \end{aligned}$$

Therefore, $\frac{dW}{dt} + \mu W \leq 2\beta(1 + m)$. From the theory of differential inequalities, we get

$$0 \leq W(x_1, x_2) \leq \frac{2\beta(1 + m)}{\mu} + \frac{W(x_1(0), x_2(0))}{e^{\mu t}}$$

and for $t \rightarrow \infty, 0 \leq W(x_1, x_2) \leq \frac{2\beta(1+m)}{\mu}$.

Since W is a linear combination of x_1 and x_2 and $W(x_1, x_2)$ is bounded for all values x_1, x_2 so x_1 and x_2 is also bounded. Hence, all the solutions of (5) enter inside the region,

$$B = \{(x_1, x_2) : 0 \leq W(x_1, x_2) \leq \frac{2\beta(1 + m)}{\mu} + \epsilon, \text{ for any } \epsilon > 0\}.$$

Hence the theorem is proved. \square

4. Extinction Criteria of the System (5)

Researchers are working continuously to maintain the ecological balance in nature. Consequently, we are trying to find some extinction criteria for both the prey and predator species.

Let us take $\bar{x}_1 = \limsup_{t \rightarrow \infty} x_1(t)$ and $\underline{x}_2 = \liminf_{t \rightarrow \infty} x_2(t)$. Then from Theorem 3.2 we get $\bar{x}_1 \leq 1$.

Theorem 4.1. *If $\bar{x}_1 < m$, then $\lim_{t \rightarrow \infty} x_1(t) = 0$.*

Proof. Let, $\lim_{t \rightarrow \infty} x_1(t) = \rho > 0$. Then for any ϵ fulfilling $0 < \epsilon < m - \bar{x}_1$ there subsists $t_\epsilon > 0$ so that $x_1(t) < \bar{x}_1 + \epsilon$ for any $t > t_\epsilon$ (from the definition of \bar{x}_1). Now for $t > t_\epsilon$, the first equation of (5) implies,

$$\begin{aligned} x_1(t) &= x_1(0) \exp\left[\int_0^t \{(1 - x_1(u))(x_1(u) - m) - \frac{\alpha \sqrt{x_2(u)}}{\sqrt{x_1(u)}} - q_1 E\} du\right] \\ &< x_1(0) \exp\left[\int_0^t (\bar{x}_1 + \epsilon - m) du\right] \\ &< x_1(0) \exp\{-(m - \bar{x}_1 - \epsilon)t\} \rightarrow 0 \text{ as } t \rightarrow \infty, \end{aligned}$$

which is a contradiction and hence the theorem is proved. \square

Theorem 4.2. *If $\underline{x}_2 > [\frac{2}{\alpha^2} \{2(1 - m) - q_1 E\}^2]$, then $\lim_{t \rightarrow \infty} x_1(t) = 0$.*

Proof. Let, $\lim_{t \rightarrow \infty} x_1(t) = \sigma > 0$. Then for any ϵ fulfilling $0 < \epsilon < 1 - m$, there exists $t_\epsilon > 0$ so that $x_1(t) < 1 + \epsilon$ for any $t > t_\epsilon$ (as $\bar{x}_1 \leq 1$). Again, for any $0 < \epsilon' < \underline{x}_2 - [\frac{2}{\alpha^2}\{2(1 - m) - q_1E\}^2]$, there exists $t_{\epsilon'} > 0$ such that $x_2(t) > \underline{x}_2 - \epsilon'$ for $t > t_{\epsilon'}$ (from definition of \underline{x}_2).

Now, for $t > \max\{t_\epsilon, t_{\epsilon'}\}$ the first equation of (5) implies,

$$\begin{aligned} \frac{dx_1}{dt} &< x_1(1 + \epsilon - m) - \alpha \sqrt{x_1} \sqrt{x_2} - q_1Ex_1 \\ &< x_1(1 + \epsilon - m) - \frac{\alpha x_1 \sqrt{x_2}}{\sqrt{1 + \epsilon}} - q_1Ex_1 \text{ as } x_1 < \sqrt{1 + \epsilon} \sqrt{x_1} \\ &< x_1\{2(1 - m) - \frac{\alpha \sqrt{x_2}}{\sqrt{2}} - q_1E\} \\ &< -\frac{\alpha x_1}{\sqrt{2}} [\sqrt{x_2} - \epsilon' - \frac{\sqrt{2}}{\alpha} \{2(1 - m) - q_1E\}] < 0, \end{aligned}$$

which implies $\lim_{t \rightarrow \infty} x_1(t) = 0$ and this is a contradiction that proves the theorem. \square

Theorem 4.3. If $\underline{x}_2 > \frac{2\beta^2}{(\delta + q_2E)^2}$, then $\lim_{t \rightarrow \infty} x_2(t) = 0$.

Proof. For any ϵ fulfilling $0 < \epsilon < 1$ there subsists $t_\epsilon > 0$ in such a way that $x_1(t) < 1 + \epsilon$ for any $t > t_\epsilon$ (as $\bar{x}_1 \leq 1$).

Also, for any ϵ' satisfying $0 < \epsilon' < \underline{x}_2 - \frac{2\beta^2}{(\delta + q_2E)^2}$ there exists $t_{\epsilon'} > 0$ such that $x_2(t) > \underline{x}_2 - \epsilon'$ for $t > t_{\epsilon'}$. Now, for $t > \max\{t_\epsilon, t_{\epsilon'}\}$, the second equation of (5) gives,

$$\begin{aligned} \frac{dx_2}{dt} &= \sqrt{x_2}(\beta \sqrt{x_1} - \delta \sqrt{x_2} - q_2E \sqrt{x_2}) \\ &< \sqrt{x_2}\{\beta \sqrt{1 + \epsilon} - \sqrt{x_2}(\delta + q_2E)\} \\ &< -(\delta + q_2E) \sqrt{x_2} \{ \sqrt{x_2} - \epsilon' - \frac{\beta \sqrt{2}}{(\delta + q_2E)} \} \\ &< 0 \text{ as } 0 < \epsilon' < \underline{x}_2 - \frac{2\beta^2}{(\delta + q_2E)^2} \end{aligned}$$

Thus, $\lim_{t \rightarrow \infty} x_2(t) = 0$ and this proves the theorem. \square

5. Equilibria and Stability Analysis of the System (5)

In this section, we discuss the existence of all possible equilibrium points with their stability. Clearly $(0, 0), \{ \frac{1+m+\sqrt{(1-m)^2-4q_1E}}{2}, 0 \}, \{ \frac{1+m-\sqrt{(1-m)^2-4q_1E}}{2}, 0 \}$ are trivial and axial equilibrium points of system (5) provided $(1 - m)^2 \geq 4q_1E$. Linearization of our proposed system about the trivial and axial equilibrium is not possible and hence cannot be studied. So, the interior equilibrium points can only be analysed here. Figure 1 presents the nullclines of our model.

Lemma 5.1. Interior equilibrium (x_1^*, x_2^*) exists if and only if the equation $x_1^2 - (1 + m)x_1 + (m + q_1E + \frac{\alpha\beta}{\delta + q_2E}) = 0$ has a positive real root.

Proof. From $\frac{dx_1}{dt} = 0$ and $\frac{dx_2}{dt} = 0$ we can easily obtain

$$x_1^2 - (1 + m)x_1 + (m + q_1E + \frac{\alpha\beta}{\delta + q_2E}) = 0$$

The positive solution of this equation ensures the existence of the interior equilibrium and vice-versa. \square

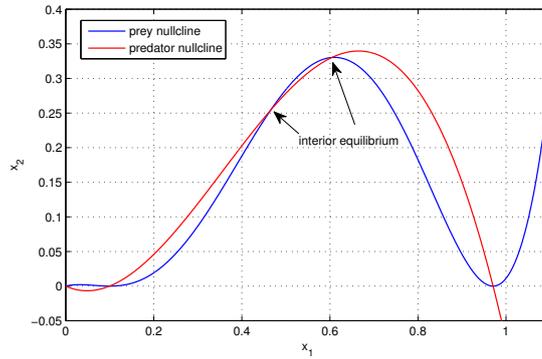


Figure 1: Nullclines of system (5) for $m = 0.07, \alpha = 0.25, \beta = 0.11, \delta = 0.118, q_1 = 0.19, q_2 = 0.22, E = 0.14$.

Theorem 5.2. *The necessary and sufficient condition for existence of the interior equilibrium (x_1^*, x_2^*) is $(1 + m)^2 \geq 4(m + q_1E + \frac{\alpha\beta}{\delta + q_2E})$.*

Proof. Let the interior equilibrium point (x_1^*, x_2^*) exists. Then Lemma 5.1 gives that $x_1^2 - (1 + m)x_1 + (m + q_1E + \frac{\alpha\beta}{\delta + q_2E}) = 0$ has a positive real root. Therefore, the discriminant of the equation must be greater than or equal to zero. Hence $(1 + m)^2 \geq 4(m + q_1E + \frac{\alpha\beta}{\delta + q_2E})$.

For the converse part, let $(1 + m)^2 \geq 4(m + q_1E + \frac{\alpha\beta}{\delta + q_2E})$ holds. Next, we have to consider the equation $x_1^2 - (1 + m)x_1 + (m + q_1E + \frac{\alpha\beta}{\delta + q_2E}) = 0$. Solving the quadratic equation we get the solutions as $x_1 = \frac{1+m}{2} \pm \frac{\sqrt{(1+m)^2 - 4(m + q_1E + \frac{\alpha\beta}{\delta + q_2E})}}{2}$. Now by our assumption, $(1 + m)^2 \geq 4(m + q_1E + \frac{\alpha\beta}{\delta + q_2E})$. Also one can observe that $\frac{1+m}{2} > 0$ as $0 < m < 1$. So, at least one of the values of x_1 must be positive. Then Lemma 5.1 ensures that the interior equilibrium exists and the theorem is proved. \square

Now, if we take $m = 0.07, \alpha = 0.25, \beta = 0.11, \delta = 0.118, q_1 = 0.37, q_2 = 0.22, E = 0.14$, then the parameter values do not satisfy the existence condition of interior equilibrium. Consequently, the interior equilibrium does not exist for the taken values. Figure 2 is the pictorial representation of this fact.

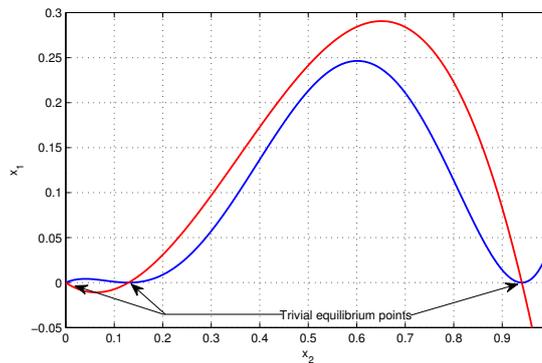


Figure 2: Behaviour of system (5) for $m = 0.07, \alpha = 0.25, \beta = 0.11, \delta = 0.118, q_1 = 0.37, q_2 = 0.22, E = 0.14$ and initial condition $(0.04, 0.001)$.

For the stability of interior equilibrium point (x_1^*, x_2^*) we consider the following theorem.

Theorem 5.3. *The interior equilibrium point (x_1^*, x_2^*) is locally asymptotically stable if and only if $3(x_1^*)^2 - 2(1 + m)x_1^* + \frac{\alpha\sqrt{x_2^*}}{2\sqrt{x_1^*}} - \frac{\beta\sqrt{x_1^*}}{2\sqrt{x_2^*}} + m + \delta + (q_1 + q_2)E > 0$ and $\{3(x_1^*)^2 - 2(1 + m)x_1^* + \frac{\alpha\sqrt{x_2^*}}{2\sqrt{x_1^*}} + m + q_1E\}(\delta + q_2E - \frac{\beta\sqrt{x_1^*}}{2\sqrt{x_2^*}}) + \frac{\alpha\beta}{4} > 0$.*

Proof. For the equilibrium point (x_1^*, x_2^*) , the Jacobian matrix is

$$J(x_1^*, x_2^*) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

where $b_{11} = x_1^*(1 - x_1^*) + (x_1^* - m)(1 - 2x_1^*) - \frac{\alpha\sqrt{x_2^*}}{2\sqrt{x_1^*}} - q_1E$, $b_{12} = -\frac{\alpha\sqrt{x_1^*}}{2\sqrt{x_2^*}}$, $b_{21} = \frac{\beta\sqrt{x_2^*}}{2\sqrt{x_1^*}}$, $b_{22} = \frac{\beta\sqrt{x_1^*}}{2\sqrt{x_2^*}} - \delta - q_2E$.

Now the characteristic equation of the matrix $J(x_1^*, x_2^*)$ is

$$\xi^2 + A\xi + B = 0,$$

where $A = -trJ(x_1^*, x_2^*) = -b_{11} - b_{22}$, $B = \det J(x_1^*, x_2^*) = b_{11}b_{22} - b_{12}b_{21}$. From the Descartes rule of sign, both the roots of $\xi^2 + A\xi + B = 0$ will be negative if and only if $A > 0$ and $B > 0$. Simplifying the expression of $-b_{11} - b_{22}$ and $b_{11}b_{22} - b_{12}b_{21}$ we get $A = 3(x_1^*)^2 - 2(1 + m)x_1^* + \frac{\alpha\sqrt{x_2^*}}{2\sqrt{x_1^*}} - \frac{\beta\sqrt{x_1^*}}{2\sqrt{x_2^*}} + m + \delta + (q_1 + q_2)E$ and $B = \{3(x_1^*)^2 - 2(1 + m)x_1^* + \frac{\alpha\sqrt{x_2^*}}{2\sqrt{x_1^*}} + m + q_1E\}(\delta + q_2E - \frac{\beta\sqrt{x_1^*}}{2\sqrt{x_2^*}}) + \frac{\alpha\beta}{4}$. This proves the theorem. \square

6. Bifurcation Analysis of the System (5)

For the occurrence of bifurcation near the interior equilibrium point (x_1^*, x_2^*) of the system (5), here we are using the Hopf bifurcation theorem [58].

Theorem 6.1. *If the interior equilibrium (x_1^*, x_2^*) exists, then Hopf bifurcation occurs at $m = m^* = x_1^* - \frac{1}{1-2x_1^*}\{\delta + (q_1 + q_2)E + \frac{\alpha\sqrt{x_2^*}}{2\sqrt{x_1^*}} - \frac{\beta\sqrt{x_1^*}}{2\sqrt{x_2^*}} - x_1^*(1 - x_1^*)\}$ provided m^* is positive with $\frac{\alpha\beta}{4} > (\frac{\beta\sqrt{x_1^*}}{2\sqrt{x_2^*}} - \delta - q_2E)^2$ and $x_1^* \neq \frac{1}{2}$.*

Proof. We can obtain that,

i) $[trJ(x_1^*, x_2^*)]_{m=m^*} = 0.$

ii) $[\det J(x_1^*, x_2^*)]_{m=m^*} = -(\frac{\beta\sqrt{x_1^*}}{2\sqrt{x_2^*}} - \delta - q_2E)^2 + \frac{\alpha\beta}{4} > 0$ as $\frac{\alpha\beta}{4} > (\frac{\beta\sqrt{x_1^*}}{2\sqrt{x_2^*}} - \delta - q_2E)^2.$

iii) If $m = m^*$, then the characteristic equation is $\xi^2 + \det J(x_1^*, x_2^*) = 0$ whose roots are purely imaginary.

iv) $[\frac{d}{dm}(trJ(x_1^*, x_2^*))]_{m=m^*} = 2x_1^* - 1 \neq 0$ as $x_1^* \neq \frac{1}{2}.$

Therefore, all the conditions of the Hopf bifurcation theorem are satisfied and this proves the theorem. \square

7. Bionomic Equilibrium Points of the System (5)

The biological equilibrium points have already been discussed in section 5. The mixture of the biological equilibrium point and economic equilibrium point is identified as bionomic equilibrium. This type of equilibrium point is attained when total revenue acquired by selling the harvested biomass is equivalent to the entirety cost consumed in harvesting it. Consider c as the constant fishing cost per unit effort, p_1 and p_2 as the constant price per unit biomass of the prey and predator species, respectively. Therefore, the net economic revenue to the society at any time is,

$$\Pi(x_1, x_2, E) = [p_1q_1x_1 + p_2q_2x_2 - c]E = 0 \tag{6}$$

The economic rent obtained from fishery becomes negative and the fishery will be closed if the fishing cost exceeds the revenue ($c > p_1q_1x_1 + p_2q_2x_2$). So, we must assume that $c < p_1q_1x_1 + p_2q_2x_2$ for the existence of

bionomic equilibrium. Then we can obtain the bionomic equilibrium points by taking the positive solutions of the following set of equations:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(1 - x_1)(x_1 - m) - \alpha \sqrt{x_1} \sqrt{x_2} - q_1Ex_1, \\ \frac{dx_2}{dt} &= \beta \sqrt{x_1} \sqrt{x_2} - (\delta + q_2E)x_2, \\ \Pi(x_1, x_2, E) &= [p_1q_1x_1 + p_2q_2x_2 - c]E = 0 \end{aligned} \tag{7}$$

Clearly $(0, 0, 0)$, $\{\frac{c}{p_1q_1}, 0, \frac{(1+m)cp_1q_1 - mp_1^2q_1^2 - c^2}{p_1^2q_1^3}\}$, $\{\frac{m+1 + \sqrt{(m-1)^2 - \frac{4\alpha\beta}{\delta}}}{2}, \frac{\beta^2}{2\delta^2}(m+1 + \sqrt{(m-1)^2 - \frac{4\alpha\beta}{\delta}}), 0\}$, $\{\frac{m+1 - \sqrt{(m-1)^2 - \frac{4\alpha\beta}{\delta}}}{2}, \frac{\beta^2}{2\delta^2}(m+1 - \sqrt{(m-1)^2 - \frac{4\alpha\beta}{\delta}}), 0\}$ are the trivial bionomic equilibrium points. Our present model also contains non-zero bionomic equilibrium point $(x_{1\infty}, x_{2\infty}, E_\infty)$ satisfying the system (7) and we can easily obtain,

$$x_{1\infty} = \frac{c(\delta + q_2E_\infty)^2}{p_1q_1(\delta + q_2E_\infty)^2 + p_2q_2\beta^2}, \text{ and } x_{2\infty} = \frac{c\beta^2}{p_1q_1(\delta + q_2E_\infty)^2 + p_2q_2\beta^2} \tag{8}$$

and E_∞ is the positive real root obtained from (8) and the first equation of (7).

8. Optimal Harvesting Policy of the System (5)

Optimal harvesting policy is subject to adoption by fishery management. Consider the current value of a continuous time stream of revenues as J . Then J is presented as:

$$J(E) = \int_0^\infty e^{-\theta t} (p_1q_1x_1 + p_2q_2x_2 - c)E dt \tag{9}$$

where θ represents the instantaneous annual discount rate [61] at time t . Now our major aim is to maximize $J(E)$ subject to the system (5) and the constraint $0 \leq E(t) \leq E_{\max}$.

The optimization of $J(E)$ can be easily done by Pontryagin’s maximal principle. Now we shall find the value of $E(t)$ such that $J(E)$ is maximized. The expression for the Hamiltonian for this optimal control problem can be written as:

$$H = e^{-\theta t} (p_1q_1x_1 + p_2q_2x_2 - c)E + \lambda_1\{x_1(1 - x_1)(x_1 - m) - \alpha \sqrt{x_1} \sqrt{x_2} - q_1Ex_1\} + \lambda_2\{\beta \sqrt{x_1} \sqrt{x_2} - (\delta + q_2E)x_2\}$$

where $\lambda_1(t)$ as well as $\lambda_2(t)$ means the adjoint variables. Their corresponding equations are as follow:

$$\begin{aligned} -\frac{d\lambda_1}{dt} &= \frac{\partial H}{\partial x_1} = Ep_1q_1e^{-\theta t} + \lambda_1\{(1 - 2x_1)(x_1 - m) + x_1(1 - x_1) - \frac{\alpha \sqrt{x_2}}{2 \sqrt{x_1}} - q_1E\} + \lambda_2 \frac{\beta \sqrt{x_2}}{2 \sqrt{x_1}} \\ -\frac{d\lambda_2}{dt} &= \frac{\partial H}{\partial x_2} = Ep_2q_2e^{-\theta t} - \lambda_1 \frac{\alpha \sqrt{x_1}}{2 \sqrt{x_2}} + \lambda_2\{\frac{\beta \sqrt{x_1}}{2 \sqrt{x_2}} - \delta - q_2E\} \end{aligned}$$

Simplifying we get,

$$\frac{d\lambda_1}{dt} + \{2(1 + m)x_1 - 3x_1^2 - m - \frac{\alpha \sqrt{x_2}}{2 \sqrt{x_1}} - q_1E\}\lambda_1 + \frac{\beta \sqrt{x_2}}{2 \sqrt{x_1}}\lambda_2 = -Ep_1q_1e^{-\theta t} \tag{10}$$

and

$$\frac{d\lambda_2}{dt} + \{\frac{\beta \sqrt{x_1}}{2 \sqrt{x_2}} - \delta - q_2E\}\lambda_2 - \frac{\alpha \sqrt{x_1}}{2 \sqrt{x_2}}\lambda_1 = -Ep_2q_2e^{-\theta t} \tag{11}$$

Now from (10) and (11) we obtain,

$$[D^2 + (V + R)D + RV + US]\lambda_1 = \{Sp_2q_2 - (D + V)p_1q_1\}Ee^{-\theta t} \tag{12}$$

where, $D \equiv \frac{d}{dt}$, $U = \frac{\alpha\sqrt{x_1}}{2\sqrt{x_2}}$, $R = \{2(1 + m)x_1 - 3x_1^2 - m - \frac{\alpha\sqrt{x_2}}{2\sqrt{x_1}} - q_1E\}$, $V = \{\frac{\beta\sqrt{x_1}}{2\sqrt{x_2}} - \delta - q_2E\}$, $S = \frac{\beta\sqrt{x_2}}{2\sqrt{x_1}}$.

So, solving (12) we can easily evaluate that $\lambda_1 = C_1e^{l_1t} + C_2e^{l_2t} + \frac{M_1}{N}e^{-\theta t}$, where C_1 and C_2 are integral constants, l_1 and l_2 are the roots of the equation $l^2 + (V + R)l + RV + US = 0$, $M_1 = \{Sp_2q_2 + (\theta - V)p_1q_1\}E$ and $N = \theta^2 - (V + R)\theta + RV + US$.

Clearly, λ_1 is bounded if either $l_i < 0$ or $C_i = 0$, ($i = 1, 2$). But it is quite difficult to find l_1 and l_2 . So, we have to assume that $C_1 = 0 = C_2$.

Then $\lambda_1 = \frac{M_1}{N}e^{-\theta t}$ where $M_1 = \{\frac{\beta p_2 q_2 \sqrt{x_2}}{2\sqrt{x_1}} + (\theta + \delta + q_2E - \frac{\beta\sqrt{x_1}}{2\sqrt{x_2}})p_1q_1\}E$ and $N = \theta^2 + \{3x_1^2 - 2(1 + m)x_1 + \frac{\alpha\sqrt{x_2}}{2\sqrt{x_1}} - \frac{\beta\sqrt{x_1}}{2\sqrt{x_2}} + m + \delta + (q_1 + q_2)E\}\theta + \{2(1 + m)x_1 - 3x_1^2 - \frac{\alpha\sqrt{x_2}}{2\sqrt{x_1}} - m - q_1E\}(\frac{\beta\sqrt{x_1}}{2\sqrt{x_2}} - \delta - q_2E) + \frac{\alpha\beta}{4}$.

Proceeding in a similar way, we get $\lambda_2 = \frac{M_2}{N}e^{-\theta t}$ where $M_2 = [\{3x_1^2 - 2(1 + m)x_1 + \frac{\alpha\sqrt{x_2}}{2\sqrt{x_1}} + m + \theta + q_1E\}p_2q_2 - \frac{\alpha p_1 q_1 \sqrt{x_1}}{2\sqrt{x_2}}]E$ and N is same as earlier.

For singular control, we have

$$\frac{\partial H}{\partial E} = e^{-\theta t}(p_1q_1x_1 + p_2q_2x_2 - c) - \lambda_1q_1x_1 - \lambda_2q_2x_2 = 0$$

Therefore,

$$e^{-\theta t}(p_1q_1x_1 + p_2q_2x_2 - c) = \lambda_1q_1x_1 + \lambda_2q_2x_2$$

Putting the values of λ_1 and λ_2 give $e^{-\theta t}(p_1q_1x_1 + p_2q_2x_2 - c) = (M_1q_1x_1 + M_2q_2x_2)\frac{e^{-\theta t}}{N}$ which implies $(p_1q_1x_1 + p_2q_2x_2 - c) = \frac{1}{N}(M_1q_1x_1 + M_2q_2x_2)$. Again, we have

$$\begin{aligned} \Pi(x_1, x_2, E) &= [p_1q_1x_1 + p_2q_2x_2 - c]E \\ \Rightarrow \Pi(x_1, x_2, E) &= (M_1q_1x_1 + M_2q_2x_2)\frac{E}{N} \end{aligned}$$

M_1, M_2 are of $O(\theta)$ and N is of $O(\theta^2)$ which implies that Π is of $O(\theta^{-1})$. Consequently Π is diminishing for all values of $\theta \geq 0$. This concludes that the maximization of Π is occurred at $\theta = 0$.

9. Numerical Analysis

We cannot deny the importance of real-world data. But the collection of data from the real field is very tough and also time taking. The economic problem may be another significant factor to affect this. Thus we are taking some hypothetical data here for verification of our analytical findings. Numerical validation always has a significant impact on ecological development through this type of model. So for simulations of this paper, we preferred a qualitative rather than a quantitative point of view. We have done these simulations with the help of MATLAB and MATHEMATICA software.

First of all, we take the values of our used parameters as $m = 0.07$, $\alpha = 0.25$, $\beta = 0.11$, $\delta = 0.118$, $q_1 = 0.19$, $q_2 = 0.22$, $E = 0.14$ and $(x_1(0), x_2(0)) = (0.04, 0.001)$. Then $x_1(0) = 0.04 < 0.07 = m$ and hence the condition of Theorem 4.1 is satisfied. Therefore, the prey population goes to extinction. Figure 3 depicts the fact graphically.

If we take the parameter values as previous and the initial condition $(x_1(0), x_2(0)) = (0.4, 0.1)$, then the populations behave like Figure 4.

Next, taking the parameter values as previous, we get $(1+m)^2 = 1.1449 > 1.1256 = 4(m+q_1E + \frac{\alpha\beta}{\delta+q_2E})$. Then by Theorem 5.2, the interior equilibrium exists and we find two interior equilibrium points $(0.6044, 0.3303)$ and $(0.4656, 0.2545)$.

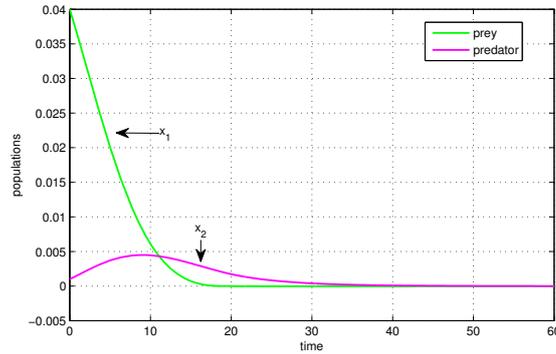


Figure 3: Extinction behaviour of system (5) for low prey density.

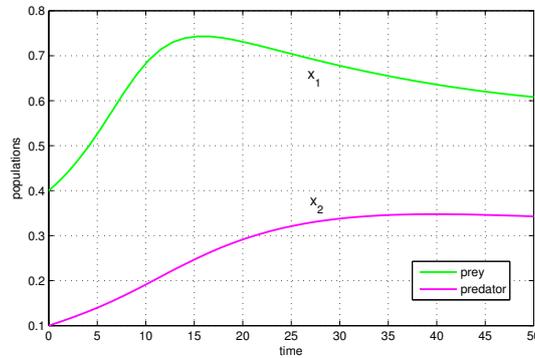


Figure 4: Stable behaviour of system (5).

For the interior equilibrium point $(x_1^*, x_2^*) = (0.6044, 0.3303)$, we evaluate

$$3(x_1^*)^2 - 2(1 + m)x_1^* + \frac{\alpha \sqrt{x_2^*}}{2\sqrt{x_1^*}} - \frac{\beta \sqrt{x_1^*}}{2\sqrt{x_2^*}} + m + \delta + (q_1 + q_2)E = 0.0658539 > 0,$$

$$\{3(x_1^*)^2 - 2(1 + m)x_1^* + \frac{\alpha \sqrt{x_2^*}}{2\sqrt{x_1^*}} + m + q_1E\}(\delta + q_2E - \frac{\beta \sqrt{x_1^*}}{2\sqrt{x_2^*}}) + \frac{\alpha\beta}{4} = 0.00623917 > 0.$$

Again, for the interior equilibrium point $(x_1^*, x_2^*) = (0.4656, 0.2545)$,

$$\{3(x_1^*)^2 - 2(1 + m)x_1^* + \frac{\alpha \sqrt{x_2^*}}{2\sqrt{x_1^*}} + m + q_1E\}(\delta + q_2E - \frac{\beta \sqrt{x_1^*}}{2\sqrt{x_2^*}}) + \frac{\alpha\beta}{4} = -0.00480677 < 0.$$

So, according to Theorem 5.3, $(0.6044, 0.3303)$ is stable and $(0.4656, 0.2545)$ is unstable equilibrium points.

Phase-space trajectories corresponding to the stabilities of the populations to a given set of parameter values are presented in Figure 5.

Further, if we take the values of the parameters as in Figure 3 and the initial condition as $(x_1(0), x_2(0)) = (10000, 108)$, then $x_2 = 108 > 107.563 = [\frac{2}{\alpha^2} \{2(1 - m) - q_1E\}^2]$. So by Theorem 4.2, the prey species will be washed out from the system. This situation is depicted in Figure 6.

Now, if we take the parameters as $m = 0.07, \alpha = 0.25, \beta = 0.11, \delta = 0.118, q_1 = 0.19, q_2 = 0.22, E = 0.14$ and initial condition $(x_1(0), x_2(0)) = (1.1, 0.1)$, then both the populations converge to their respective equilibrium state. The time series plot of the system is shown in Figure 7.

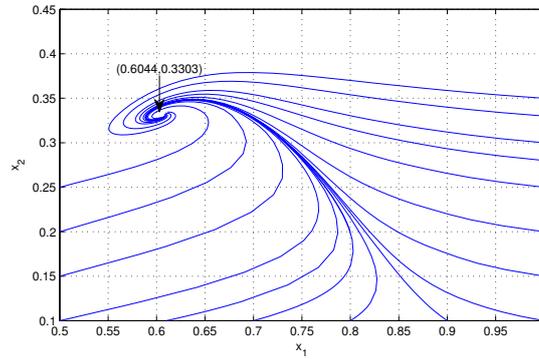


Figure 5: Phase portrait of the stable equilibrium point $(0.6044, 0.3303)$.

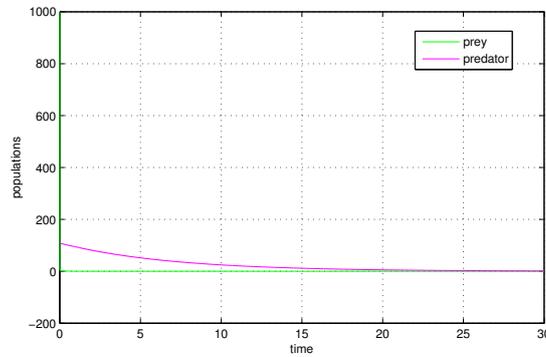


Figure 6: Extinction behaviour of system (5) for high predator density.

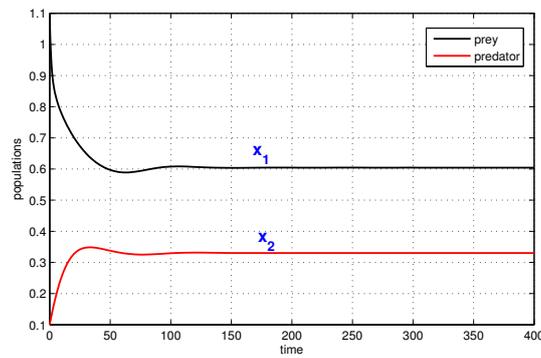


Figure 7: Steady state of the populations converging $(0.6044, 0.3303)$.

Figure 7 shows that the prey and predator populations settle down to their respective steady-state level as time progresses.

It is also observed that our assumed parameter values satisfy all the conditions of Theorem 6.1 for the interior equilibrium point $(0.6044, 0.3303)$. Hence, there exists a simple Hopf- bifurcation with m as a bifurcation parameter. Consequently, we have both stable and unstable behaviour near $m = 0.385$. The bifurcation diagram of the system is presented in Figure 8.

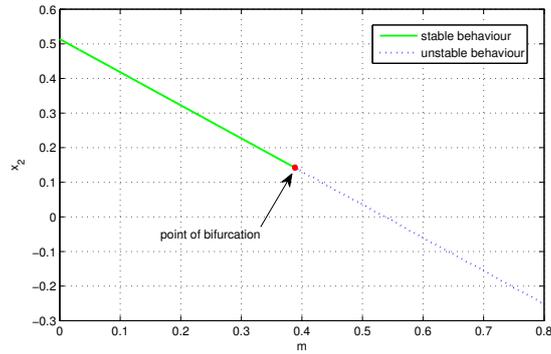


Figure 8: Bifurcation behaviour of system (5) for m .

Figure 8 shows stable behavior (plotted in green line) as m progresses up to 0.385; after that the system becomes unstable (plotted blue dotted line).

Now, we shall try to throw some light on the sensitivity of our used parameter values. Consequently, we analyse the different behaviour of our model system through this. Theorem 5.2 ensures the existence of interior equilibrium point, and Theorem 5.3 states its stability. In this study, we observed the presence of Hopf-bifurcation by Theorem 6.1. Table 1 and Table 2 reflect the results obtained from the theorems. Here, we noticed that the system (5) always gives either an infeasible solution or two solutions together. Sensitivity analysis of our proposed model is provided in the following two tables:

Table-1: Sensitivity of parameters m, α, β, q_1 on Stability and Bifurcation				
Parameter	Change (%)	Interior equilibrium point	Stability	Hopf Bifurcation point
m	-15%	(0.628356, 0.343388)	Stable	0.472078
		(0.431144, 0.235614)	Unstable	0.807672
	14%	(0.556639, 0.304196)	Stable	0.0853513
		(0.523161, 0.285901)	Unstable	does not exist
	15%	infeasible	–	–
α	-3%	(0.636772, 0.347987)	Stable	0.488131
		(0.433228, 0.236753)	Unstable	0.844387
	2%	(0.568421, 0.310634)	Stable	0.202561
		(0.501579, 0.274106)	Unstable	does not exist
	3%	infeasible	–	–
β	-3%	(0.636772, 0.327421)	Stable	0.488131
		(0.433228, 0.222761)	Unstable	0.844387
	2%	(0.568421, 0.323184)	Stable	0.202561
		(0.501579, 0.28518)	Unstable	does not exist
	3%	infeasible	–	–
q_1	-19%	(0.634334, 0.346655)	Stable	0.472042
		(0.435666, 0.238086)	Unstable	0.882627
	18%	(0.540017, 0.295112)	Unstable	does not exist
		(0.529983, 0.289629)	Unstable	does not exist
	19%	infeasible	–	–

From Table 1 we observe that if we increase the value of m by 15% and above, the solution of the system becomes infeasible; otherwise, we get a feasible solution. Similarly, for α, β and q_1 we obtain the same results for 3%, 3% and 19% respectively.

Parameter	Change (%)	Interior equilibrium point	Stability	Hopf Bifurcation point
δ	-4%	infeasible	–	–
	-3%	(0.552587, 0.316879)	Unstable	0.0453591
		(0.517413, 0.296709)	Unstable	does not exist
	4%	(0.637446, 0.327265)	Stable	0.498542
		(0.432554, 0.222073)	Unstable	0.821946
q_2	-13%	infeasible	–	–
	-12%	(0.545284, 0.313364)	Unstable	does not exist
		(0.524716, 0.301544)	Unstable	does not exist
	13%	(0.633264, 0.328172)	Stable	0.485986
		(0.436736, 0.226326)	Unstable	0.8557

From Table 2 we observe that, if we decrease the value of δ by 4% and above, then the solution of the system becomes infeasible; otherwise, we get a feasible solution. A similar observation has been seen for q_2 . In this case, if we decrease the value of q_2 by 13% and above, then a feasible solution becomes infeasible. From Table 1 and Table 2 we conclude that the parameters m, α, β and q_1 have same tendency regarding existence of interior equilibrium and the parameters δ, q_2 have the opposite tendency for the same.

From Tables 1 and 2, we get the numerical idea with the stability and existence of Hopf-bifurcation for the equilibrium solutions. For the pictorial representation of the sensitivities, let us take the parameter values as $\alpha = 0.25, \beta = 0.11, \delta = 0.118, q_1 = 0.19, q_2 = 0.22, E = 0.14$. Then the changing values of m give different behaviour of the equilibrium curve which is depicted through Figure 9. Similarly, the sensitivity of the equilibrium curve to q_1 is presented through Figure 10.

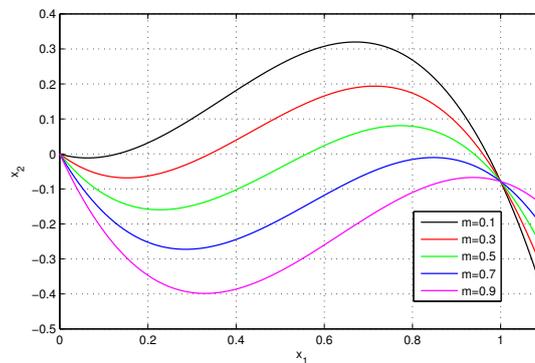


Figure 9: Behaviour of the equilibrium curve for different m .

From Figure 9 we observe that the increasing value of m leads the equilibrium to an infeasible solution. Conversely, if we decrease the value of m , then we always get a feasible solution. Similarly, Figure 10 gives a sensitivity of q_1 on the equilibrium curve. Therefore, the catchability coefficient of prey q_1 behaves the same as m for feasibility. But we find that q_1 brings most of the solutions to infeasibility if we increase its value. So, we have to be very conscious about the harvesting policy.

The effect of death rate (d) on the equilibrium curve is presented in Figure 11. In our present paper, we assumed the harvesting for both the prey and predator species. So, we must have to study the effect of E on our proposed model system. To observe the effect of harvesting on the phase portrait of the system (5), we assume the parameter values as $m = 0.07, \alpha = 0.25, \beta = 0.11, \delta = 0.118, q_1 = 0.19, q_2 = 0.22$. For $E = 0.1$, the phase portrait of the system (5) based on the above set of parameter values is depicted in Figure 12.

Figure 11 shows that the predator’s death rate can increase for the low density of prey. Figure 12 shows that starting with different initial conditions, the curves behave differently up to the stable equilibrium

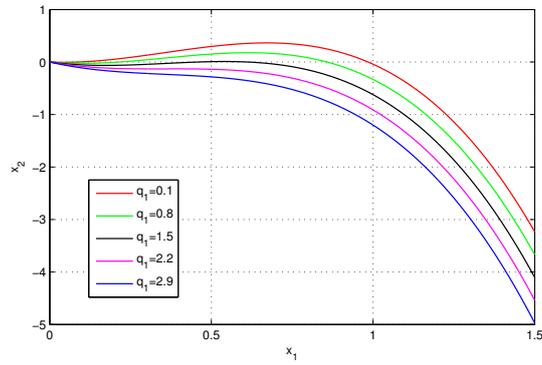


Figure 10: Behaviour of the equilibrium curve for different q_1 .

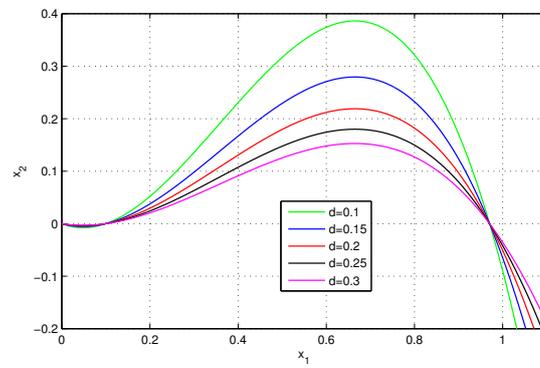


Figure 11: Effects of d on the equilibrium curve.

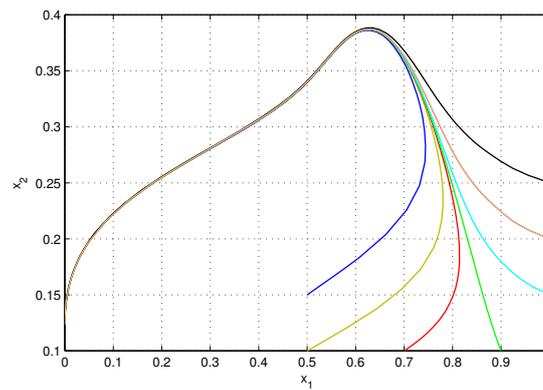


Figure 12: Phaseportrait of the system (5) for different initial condition when $E = 0.1$.

(0.5632, 0.3477). But from somewhere after the stable point the curves go through a particular path. This level of harvesting strategy may be acceptable for balanced ecology.

The phase-space trajectories corresponding to $E = 0.2$ and $E = 0.3$ are depicted through Figure 13 and Figure 14, respectively.

From Figure 13, we observe that all trajectories converge to the stable equilibrium point (0.627, 0.289)

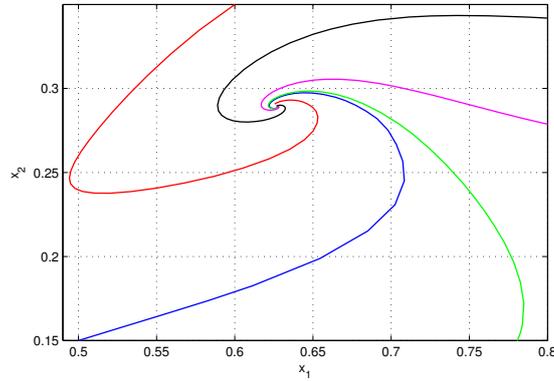


Figure 13: Phase portrait for system (5) when $E = 0.2$.

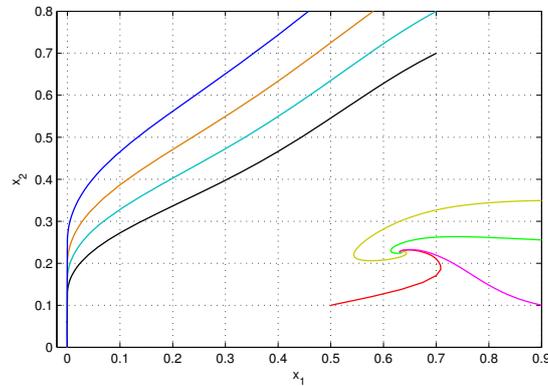


Figure 14: Phase portrait for system (5) when $E = 0.3$.

starting with different initial conditions. Therefore, the harvesting strategy of this situation is acceptable also. Figure 14 depicts the stable and unstable behavior of our proposed model system. The stable and unstable behaviour (Figure 14) of our model system demands the replanning of the harvesting policy for $E = 0.3$.

The phase-space trajectories corresponding to $E = 0.4$ is presented in Figure 15. Also, the time series plot of our system for different harvesting efforts based on the parameter values as $m = 0.07$, $\alpha = 0.25$, $\beta = 0.11$, $\delta = 0.118$, $q_1 = 0.19$, $q_2 = 0.22$ and initial condition as $(x_1(0), x_2(0)) = (0.35, 0.1)$ is depicted in Figure 16.

From Figure 15 we observe that the stability differs for a little change of parameter value. Therefore, modification in harvesting policy is also needed for $E = 0.4$. Figure 16 depicts that the populations behave differently with different harvesting efforts. Consequently, we must have to follow the optimal harvesting strategy for balancing ecology.

10. Discussion and Concluding Remarks

Modelling of the prey-predator system has undergone different developments in Biomathematics. Although, for balance in nature, ecology demands more and more constructive steps regarding this. Herd behaviour and Allee effect are natural phenomena of species. But, for any ecosystem, harvesting is a significant threat commonly created by human beings. So, our theorems regarding the threat of extinction

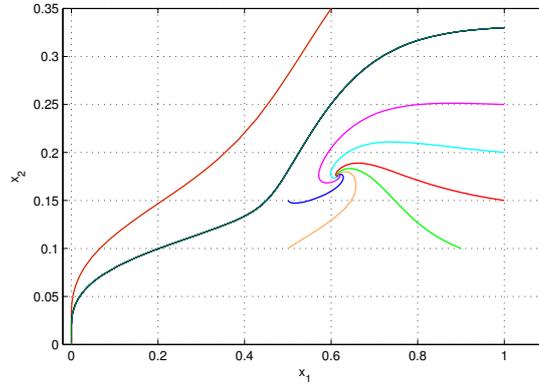


Figure 15: Different behaviour of phaseportrait for $E = 0.4$.

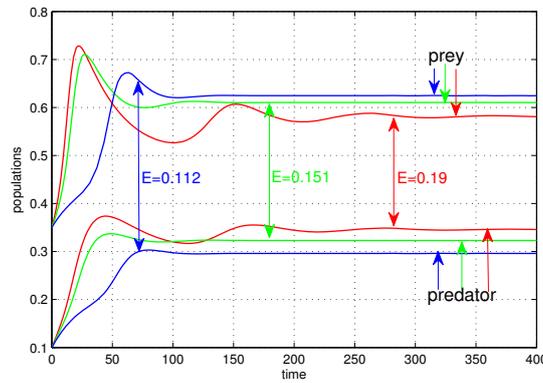


Figure 16: Behaviour of the populations with different harvesting efforts.

for prey and predator species may help ecological balance in nature. Giving importance to harvesting, we obtained policy for optimal harvesting to maintain the sustainable development of ecosystems. Our numerical analysis with a graphical view relates this study to biological importance. Sensitivity analysis can help ecologists regarding ecological balance. This study observed that the Allee effect and harvesting together are threatening factors for prey populations. Harvesting on predator species also plays a vital role in the extinction of population density.

As a consequence, we can say that our considered environment of ecosystem needs prudent initiatives. Otherwise, the ecology may be unbalanced. So, one can use our paper’s methods and results to form patterns in the prey-predator modelling, and hence we hope that our present model can help in balancing ecology. Moreover, this paper initiates future work with time delay, stochastic model, different three species models, and so on. But more complex model demands further study in this field. Comparative study between ratio-dependent and prey-dependent models can be developed with a different point of view. New models in a different environment on prey-predator systems can be formed and analysed in this way.

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References

- [1] J. Segura, F. M. Hilker, D. Franco, Adaptive threshold harvesting and the suppression of transients, *J. Theor. Biol.*, 395 (2016) 103-114.
- [2] A. Cuspilici, P. Monforte, M. A. Ragusa, Study of Saharan dust influence on PM10 measures in Sicily from 2013 to 2015, *Ecological Indicators* 76 (2017) 297-303.
- [3] A. Duro, V. Piccione, M. A. Ragusa, V. Veneziano, New Environmentally Sensitive Patch Index- ESPI- for MEDALUS protocol, AIP conference Proceedings, Vol-1637 (2014) 305-312.
- [4] M.I.D.S. Costa, P. V. Esteves, L.D.B. Faria, L. Anjos, Prey dynamics under generalist predator culling in stage structured models, *Math. Biosci.*, 285 (2017) 68-74.
- [5] K. Sarkar, S. Khajanchi, Impact of fear effect on the growth of prey in a predator-prey interaction model, *Ecol. Complex.*, 42, 100826, 2020.
- [6] H. Kim, J. H. Choi, Exact Solutions of a Diffusive Predator–Prey System by the Generalized Riccati Equation, *B. Malays. Matht. Soi. So.*, 39 (2016) 1125-1143.
- [7] M. I. S. Costa, L. dos Anjos, Integrated Pest Management in a Predator-Prey System with Allee Effects, *Neotrop. Entomol.*, 44 (2015) 385-391.
- [8] Z. Hou, On permanence of Lotka-Volterra systems with delays and variable intrinsic growth rates. *Nonlinear Anal.*, 14 (2013) 960-975.
- [9] P. A. Braza, The bifurcation structure of the Holling Tanner model for predator-prey interactions using two-timing, *SIAM. J. Appl. Math.*, 63 (3) (2003) 889-904.
- [10] S. B. Hsu, T. W. Hwang, Y. Kuang, Global analysis of the Michaelis-Menten-type ratio dependent predator-prey system, *J. Math. Biol.* 42 (2001) 490-506.
- [11] P. A. Abrams, L. R. Ginzburg, The nature of predation: prey dependent, ratio dependent or neither?, *Trends. Ecol. Evol.* 15 (2000) 337-341.
- [12] J. Choi, Y. J. Kim, Predator–prey equations with constant harvesting and planting, *J. Theor. Biol.*, 458 (2018) 47-57.
- [13] D. Pal, G. S. Mahapatra, G. P. Samanta, New approach for stability and bifurcation analysis on predator–prey harvesting model for interval biological parameters with time delays, *Comp. Appl. Math.*, 37 (2018) 3145-3171.
- [14] D. Pal, G. S. Mahapatra, G. P. Samanta, A study of bifurcation of prey–predator model with time delay and harvesting using fuzzy parameters, *J. Biol. Syst.*, 26 (2018) 339-372.
- [15] Y. Liu, T. Zhang, X. Liu, Investigating the interactions between Allee effect and harvesting behaviour of a single species model: An evolutionary dynamics approach, *Physica A*, 549, 124323, 2020.
- [16] C. Wei, J. Liu, L. Chen, Homoclinic bifurcation of a ratio-dependent predator–prey system with impulsive harvesting, *Nonlinear Dyn.*, 89 (2017) 2001-2012.
- [17] D. Pal, G.S. Mahapatra, A bioeconomic modeling of two-prey and one-predator fishery model with optimal harvesting policy through hybridization approach, *Appl. Math. Comput.*, 242 (2014) 748-763.
- [18] M. D. Asfaw, S. M. Kassa, E. M. Lungu, Co-existence thresholds in the dynamics of the plant-herbivore interaction with Allee effect and harvest, *Int. J. Biomath.*, 11, 1850057, 2018.
- [19] M. H. Holden, E. McDonald-Madden, High prices for rare species can drive large populations extinct: the anthropogenic Allee effect revisited, *J. Theor. Biol.*, 429 (2017) 170-180.
- [20] N. Martínez-Jeraldo, P. Aguirre, Allee effect acting on the prey species in a Leslie-Gower predation model, *Nonlinear Anal. Real World Appl.*, 45 (2019) 895-917.
- [21] L. Yang, S. Zhong, Dynamics of an impulsive diffusive ecological model with distributed delay and additive Allee effect, *J. Appl. Math. Comput.*, 48 (2015) 1-23.
- [22] C. Rebelo, C. Soaresina, Coexistence in seasonally varying predator–prey systems with Allee effect, *Nonlinear Anal. Real World Appl.*, 55, 103140, 2020.
- [23] C. Arancibia-Ibarra, The basins of attraction in a modified May–Holling–Tanner predator-prey model with Allee affect, *Nonlinear Anal. Theory Methods Appl.*, 185 (2019) 15-28.
- [24] M. Sen, M. Banerjee, A. Morozov, Bifurcation analysis of a ratio-dependent prey-predator model with the Allee effect, *Ecol. Complex*, 11 (2012) 12-27.
- [25] K. Manna, M. Banerjee, Stationary, non-stationary and invasive patterns for a prey-predator system with additive Allee effect in prey growth, *Ecol. Complex*, 36 (2018) 206-217.
- [26] C. Wei, L. Chen, Periodic solution and heteroclinic bifurcation in a predator-prey system with Allee effect and impulsive harvesting, *Nonlinear Dyn.*, 76(2) (2014) 1109-1117.
- [27] E. N. Bodine, A. E. Yust, Predator–prey dynamics with intraspecific competition and an Allee effect in the predator population, *Lett. Biomath.*, 4(1) (2017) 23-38.
- [28] S. Elaydi, E. Kwessi, G. Livadiotis, Hierarchical competition models with the Allee effect III: multispecies, *J. Biol. Dynam.*, 12(1) (2018) 271-287.
- [29] M. Sen, M. Banerjee, Y. Takeuchi, Influence of Allee effect in prey populations on the dynamics of two-prey-one-predator model, *Math. Biosci. Eng.*, 15(4) (2018) 883-904.
- [30] K. Du, R. Peng, N. Sun, The role of protection zone on species spreading governed by a reaction-diffusion model with strong Allee effect, *J. Differ. Equ.*, 266 (2019) 7327-7356.
- [31] H. M. Hastings, M. Radin, T. Wiandt, Fishing quotas, induced allee effect and fluctuation-driven extinction, *Nonlinear Dynamics Psychol Life Sci.*, 21 (2017) 1-17.
- [32] D. Wu, H. Zhao, Y. Yuan, Complex dynamics of a diffusive predator-prey model with strong Allee effect and threshold harvesting, *J. Math. Anal. Appl.*, 469 (2019) 982-1014.

- [33] C. Liu, N. Lu, Q. Zhang, Dynamical analysis in a hybrid bioeconomic system with multiple time delays and strong Allee effect, *Math. Comput. Simulat.*, 136 (2017) 104-131.
- [34] S. Biswas, D. Pal, G. S. Mahapatra, G. P. Samanta, Dynamics of a Prey-Predator system with herd behaviour in both and strong Allee effect in prey, *Biophysics*, 65(5) (2020) 826-835.
- [35] J. C. Marques, H. Malchow, L. A. D. Rodrigues, D. C. Mistro, The Effects of Stochasticity on Pattern Formation in a Space- and Time-Discrete Predator-Prey System with Strong Allee Effect in the Prey, *Bull. Math. Biol.*, 81 (2019) 1369-1393.
- [36] L. Berec, E. Angulo, F. Courchamp, Multiple Allee effects and population management, *Trends Ecol. Evol.*, 22 (2006) 185-191.
- [37] W. Ni, M. Wang, Dynamical properties of a Leslie-Gower prey-predator model with strong Allee effect in prey, *Discrete and Continuous Dynamical Systems - Series B*, 22(9) (2017) 3409-3420.
- [38] F. Rao, Y. Kang, The complex dynamics of a diffusive prey-predator model with an Allee effect in prey, *Ecol. Complex*, 28 (2016) 123-144.
- [39] W. Ni, M. Wang, Dynamics and patterns of a diffusive Leslie-Gower prey-predator model with strong Allee effect in prey, *J. Differ. Equ.*, 261(7) (2016) 4244-4274.
- [40] N. Min, M. Wang, Dynamics of a diffusive prey-predator system with strong allee effect growth rate and a protection zone for the prey, *Discrete Cont. Dyn.-B*, 23(4) (2018) 1721-1737.
- [41] E. González-Olivares, A. Rojas-Palma, Multiple Limit Cycles in a Gause type predator-prey model with Holling type III functional response and Allee effect on prey, *Bull. Math. Biol.*, 73 (2011) 1378-1397.
- [42] X. Lai, S. Liu, R. Lin, Rich dynamical behaviours for predator-prey model with weak Allee effect, *Applicable Analysis*, 89(8) (2010) 1271-1292.
- [43] T. Yu, Y. Tian H. Guo, X. Song, Dynamical analysis of an integrated pest management predator-prey model with weak Allee effect, *J. Biol. Dynam.*, 13(1) (2019) 218-244.
- [44] S.K. Sasmal, J. Chattopadhyay, An eco-epidemiological system with infected prey and predator subject to the weak Allee effect, *Math. Biosci.*, 246(2) (2013) 260-271.
- [45] D. Jana, E. M. Elsayed, Interplay between strong Allee effect, harvesting and hydra effect of a single population discrete-time system, *Int. J. Biomath.*, 9, 1650004, 2016.
- [46] V. Rivera, P. Aguirre, Study of a Tritrophic Food Chain Model with Non-differentiable Functional Response, *Acta Appl. Math.*, 165 (2020) 19-43.
- [47] G. Buffoni, M. Groppi, C. Soresina, Dynamics of predator-prey models with a strong Allee effect on the prey and predator-dependent trophic functions, *Nonlinear Anal. Real World Appl.*, 30 (2016) 143-169.
- [48] K. Ryu, W. Ko, M. Haque, Bifurcation analysis in a predator-prey system with a functional response increasing in both predator and prey densities, *Nonlinear Dyn.*, 94 (2018) 1639-1656.
- [49] D. Indrajaya, A. Suryanto, A. R. Alghofari, Dynamics of modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response and additive Allee effect, *Int. J. Ecol. Dev.*, 31 (2016) 60-71.
- [50] E. González-Olivares, L. M. Gallego-Berrió, B. González-Yañez, A. Rojas-Palma, Consequences of weak Allee effect on prey in the May-Holling-Tanner predator-prey model, *Math. Methods Appl. Sci.*, 38 (2015) 5183-5196.
- [51] M. Anacleto, C. Vidal, Dynamics of a delayed predator-prey model with Allee effect and Holling type II functional response, *Math. Method Appl. Sci.*, 43 (2020) 5708-5728.
- [52] C. Cosner, D. L. DeAngelis, J. S. Ault, D. E. Olson, Effects of spatial grouping on the functional response of predators, *Theor. pop. Biol.*, 56 (1999) 65-75.
- [53] J. Chattopadhyay, S. Chatterjee, E. Venturino, Patchy agglomeration as a transition from monospecies to recurrent plankton blooms, *J. Theor. Biol.*, 253 (2008) 289-295.
- [54] V. Ajraldi, M. Pittavino, E. Venturino, Modelling herd behaviour in population systems, *Nonlinear Anal. RWA*, 12 (2011) 2319-2338.
- [55] B. W. Kooi, E. Venturino, Ecoepidemic predator-prey model with feeding satiation, prey herd behavior and abandoned infected prey, *Math. Biosci.*, 274 (2016) 58-72.
- [56] G.S. Mahapatra, P. Santra, Prey-predator model for Optimal Harvesting with Functional Response Incorporating Prey Refuge, *Int. J. Biomath.*, 09, ID1650014, 2016.
- [57] N. Min, M. Wang, Hopf bifurcation and steady-state bifurcation for a leslie-gower prey-predator model with strong allee effect in prey, *Discrete Contin. Dyn.-A*, 39(2) (2019) 1071-1099.
- [58] J. Banerjee, S.K. Sasmal, R.K. Layek, Supercritical and subcritical Hopf-bifurcations in a two-delayed prey-predator system with density-dependent mortality of predator and strong Allee effect in prey, *Biosyst.*, 180 (2019) 19-37.
- [59] D. Pal, S. Biswas, G. S. Mahapatra, G. P. Samanta, A prey-predator dynamics with square root functional responses and strong Allee effect, *J. Appl. Nonlinear Dyn.*, 8 (3) (2019) 419-433.
- [60] P.J. Pal, T. Saha, Qualitative analysis of a predator-prey system with double Allee effect in prey, *Chaos Solitons Fract.*, 73 (2015) 36-63.
- [61] D. L. Ragoin, G. Brown, Harvest policies and non-market valuation in a predator-prey system, *J. Environ. Econ. Manag.* 12 (1985) 155-168.