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Existence result for fractional q-difference equations on the half-line

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Abstract. In this paper, we obtain an existence result for the integral boundary value problems of nonlinear fractional q-difference equations on the half-line using Schauder's fixed point theorem.

1. The first section

Quantum calculus, roughly speaking, is ordinary calculus without limits. By doing so, we enlarge the class of functions that we deal with, by considering nondifferentiable functions and apply quantum derivatives on them. There are several types of quantum calculus such as h-calculus which is known as the calculus of finite differences, the q-calculus and Hahn's calculus. In this study we are concerned with the q-calculus. The q-difference calculus were first developed by Jackson [1], while basic definitions and properties can be found in the papers [4,5] and the book [15].

The q-difference equations have received a considerable interest of many mathematicians from many aspects, theoretical and practical. Specifically, q-difference equations have been widely used in mathematical physical problems, dynamical system and quantum models [19], q-analogues of mathematical physical problems including heat and wave equations [23], sampling theory of signal analysis [24, 25].

In recent years, some boundary value problems with fractional q-differences have begun to be addressed by many authors [2,3, 6-14, 16-18, 20-22, 26-29, 32-34]. They obtained many results as regards the existence and multiplicity of nontrivial solutions, positive solutions, negative solutions and extremal solutions by applying some well-known tools of fixed point theory such as the Banach contraction principle, the Guo–Krasnosel'skii fixed point theorem on cones, monotone iterative methods and Leray–Schauder degree theory.

In 2010, Su and Zhang [30] considered the following fractional boundary value problem:

$$D^{\alpha}_{0^+}u(t) = f(t, u(t), D^{\alpha-1}_{0^+}u(t)), \qquad t \in J := [0, \infty),$$

$$u(0) = 0, \quad D^{\alpha-1}_{0^+}u(\infty) = u_\infty, \quad u_\infty \in \mathbb{R},$$

where $1 < \alpha \leq 2$, $f \in C(J\times\mathbb{R}\times\mathbb{R}, \mathbb{R})$, $D^{\alpha}_{0^+}$ and $D^{\alpha-1}_{0^+}$ are the standard Riemann-Liouville fractional derivatives and $D^{\alpha-1}_{0^+}u(\infty) := \lim_{t\to+\infty} D^{\alpha-1}_{0^+}u(t)$. By using Schauder's fixed point theorem, they proved the existence result of an unbounded solution for the problem.

In this paper we will consider the following boundary value problem of fractional q-difference equations

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$$D_q^\alpha u(t) + f(t, u(t), D_q^{\alpha-1} u(t)) = 0, \qquad 0 < t < \infty, \qquad (1.1)$$

$$u(0) = 0, \quad D_q^{\alpha-1} u(\infty) = \gamma D_q^\beta u(\eta) \qquad (1.2)$$

where D_q^α is a fractional q-derivative of Riemann-Liouville type, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 < q < 1$.

The paper is organized as follows. In Section 2 , we introduce some definitions of q-fractional integral and differential operator together with some basic properties and lemmas to prove our main results. In Section 3, we investigate the existence of positive solutions for boundary value problem (1.1)-(1.2) by a fixed point theorem in cones. Moreover, an example is given to illustrate our main result.

2. The second section

In this section, we list some useful definitions and preliminaries, which are useful for the proof of the main results.

Definition 2.1 [5] Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The fractional q-integral of the Riemann-Liouville type is

$$(I_q^0 f)(x) = f(x)$$

and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t, \qquad x \in [0, 1]$$

where

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}.$$

Definition 2.2 [18] The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by

$$(D_q^\alpha f)(x) = f(x)$$

and

$$(D_q^\alpha f)(x) = \left(D_q^p I_q^{p-\alpha} f\right)(x), \quad \alpha > 0,$$

where p is the smallest integer greater than or equal to α.

Lemma 2.1 [3] *Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds.*

$$\left(I_q^\alpha D_q^p f\right)(x) = \left(D_q^p I_q^\alpha f\right)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha + k - p + 1)} (D_q^k f)(0).$$

Remark 2.1 *It is well known that* $(I_{a^+}^\alpha f)(a) = 0$ *for* $f(t) \in C[a, b]$, $\alpha > 0$ *and* $I_{a^+}^\alpha : C[a, b] \to C[a, b]$ *for* $\alpha > 0$. The next result is important to prove our main result.

Theorem 2.1 (Arzela-Ascoli Theorem) *For $A \subset C[a, b]$. A is compact if and only if A is closed, bounded and equicontinuous.*

Lemma 2.2 *Let $h : [0,\infty] \to [0, \infty)$ be a given continuous function. Then the boundary value problem*

$$D_q^{\alpha} u(t) + h(t) = 0, \quad 0 < t < \infty, \qquad (2.1)$$

$$u(0) = 0, \quad D_q^{\alpha-1} u(\infty) = \gamma D_q^{\beta} u(\eta) \qquad (2.2)$$

where D_q^{α} is a fractional q-derivative of Riemann-Liouville type, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 < q < 1$, is equivalent to

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s + \frac{t^{\alpha-1}}{\Delta \Gamma_q(\alpha)} \left\{ -\frac{\gamma}{\Gamma_q(\alpha-\beta)} \int_0^{\eta} (\eta-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \int_0^{\infty} h(s) d_q s \right\} \\ &= \int_0^{\infty} G(t,qs) h(s) d_q s \end{aligned}$$

where

$$G(t,qs) = \begin{cases} G_1(t,qs) , & t < \eta \\ G_2(t,qs) , & \eta < t, \end{cases}$$

$$G_1(t,qs) = \begin{cases} -\frac{1}{\Gamma_q(\alpha)} (t-qs)^{(\alpha-1)} + \left(1 - \frac{\gamma}{\Gamma_q(\alpha-\beta)} (\eta-qs)^{(\alpha-\beta-1)}\right) \frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & , s < t \\ \left(1 - \frac{\gamma}{\Gamma_q(\alpha-\beta)} (\eta-qs)^{(\alpha-\beta-1)}\right) \frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & , t < s < \eta \\ \frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & , s > \eta \end{cases}$$

and

$$G_2(t,qs) = \begin{cases} -\frac{1}{\Gamma_q(\alpha)} (t-qs)^{(\alpha-1)} + \left(1 - \frac{\gamma}{\Gamma_q(\alpha-\beta)} (\eta-qs)^{(\alpha-\beta-1)}\right) \frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & , s < \eta \\ -\frac{1}{\Gamma_q(\alpha)} (t-qs)^{(\alpha-1)} + \frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & , \eta < s < t \\ \frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & , s > t \end{cases}$$

such that

$$\Delta = 1 - \frac{\gamma}{\Gamma_q(\alpha-\beta)} \eta^{\alpha-\beta-1} \neq 0$$

Proof: From *Lemma 2.1*, we have

$$u(t) = -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

Since $u(0) = 0$ we get $c_2 = 0$.
Thus, we have

$$u(t) = -I_q^{\alpha} h(t) + c_1 t^{\alpha-1}.$$

Since

$$\begin{aligned} D_q^{\beta} u(t) &= -I_q^{\alpha-\beta} h(t) + c_1 \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)} t^{\alpha-\beta-1} \\ &= -\frac{1}{\Gamma_q(\alpha-\beta)} \int_0^t (t-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + c_1 \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)} t^{\alpha-\beta-1} \end{aligned}$$

and

$$D_q^{\alpha-1} u(t) = -\frac{1}{\Gamma_q(1)} \int_0^t h(s)\, d_q s + c_1 \frac{\Gamma_q(\alpha)}{\Gamma_q(1)} = -\int_0^t h(s)\, d_q s + c_1 \Gamma_q(\alpha),$$

we get

$$-\int_0^{\infty} h(s)\, d_q s + c_1 \Gamma_q(\alpha) = \gamma \left\{ -\frac{1}{\Gamma_q(\alpha-\beta)} \int_0^{\eta} (\eta-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + c_1 \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)} \eta^{\alpha-\beta-1} \right\},$$

using the second boundary condition. Thus we get

$$c_1 \left\{ \Gamma_q(\alpha) - \gamma \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)} \eta^{\alpha-\beta-1} \right\} = -\frac{\gamma}{\Gamma_q(\alpha-\beta)} \int_0^{\eta} (\eta-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \int_0^{\infty} h(s) d_q s.$$

If we define

$$\Delta := 1 - \frac{\gamma}{\Gamma_q(\alpha-\beta)} \eta^{\alpha-\beta-1},$$

we have

$$c_1 = \frac{1}{\Delta \Gamma_q(\alpha)} \left\{ -\frac{\gamma}{\Gamma_q(\alpha-\beta)} \int_0^{\eta} (\eta-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \int_0^{\infty} h(s) d_q s \right\}.$$

So, we can get

$$u(t) = -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s + \frac{t^{\alpha-1}}{\Delta \Gamma_q(\alpha)} \left\{ -\frac{\gamma}{\Gamma_q(\alpha-\beta)} \int_0^{\eta} (\eta-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \int_0^{\infty} h(s) d_q s \right\}.$$

If $t < \eta$, we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s \\ &+ \frac{t^{\alpha-1}}{\Delta \Gamma_q(\alpha)} \left\{ -\frac{\gamma}{\Gamma_q(\alpha-\beta)} \left[\int_0^t (\eta-qs)^{(\alpha-\beta-1)} h(s)\, d_q s + \int_t^{\eta} (\eta-qs)^{(\alpha-\beta-1)} h(s)\, d_q s \right] \right. \\ &+ \left. \int_0^t h(s)\, d_q s + \int_t^{\eta} h(s) d_q s + \int_{\eta}^{\infty} h(s) d_q s \right\} \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^t \left[-(t-qs)^{(\alpha-1)} - \frac{\gamma t^{\alpha-1}}{\Delta \Gamma_q(\alpha-\beta)} (\eta-qs)^{(\alpha-\beta-1)} + \frac{t^{\alpha-1}}{\Delta} \right] h(s)\, d_q s \\ &+ \frac{t^{\alpha-1}}{\Delta \Gamma_q(\alpha)} \int_t^{\eta} \left[1 - \frac{\gamma}{\Gamma_q(\alpha-\beta)} (\eta-qs)^{(\alpha-\beta-1)} \right] h(s)\, d_q s + \frac{t^{\alpha-1}}{\Delta \Gamma_q(\alpha)} \int_{\eta}^{\infty} h(s) d_q s \\ &= \int_0^{\infty} G_1(t, qs) h(s) d_q s, \quad \text{that} \end{aligned}$$

$$G_1(t,qs)=\begin{cases} -\frac{1}{\Gamma_q(\alpha)}(t-qs)^{(\alpha-1)}+\left(1-\frac{\gamma}{\Gamma_q(\alpha-\beta)}(\eta-qs)^{(\alpha-\beta-1)}\right)\frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & ,\ s<t \\ \left(1-\frac{\gamma}{\Gamma_q(\alpha-\beta)}(\eta-qs)^{(\alpha-\beta-1)}\right)\frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & ,\ t<s<\eta \\ \frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & ,\ s>\eta. \end{cases}$$

If $\eta<t$, we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma_q(\alpha)}\left\{\int_0^\eta (t-qs)^{(\alpha-1)}h(s)\,d_qs+\int_\eta^t (t-qs)^{(\alpha-1)}h(s)\,d_qs\right\} \\ &+ \frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)}\left\{\int_0^\eta h(s)\,d_qs+\int_\eta^t h(s)\,d_qs+\int_t^\infty h(s)\,d_qs\right\}-\frac{\gamma t^{\alpha-1}}{\Delta\Gamma_q(\alpha)\Gamma_q(\alpha-\beta)}\int_0^\eta (\eta-qs)^{(\alpha-\beta-1)}h(s)\,d_qs \\ &= \int_0^\eta\left[-\frac{1}{\Gamma_q(\alpha)}(t-qs)^{(\alpha-1)}+\frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)}-\frac{\gamma t^{\alpha-1}}{\Delta\Gamma_q(\alpha)\Gamma_q(\alpha-\beta)}(\eta-qs)^{(\alpha-\beta-1)}\right]h(s)\,d_qs \\ &+ \int_\eta^t\left[-\frac{1}{\Gamma_q(\alpha)}(t-qs)^{(\alpha-1)}+\frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)}\right]h(s)\,d_qs+\int_t^\infty\frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)}h(s)\,d_qs \\ &= \int_0^\infty G_2(t,qs)h(s)d_qs,\ \text{ that} \end{aligned}$$

$$G_2(t,qs)=\begin{cases} -\frac{1}{\Gamma_q(\alpha)}(t-qs)^{(\alpha-1)}+\left(1-\frac{\gamma}{\Gamma_q(\alpha-\beta)}(\eta-qs)^{(\alpha-\beta-1)}\right)\frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & ,\ s<\eta \\ -\frac{1}{\Gamma_q(\alpha)}(t-qs)^{(\alpha-1)}+\frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & ,\ \eta<s<t \\ \frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)} & ,\ s>t. \end{cases}$$

□

Define the space

$$X=\left\{u(t)\in C(J,\mathbb{R}):\sup_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}}<+\infty\right\}$$

with the norm $\|u\|_X=\sup_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}}$ and the space

$$Y=\left\{u(t)\in X:D_q^{\alpha-1}u(t)\in C(J,\mathbb{R}),\sup_{t\in J}\left|D_q^{\alpha-1}u(t)\right|<+\infty\right\}$$

with the norm $\|u\|_Y=\max\left\{\sup_{t\in J}\frac{|u(t)|}{1+t^{\alpha-1}}\ ,\ \sup_{t\in J}\left|D_q^{\alpha-1}u(t)\right|\ \right\}$.

Lemma 2.3 *$(X, \|.\|_X)$ is a Banach space.*

Proof: Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(X, \|.\|_X)$, then $\forall \varepsilon > 0, \exists N > 0$ such that

$$\left|\frac{u_n(t)}{1+t^{\alpha-1}} - \frac{u_m(t)}{1+t^{\alpha-1}}\right| < \varepsilon$$

for any $t \in J$ and $n, m > N$. Thus, $\{u_n\}_{n=1}^{\infty}$ converges uniformly to a function $\frac{v(t)}{1+t^{\alpha-1}}$ and we can verify easily that $v(t) \in X$. Then $(X, \|.\|_X)$ is a Banach space. □

Lemma 2.4 *$(Y, \|.\|_Y)$ is a Banach space.*

Proof: Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(Y, \|.\|_Y)$, then $\{u_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $(X, \|.\|_X)$.

Thus

$$\lim_{n\to\infty} \frac{u_n(t)}{1+t^{\alpha-1}} = \frac{u(t)}{1+t^{\alpha-1}}$$

and

$$u(t) \in X.$$

Moreover

$$\lim_{n\to\infty} D_q^{\alpha-1} u_n = v(t) \quad \text{and} \quad \sup_{t\in J} \left|D_q^{\alpha-1} u(t)\right| < +\infty.$$

Next we need to ensure that $v(t) = D_q^{\alpha-1} u(t)$.

In view of Lebesgue's Dominated Convergence Theorem and uniform convergence of $\left\{D_q^{\alpha-1} u_n(t)\right\}_{n=1}^{\infty}$, there exists a positive constant $M > 0$ such that $\frac{|u_n(t)|}{1+t^{\alpha-1}} \le M, \quad n = 1, 2, \ldots$

Then

$$v(t) = \lim_{n\to\infty} D_q^{\alpha-1} u_n(t) = \frac{1}{\Gamma_q(2-\alpha)} \lim_{n\to\infty} \frac{d}{dt} \int_0^t (t-qs)^{(1-\alpha)} u_n(s) d_q s$$

together with

$$\int_0^t (t-qs)^{(1-\alpha)} \left(1+s^{\alpha-1}\right) \frac{u_n(s)}{1+s^{\alpha-1}} d_q s \le M \int_0^t (t-qs)^{(1-\alpha)} \left(1+s^{\alpha-1}\right) d_q s$$

$$= M \left[\int_0^1 t^{2-\alpha} (1-qu)^{(1-\alpha)} d_q u + t \int_0^1 u^{\alpha-1} (1-qu)^{(1-\alpha)} d_q u\right].$$

The q-beta function defined by the usual formula

$$B_q(t, s) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}$$

has the q-integral representation, which is a q-analogue of Euler's formula:

$$B_q(t, s) = \int_0^1 u^{t-1} (1-qu)^{(s-1)} d_q u, \qquad t, s > 0.$$

Thus we get

$$M\left[t^{2-\alpha}\int_0^1 (1-qu)^{(1-\alpha)}d_qu + t\int_0^1 u^{\alpha-1}(1-qu)^{(1-\alpha)}d_qu\right] \le Mt^{2-\alpha} + B_q(\alpha, 2-\alpha)Mt$$

that ensures $v(t) = D_q^{\alpha-1}u(t)$. Thus $(Y, \|.\|_Y)$ is a Banach space. □

Because the Arzela-Ascoli Theorem fails to work in Y, we need a modified compactness criterion to prove the compactness of the operator.

Lemma 2.5 *Let $Z \subseteq Y$ be a bounded set. Then Z is relatively compact in Y if the following conditions hold:*
(1) *For any $u(t) \in Z$, $\frac{u(t)}{1+t^{\alpha-1}}$ and $D_q^{\alpha-1}u(t)$ are equicontinuous on any compact interval of J.*

(2) *Given $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that*

$$\left|\frac{u(t_1)}{1+{t_1}^{\alpha-1}} - \frac{u(t_2)}{1+{t_2}^{\alpha-1}}\right| < \varepsilon$$

and

$$\left|D_q^{\alpha-1}u(t_1) - D_q^{\alpha-1}u(t_2)\right| < \varepsilon$$

for any $t_1, t_2 \ge T$ and $u(t) \in Z$.

Proof: We need to prove that Z is totally bounded. First we consider the case $t \in [0, T]$.

Define

$$Z_{[0,T]} = \{u(t) : u(t) \in Z, t \in [0, T]\}.$$

Then clearly, $Z_{[0,T]}$ with the norm

$$\|u\|_\infty = \sup_{t\in[0,T]} \frac{|u(t)|}{1+t^{\alpha-1}}$$

is a Banach space in our problem $\alpha \in (1, 2]$. Then condition *(1)* combined with the Arzela-Ascoli theorem indicates that $Z_{[0,T]}$ is relatively compact hence $Z_{[0,T]}$ is totally bounded, namely, for any $\varepsilon > 0$, there exist finitely many balls $B_\varepsilon(u_i)$ such that

$$Z_{[0,T]} \subset \bigcup_{i=1}^{n} B_\varepsilon(u_i)$$

where

$$B_\varepsilon(u_i) = \left\{u(t) \in Z_{[0,T]} : \|u - u_i\|_\infty = \sup_{t\in[0,T]}\left|\frac{u(t)}{1+t^{\alpha-1}} - \frac{u_i(t)}{1+t^{\alpha-1}}\right| < \varepsilon\right\}.$$

Similarly the space $Z_{[0,T]}^{\alpha-1} = \left\{D_q^{\alpha-1}u(t) : u(t) \in Z_{[0,T]}\right\}$ with the norm $\left\|D_q^{\alpha-1}u\right\|_\infty = \sup_{t\in[0,T]}\left|D_q^{\alpha-1}u(t)\right|$ is a Banach space and can be covered by finitely many balls $B_\varepsilon\left(D_q^{\alpha-1}v_j\right)$, that is

$$Z_{[0,T]}^{\alpha-1} \subset \bigcup_{j=1}^{m} B_\varepsilon(D_q^{\alpha-1}v_j), v_j \in Z_{[0,T]}$$

where

$$B_\varepsilon\left(D_q^{\alpha-1}v_j\right)=\left\{D_q^{\alpha-1}u\left(t\right)\in Z_{[0,T]}^{\alpha-1}:\left\|D_q^{\alpha-1}u-D_q^{\alpha-1}v_j\right\|_\infty=\sup_{t\in[0,T]}\left|D_q^{\alpha-1}u\left(t\right)-D_q^{\alpha-1}v_j\left(t\right)\right|<\varepsilon\right\}.$$

Next we define

$$Z_{ij}=\left\{u\left(t\right)\in Z:u_{[0,T]}\in B_\varepsilon\left(u_i\right),D_q^{\alpha-1}\,u_{[0,T]}\in B_\varepsilon\left(D_q^{\alpha-1}v_j\right)\right\}.$$

It is clear that

$$Z_{[0,T]}\subset\bigcup_{\substack{1\le i\le n\\1\le j\le m}}Z_{ij[0,T]},$$

because of the definition of the set Z_{ij} and the relation $Z_{[0,T]}\subset\bigcup_{i=1}^n B_\varepsilon(u_i)$.

Now we take $u_{ij}\in Z_{ij}$ then Z can be covered by the balls $B_{4\varepsilon}\left(u_{ij}\right),\ i=1,2,\ldots,n,\ \ j=1,2,\ldots,m$ where

$$B_{4\varepsilon}\left(u_{ij}\right)=\left\{u\left(t\right)\in Z:\ \left\|u-u_{ij}\right\|_Y<4\varepsilon\right\}.$$

In fact, for $t\in[0,T]$,

$$\left|\frac{u\left(t\right)}{1+t^{\alpha-1}}-\frac{u_{ij}\left(t\right)}{1+t^{\alpha-1}}\right|\le\left|\frac{u\left(t\right)}{1+t^{\alpha-1}}-\frac{u_i\left(t\right)}{1+t^{\alpha-1}}\right|+\left|\frac{u_i\left(t\right)}{1+t^{\alpha-1}}-\frac{u_{ij}\left(t\right)}{1+t^{\alpha-1}}\right|<2\varepsilon$$

and

$$\left|D_q^{\alpha-1}u\left(t\right)-D_q^{\alpha-1}u_{ij}\left(t\right)\right|\le\left|D_q^{\alpha-1}u\left(t\right)-D_q^{\alpha-1}v_j\left(t\right)\right|+\left|D_q^{\alpha-1}v_j\left(t\right)-D_q^{\alpha-1}u_{ij}\left(t\right)\right|<2\varepsilon$$

For $t\in[T,+\infty)$, we have

$$\begin{aligned}\left|\frac{u\left(t\right)}{1+t^{\alpha-1}}-\frac{u_{ij}\left(t\right)}{1+t^{\alpha-1}}\right| &\le \left|\frac{u\left(t\right)}{1+t^{\alpha-1}}-\frac{u\left(T\right)}{1+T^{\alpha-1}}\right|+\left|\frac{u\left(T\right)}{1+T^{\alpha-1}}-\frac{u_{ij}\left(T\right)}{1+T^{\alpha-1}}\right|+\left|\frac{u_{ij}\left(T\right)}{1+T^{\alpha-1}}-\frac{u_{ij}\left(t\right)}{1+t^{\alpha-1}}\right|\\ &< \varepsilon+2\varepsilon+\varepsilon=4\varepsilon\end{aligned}$$

and

$$\begin{aligned}\left|D_q^{\alpha-1}u\left(t\right)-D_q^{\alpha-1}u_{ij}\left(t\right)\right| &\le \left|D_q^{\alpha-1}u\left(t\right)-D_q^{\alpha-1}u\left(T\right)\right|+\left|D_q^{\alpha-1}u\left(T\right)-D_q^{\alpha-1}u_{ij}\left(T\right)\right|\\ &+ \left|D_q^{\alpha-1}u_{ij}\left(T\right)-D_q^{\alpha-1}u_{ij}\left(t\right)\right|\\ &< \varepsilon+2\varepsilon+\varepsilon=4\varepsilon.\end{aligned}$$

These ensure that

$$\left\|u(t)-u_{ij}(t)\right\|_Y<4\varepsilon$$

Therefore, Z is totally bounded and *Lemma 2.5* is proved. □

3. Main Results

In this section, we prove the existence result of an unbounded solution for (1.1) − (1.2) and give an example illustrating the usefulness of our result. First, let us state a condition:
(H) there exist nonnegative functions $a(t)$, $b(t)$, $c(t) \in L^1(J)$ such that

$$\left|f(t,x,y)\right| \leq a(t)\left|x\right| + b(t)\left|y\right| + c(t)$$

and

$$\int_0^{+\infty}\left[\left(1+t^{\alpha-1}\right)a(t)+b(t)\right]d_qt < \frac{\Gamma_q(\alpha)}{k}$$

$$\int_0^{+\infty} c(t)d_qt < +\infty$$

where

$$k = 1 + \frac{1}{\Delta} + \frac{\gamma\eta^{\alpha-\beta-1}}{\Gamma_q(\alpha-\beta)}.$$

Theorem 3.1 *Assume that $f: J\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ is continuous and the condition* (H) *is satisfied. Then there exists at least one function $u(t)\in Y$ solving the problem* (1.1) − (1.2) *.*

Proof: Defining the operator A by

$$\begin{aligned} Au(t) = &-\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f\left(s,u(s),D_q^{\alpha-1}u(s)\right)d_qs \\ &+\frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)}\left\{-\frac{\gamma}{\Gamma_q(\alpha-\beta)}\int_0^\eta (\eta-qs)^{(\alpha-\beta-1)} f\left(s,u(s),D_q^{\alpha-1}u(s)\right)d_qs \right.\\ &\left.+\int_0^\infty f\left(s,u(s),D_q^{\alpha-1}u(s)\right)d_qs\right\}, \end{aligned} \tag{1.3}$$

we get

$$\begin{aligned} D_q^{\alpha-1}Au(t) = &\int_0^t f\left(s,u(s),D_q^{\alpha-1}u(s)\right)d_qs \\ &+\frac{1}{\Delta}\left\{-\frac{\gamma}{\Gamma_q(\alpha-\beta)}\int_0^\eta (\eta-qs)^{(\alpha-\beta-1)} f\left(s,u(s),D_q^{\alpha-1}u(s)\right)d_qs\right. \\ &\left.+\int_0^\infty f\left(s,u(s),D_q^{\alpha-1}u(s)\right)d_qs\right\}. \end{aligned} \tag{1.4}$$

Together with (1.3), (1.4), *Remark 2.1* and the continuity of f, we can conclude that $Au(t)$ and $D_q^{\alpha-1}Au(t)$ are continuous on J.

In what follows we divide the proof into several steps.

Step 1. Choose

$$R \geq \frac{k\int_0^\infty c(t)d_qt}{\Gamma_q(\alpha) - k\int_0^\infty\left[(1+t^{\alpha-1})a(t)+b(t)\right]d_qt}$$

and let

$$U = \{u(t) \in Y : \|u(t)\|_Y \leq R\}.$$

Then, $A : U \to U$.
Indeed, for any $u(t) \in U$, by (1.3), (1.4) and the condition (H), we get

$$\begin{aligned}
\frac{|Au(t)|}{1+t^{\alpha-1}} &\leq \frac{1}{1+t^{\alpha-1}} \left\{ \frac{1}{\Gamma_q(\alpha)} \int_0^\infty (t-qs)^{(\alpha-1)} \left| f\left(s, u(s), D_q^{\alpha-1}u(s)\right) \right| d_q s \right. \\
&+ \frac{t^{\alpha-1}}{\Delta \Gamma_q(\alpha)} \left\{ \frac{\gamma}{\Gamma_q(\alpha-\beta)} \int_0^\infty (\eta-qs)^{(\alpha-\beta-1)} \left| f\left(s, u(s), D_q^{\alpha-1}u(s)\right) \right| d_q s \right. \\
&\left.\left. + \int_0^\infty \left| f\left(s, u(s), D_q^{\alpha-1}u(s)\right) \right| d_q s \right\} \right\} \\
&\leq \frac{1}{\Gamma_q(\alpha)} \int_0^\infty \left| f\left(t, u(t), D_q^{\alpha-1}u(t)\right) \right| d_q t \\
&+ \frac{\gamma}{\Delta \Gamma_q(\alpha) \Gamma_q(\alpha-\beta)} \eta^{\alpha-\beta-1} \int_0^\infty \left| f\left(t, u(t), D_q^{\alpha-1}u(t)\right) \right| d_q t + \frac{1}{\Delta \Gamma_q(\alpha)} \int_0^\infty \left| f\left(t, u(t), D_q^{\alpha-1}u(t)\right) \right| d_q t \\
&\leq \frac{1}{\Gamma_q(\alpha)} \left(1 + \frac{1}{\Delta} + \frac{\gamma \eta^{\alpha-\beta-1}}{\Delta \Gamma_q(\alpha-\beta)} \right) \left\{ \|u\|_Y \int_0^{+\infty} \left[\left(1+t^{\alpha-1}\right) a(t) + b(t) \right] d_q t + \int_0^{+\infty} c(t) d_q t \right\} \leq R,
\end{aligned}$$

and

$$\begin{aligned}
\left| D_q^{\alpha-1} Au(t) \right| &= \left| \int_0^t f\left(s, u(s), D_q^{\alpha-1}u(s)\right) d_q s \right. \\
&+ \frac{1}{\Delta} \left\{ -\frac{\gamma}{\Gamma_q(\alpha-\beta)} \int_0^\eta (\eta-qs)^{(\alpha-\beta-1)} f\left(s, u(s), D_q^{\alpha-1}u(s)\right) d_q s \right. \\
&\left.\left. + \int_0^\infty f\left(s, u(s), D_q^{\alpha-1}u(s)\right) d_q s \right\} \right| \\
&\leq \int_0^\infty \left| f\left(t, u(t), D_q^{\alpha-1}u(t)\right) \right| d_q t + \frac{1}{\Delta} \int_0^\infty \left| f\left(t, u(t), D_q^{\alpha-1}u(t)\right) \right| d_q t \\
&+ \frac{\gamma}{\Delta \Gamma_q(\alpha-\beta)} \eta^{\alpha-\beta-1} \int_0^\infty \left| f\left(t, u(t), D_q^{\alpha-1}u(t)\right) \right| d_q t \\
&\leq \left(1 + \frac{1}{\Delta} + \frac{\gamma \eta^{\alpha-\beta-1}}{\Delta \Gamma_q(\alpha-\beta)} \right) \left\{ \|u\|_Y \int_0^{+\infty} \left[\left(1+t^{\alpha-1}\right) a(t) + b(t) \right] d_q t + \int_0^{+\infty} c(t) d_q t \right\} \\
&\leq \Gamma_q(\alpha) R \leq R,
\end{aligned}$$

since

$$\begin{aligned}
\left| f(t, u(t), D_q^{\alpha-1}u(t) \right| &\leq a(t) |u(t)| + b(t) \left| D_q^{\alpha-1}u(t) \right| + c(t) \\
&\leq a(t) \left(1+t^{\alpha-1}\right) \|u\|_Y + b(t) \|u\|_Y + c(t)
\end{aligned}$$

Hence, $\|Au(t)\|_Y \leq R$ and this shows that $A : U \to U$.

Step 2. Let V be a subset of U. We apply *Lemma 2.5* to verify that AV is relatively compact.

Let $I \subset J$ be a compact interval, $t_1, t_2 \in I$ and $t_1 < t_2$, then for any $u(t) \in V$, we have

$$\begin{aligned}
\left|\frac{Au(t_2)}{1+{t_2}^{\alpha-1}} - \frac{Au(t_1)}{1+{t_1}^{\alpha-1}}\right| \le & \frac{1}{\Gamma_q(\alpha)}\left|\int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{1+{t_2}^{\alpha-1}} f\left(s,u(s),D_q^{\alpha-1}u(s)\right) d_qs \right.\\
& \left. - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{1+{t_1}^{\alpha-1}} f\left(s,u(s),D_q^{\alpha-1}u(s)\right) \right|\\
& + \frac{1}{\Delta\Gamma_q(\alpha)}\left|\frac{t_2^{\alpha-1}}{1+{t_2}^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+{t_1}^{\alpha-1}}\right| \int_0^{\infty}\left|f\left(s,u(s),D_q^{\alpha-1}u(s)\right)\right| d_qs\\
& \le \frac{1}{\Gamma_q(\alpha)}\left|\int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{1+{t_2}^{\alpha-1}} f\left(s,u(s),D_q^{\alpha-1}u(s)\right) d_qs \right.\\
& \left. - \int_0^{t_1} \frac{(t_2-qs)^{(\alpha-1)}}{1+{t_2}^{\alpha-1}} f\left(s,u(s),D_q^{\alpha-1}u(s)\right) \right|\\
& + \frac{1}{\Gamma_q(\alpha)}\left|\int_0^{t_1} \frac{(t_2-qs)^{(\alpha-1)}}{1+{t_2}^{\alpha-1}} f\left(s,u(s),D_q^{\alpha-1}u(s)\right) d_qs \right.\\
& \left. - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{1+{t_1}^{\alpha-1}} f\left(s,u(s),D_q^{\alpha-1}u(s)\right) \right|\\
& + \frac{1}{\Delta\Gamma_q(\alpha)}\left|\frac{t_2^{\alpha-1}}{1+{t_2}^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+{t_1}^{\alpha-1}}\right| \int_0^{\infty}\left|f\left(s,u(s),D_q^{\alpha-1}u(s)\right)\right| d_qs\\
& \le \frac{1}{\Gamma_q(\alpha)}\int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{1+{t_2}^{\alpha-1}} \left|f\left(s,u(s),D_q^{\alpha-1}u(s)\right)\right| d_qs\\
& + \frac{1}{\Gamma_q(\alpha)}\int_0^{t_1} \left|\frac{(t_2-qs)^{(\alpha-1)}}{1+{t_2}^{\alpha-1}} - \frac{(t_1-qs)^{(\alpha-1)}}{1+{t_1}^{\alpha-1}}\right| \left|f\left(s,u(s),D_q^{\alpha-1}u(s)\right)\right| d_qs\\
& + \frac{1}{\Delta\Gamma_q(\alpha)}\left|\frac{t_2^{\alpha-1}}{1+{t_2}^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+{t_1}^{\alpha-1}}\right| \int_0^{\infty}\left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right| d_qt
\end{aligned}$$

and

$$\left|D_q^{\alpha-1}Au(t_2) - D_q^{\alpha-1}Au(t_1)\right| \le \left|\int_0^{t_2} f\left(s,u(s),D_q^{\alpha-1}u(s)\right) d_qs - \int_0^{t_1} f\left(s,u(s),D_q^{\alpha-1}u(s)\right) d_qs\right|$$

$$\le \int_{t_1}^{t_2} \left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right) d_qt\right|.$$

Note that for any $u(t) \in V$, we have $f\left(t,u(t),D_q^{\alpha-1}u(t)\right)$ is bounded on I. Then it is easy to see that $\frac{Au(t)}{1+t^{\alpha-1}}$ and $D_q^{\alpha-1}Au(t)$ are equicontinuous on I.

Next we show that for any $u(t) \in V$, $\frac{Au(t)}{1+t^{\alpha-1}}$ and $D_q^{\alpha-1}Au(t)$ satisfy the condition *(2)* of *Lemma 2.5.* Considering the condition (H), for given $\varepsilon > 0$, there exists a constant $L > 0$ such that

$$\int_L^{\infty} \left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right| d_qt < \varepsilon$$

On the other hand, since $\lim_{t\to\infty} \frac{t^{\alpha-1}}{1+t^{\alpha-1}} = 1$, there exists a constant $T_1 > 0$ such that for any $t_1, t_2 \ge T_1$.

$$\left|\frac{t_1^{\alpha-1}}{1+t_1{}^{\alpha-1}}-\frac{t_2^{\alpha-1}}{1+t_2{}^{\alpha-1}}\right| \le \left|1-\frac{t_1^{\alpha-1}}{1+t_1{}^{\alpha-1}}\right|+\left|1-\frac{t_2^{\alpha-1}}{1+t_2{}^{\alpha-1}}\right| < \varepsilon$$

Similarly, in view of $\lim_{t\to\infty}\frac{(t-L)^{\alpha-1}}{1+t^{\alpha-1}}=1$, there exists a constant $T_2 > L > 0$ such that for any $t_1, t_2 \ge T_2$ and $0 \le s \le L$,

$$\begin{aligned}\left|\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1{}^{\alpha-1}}-\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2{}^{\alpha-1}}\right| &\le \left[1-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1{}^{\alpha-1}}\right]+\left[1-\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2{}^{\alpha-1}}\right]\\ &\le \left[1-\frac{(t_1-L)^{(\alpha-1)}}{1+t_1{}^{\alpha-1}}\right]+\left[1-\frac{(t_2-L)^{(\alpha-1)}}{1+t_2{}^{\alpha-1}}\right] < \varepsilon\end{aligned}$$

Now choose $T > max\{T_1, T_2\}$, then for $t_1, t_2 \ge T$, we have

$$\begin{aligned}\left|\frac{Au(t_2)}{1+t_2{}^{\alpha-1}}-\frac{Au(t_1)}{1+t_1{}^{\alpha-1}}\right| &\le \frac{1}{\Gamma_q(\alpha)}\int_0^L\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2{}^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1{}^{\alpha-1}}\right|\left|f\left(s,u(s),D_q^{\alpha-1}u(s)\right)\right|d_qs\\ &+\frac{1}{\Gamma_q(\alpha)}\int_L^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2{}^{\alpha-1}}\left|f\left(s,u(s),D_q^{\alpha-1}u(s)\right)\right|d_qs\\ &+\frac{1}{\Gamma_q(\alpha)}\int_L^{t_1}\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2{}^{\alpha-1}}\left|f\left(s,u(s),D_q^{\alpha-1}u(s)\right)\right|d_qs\\ &+\frac{1}{\Delta\Gamma_q(\alpha)}\left|\frac{t_2^{\alpha-1}}{1+t_2{}^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1{}^{\alpha-1}}\right|\int_0^\infty\left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt\\ &\le \frac{\max_{t\in[0,L],u\in V}\left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|}{\Gamma_q(\alpha)}\int_0^L\left|\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2{}^{\alpha-1}}-\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1{}^{\alpha-1}}\right|d_qs\\ &+\frac{1}{\Gamma_q(\alpha)}\int_L^\infty\left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt+\frac{1}{\Gamma_q(\alpha)}\int_L^\infty\left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt\\ &+\frac{1}{\Delta\Gamma_q(\alpha)}\left|\frac{t_2^{\alpha-1}}{1+t_2{}^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1{}^{\alpha-1}}\right|\int_0^\infty\left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt\\ &\le \frac{\max_{t\in[0,L],u\in V}\left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|}{\Gamma_q(\alpha)}L\varepsilon+\frac{2\varepsilon}{\Gamma_q(\alpha)}+\frac{\varepsilon}{\Delta\Gamma_q(\alpha)}\int_0^\infty\left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt\end{aligned}$$

and

$$\left|D_q^{\alpha-1}Au(t_2)-D_q^{\alpha-1}Au(t_1)\right| \le \int_{t_1}^{t_2}\left|f\left(s,u(s),D_q^{\alpha-1}u(s)\right)d_qs\right| \le \int_L^\infty\left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right)d_qt\right| < \varepsilon$$

Consequently, *Lemma 2.5* yields that AV is relatively compact.

Step 3. $A : U \to U$ is a continuous operator. Let $u_n,\ u \in U,\ n = 1, 2, \dots$ and $\|u_n - u\|_Y \to 0$ as $n \to +\infty$. Then, we have

$$\left|\frac{Au_n(t)}{1+t^{\alpha-1}}-\frac{Au(t)}{1+t^{\alpha-1}}\right|\leq\frac{1}{\Gamma_q(\alpha)}\int_0^{\infty}\left|f\left(t,u_n(t),D_q^{\alpha-1}u_n(t)\right)-f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt$$
$$+\frac{1}{\Delta\Gamma_q(\alpha)}\int_0^{\infty}\left|f\left(t,u_n(t),D_q^{\alpha-1}u_n(t)\right)-f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt$$
$$+\frac{\gamma}{\Delta\Gamma_q(\alpha)\Gamma_q(\alpha-\beta)}\eta^{\alpha-\beta-1}\int_0^{\infty}\left|f\left(t,u_n(t),D_q^{\alpha-1}u_n(t)\right)-f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt$$
$$\leq\frac{1}{\Gamma_q(\alpha)}\left(1+\frac{1}{\Delta}+\frac{\gamma\eta^{\alpha-\beta-1}}{\Delta\Gamma_q(\alpha-\beta)}\right)\int_0^{\infty}\left|f\left(t,u_n(t),D_q^{\alpha-1}u_n(t)\right)-f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt$$
$$\leq\frac{k}{\Gamma_q(\alpha)}\int_0^{\infty}\left|f\left(t,u_n(t),D_q^{\alpha-1}u_n(t)\right)\right|d_qt+\frac{k}{\Gamma_q(\alpha)}\int_0^{\infty}\left|f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt$$

$$\leq\frac{k}{\Gamma_q(\alpha)}\left\{\|u_n\|_Y\int_0^{+\infty}\left[\left(1+t^{\alpha-1}\right)a(t)+b(t)\right]d_qt+\int_0^{+\infty}c(t)d_qt\right\}$$
$$+\frac{k}{\Gamma_q(\alpha)}\left\{\|u\|_Y\int_0^{+\infty}\left[\left(1+t^{\alpha-1}\right)a(t)+b(t)\right]d_qt+\int_0^{+\infty}c(t)d_qt\right\}$$
$$\leq\frac{2kR}{\Gamma_q(\alpha)}\int_0^{+\infty}\left[\left(1+t^{\alpha-1}\right)a(t)+b(t)\right]d_qt+\frac{2k}{\Gamma_q(\alpha)}\int_0^{+\infty}c(t)d_qt$$

and

$$\left|D_q^{\alpha-1}Au_n(t)-D_q^{\alpha-1}Au(t)\right|\leq\int_0^{\infty}\left|f\left(t,u_n(t),D_q^{\alpha-1}u_n(t)\right)-f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt$$
$$+\frac{1}{\Delta}\int_0^{\infty}\left|f\left(t,u_n(t),D_q^{\alpha-1}u_n(t)\right)-f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt$$
$$+\frac{\gamma}{\Delta\Gamma_q(\alpha-\beta)}\eta^{\alpha-\beta-1}\int_0^{\infty}\left|f\left(t,u_n(t),D_q^{\alpha-1}u_n(t)\right)-f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt$$
$$\leq\left(1+\frac{1}{\Delta}+\frac{\gamma\eta^{\alpha-\beta-1}}{\Delta\Gamma_q(\alpha-\beta)}\right)\int_0^{\infty}\left|f\left(t,u_n(t),D_q^{\alpha-1}u_n(t)\right)-f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt$$
$$\leq k\int_0^{\infty}\left|f\left(t,u_n(t),D_q^{\alpha-1}u_n(t)\right)-f\left(t,u(t),D_q^{\alpha-1}u(t)\right)\right|d_qt$$
$$\leq 2kR\int_0^{+\infty}\left[\left(1+t^{\alpha-1}\right)a(t)+b(t)\right]d_qt+2k\int_0^{+\infty}c(t)d_qt.$$

Then Lebesgue's Dominated Convergence Theorem asserts the continuity of A.

Therefore, by Schauder's fixed point theorem we conclude that the problem (1.1) – (1.2) has at least one solution in U and the proof is finished. □

Example 3.1 Consider the following problem

$$D_{\frac{1}{3}}^{\frac{4}{3}}u(t)+\frac{sin\left(D_{\frac{1}{3}}^{\frac{1}{3}}u(t)\right)}{1+t^3}+\frac{\sqrt{|u(t)|\,.D_{\frac{1}{3}}^{\frac{1}{3}}u(t)}}{10e_{\frac{1}{3}}^{t}\left(1+t^{\frac{1}{3}}\right)}=0,\qquad t\in[0\,,+\infty)\,,$$

$$u(0)=0,\quad D_{\frac{1}{3}}^{\frac{1}{3}}u(\infty)=\frac{1}{4}D_{\frac{1}{3}}^{\frac{1}{2}}u\left(10^6\right),$$

where

$$\alpha = \frac{4}{3}, \ \ \eta = 10^6, \ \ q = \frac{1}{3}, \ \ \beta = \frac{1}{2}, \ \ \gamma = \frac{1}{4}$$

and

$$f(t,x,y) = \frac{1}{1+t^3} siny + \frac{\sqrt{|xy|}}{10 e_{\frac{1}{3}}^{t} \left(1 + t^{\frac{1}{3}}\right)}.$$

Obviously,

$$\left|f(t,x,y)\right| \leq \frac{1}{1+t^3} + \frac{|x|}{20 e_{\frac{1}{3}}^{t} \left(1 + t^{\frac{1}{3}}\right)} + \frac{|y|}{20 e_{\frac{1}{3}}^{t} \left(1 + t^{\frac{1}{3}}\right)} \text{ such that } a(t) = b(t) = \frac{e_{\frac{1}{3}}^{-t}}{20\left(1 + t^{\frac{1}{3}}\right)} \text{ and } c(t) = \frac{1}{1+t^3}.$$

We can easily calculate that

$$\int_0^\infty \left[\left(1 + t^{\frac{1}{3}}\right) \frac{e_{\frac{1}{3}}^{-t}}{20\left(1 + t^{\frac{1}{3}}\right)} + \frac{e_{\frac{1}{3}}^{-t}}{20\left(1 + t^{\frac{1}{3}}\right)} \right] d_{\frac{1}{3}} t < \frac{1}{10},$$

$$\int_0^\infty \frac{1}{1+t^3} d_{\frac{1}{3}} t < \infty.$$

Also, since $\Delta \cong 1$ and $k \cong 2$, we can easily see that the condition

$$\int_0^\infty \left[\left(1 + t^{\frac{1}{3}}\right) \frac{e_{\frac{1}{3}}^{-t}}{20\left(1 + t^{\frac{1}{3}}\right)} + \frac{e_{\frac{1}{3}}^{-t}}{20\left(1 + t^{\frac{1}{3}}\right)} \right] d_{\frac{1}{3}} t < \frac{\Gamma_{\frac{1}{3}}\left(\frac{4}{3}\right)}{k}$$

is hold because of

$$\frac{\Gamma_{\frac{1}{3}}\left(\frac{4}{3}\right)}{k} \cong \frac{1}{2} \Gamma_{\frac{1}{3}}\left(\frac{4}{3}\right) > \frac{1}{10}.$$

Hence, the conditions of *Theorem 3.1* are satisfied and this problem has a solution.

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