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A characterization of S-pseudospectra of linear operators in a Hilbert space

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Abstract. In this work, we introduce and study the S-pseudospectra of linear operators defined by non-strict inequality in a Hilbert space. Inspired by A. Böttcher's result [3], we prove that the S-resolvent norm of bounded linear operators is not constant in any open set of the S-resolvent set. Beside, we find a characterization of the S-pseudospectrum of bounded linear operator by means the S-spectra of all perturbed operators with perturbations that have norms strictly less than ε.

1. Introduction

The concept of pseudospectra was developed by many mathematicians. For example, we can cite J. M. Varah [12], L. N. Trefethen [10, 11], A. Jeribi [5, 6] and A. Ammar and A. Jeribi [1]. We refer the reader to L. N. Trefethen [10] for the definition pseudospectra of the closed linear operator A

$$\Sigma_\varepsilon(A) := \sigma(A) \bigcup \left\{ \lambda \in \mathbb{C} : \|(\lambda - A)^{-1}\| \geq \frac{1}{\varepsilon} \right\},$$

where $\varepsilon > 0$. By convention $\|(\lambda - A)^{-1}\| = +\infty$ if, and only if, $\lambda \in \sigma(A)$. If A is self-adjoint operator, then we have

$$\|(\lambda - A)^{-1}\| = \frac{1}{d(\lambda, \sigma(A))}, \tag{1.1}$$

where $d(\lambda, \sigma(A))$: is the distance between λ and the spectrum of A.

In [9], T. Finck and T. Ehrhardt have proved that the pseudospectra of a bounded linear operator acting in a Hilbert space, is equal to the union of the spectra of all perturbed operators with perturbations that have norms less than ε, i.e.,

$$\Sigma_\varepsilon(A) = \bigcup_{\|D\| \leq \varepsilon} \sigma(A + D).$$

Until now, a number of papers devoted to extend this notion to the S-pseudospectra that is also studied under the name pseudospectra of operator pencils (e.g [4]).

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In this work, we study some properties of the S-pseudospectrum of linear operators in a Hilbert space and we show that the S-resolvent of a bounded operator cannot have constant norm. After that, we establish a characterization of S-pseudospectrum.

We organize our paper in the following way: Section 2 contains preliminary properties that we will need to prove the main results. In Section 3, we begin giving some proprieties of S-pseudospectrum of linear operators in a Hilbert space. Beside that, we characterize the S-pseudospectrum of bounded linear operators by means of perturbation of its S-spectrum in a Hilbert space.

2. Preliminary results

The goal of this section consists in collect some results which will be needed in the sequel.

Throughout this paper, let H be a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by $\mathcal{L}(H)$ the set of all bounded linear operators from H into H. For $A \in \mathcal{L}(H)$, we will denote by $\mathcal{D}(A)$ the domain, $N(A)$ the null space and $R(A)$ the range of A.

Definition 2.1. (*i*) *Let $A \in \mathcal{L}(H)$. The linear operator A' is called the adjoint of A if $\langle Ax, y\rangle = \langle x, A'y\rangle$, for all $x, y \in H$. The operator A' is called the adjoint of A.*

(*ii*) *A densely defined operator A on H is called symmetric, if $A \subset A'$, that is, if $\mathcal{D}(A) \subset \mathcal{D}(A')$ and $Ax = A'x$, for all $x \in \mathcal{D}(A)$. Equivalently, A is symmetric if, and only if, $\langle Ax, y\rangle = \langle x, Ay\rangle$, for all $x, y \in \mathcal{D}(A)$.*

(*iii*) *A is called self-adjoint if $A = A'$ that is, if, and only if, A is symmetric and $\mathcal{D}(A) = \mathcal{D}(A')$.* ◊

Lemma 2.1. [7, Theorem 11.3] *If $A, B \in \mathcal{L}(H)$. Then,*

(*i*) $(A + B)' = A' + B'$;

(*ii*) $(\lambda A)' = \overline{\lambda} A'$, *for all* $\lambda \in \mathbb{C}$;

(*iii*) $(A\,B)' = B'\,A'$;

(*iv*) $(A')' = A$. ◊

Proposition 2.1. [7] *Let $A \in \mathcal{L}(H)$. Then,*

(*i*) *A is invertible if, and only if, its adjoint A' is invertible, and in that case*

$$(A^{-1})' = (A')^{-1}.$$

(*ii*) $A' \in \mathcal{L}(H')$ *and* $\|A'\| = \|A\|$. ◊

Proposition 2.2. *Let $A \in \mathcal{L}(H)$.*

(*i*) [8, Theorem 7.3.1] *If $\|A\| < 1$, then $(I - A)^{-1}$ exists as a bounded linear operator on X and $(I - A)^{-1} = \sum_{n=0}^{+\infty} A^n$.*

(*ii*)[6, Theorem 3.3.2] *Let $S \in \mathcal{L}(H)$ such that $S \neq A$ and $S \neq 0$ S commutes with A, then for any λ and $\lambda_0 \in \rho_S(A)$ with $|\lambda - \lambda_0| < \|(\lambda_0 S - A)^{-1} S\|^{-1}$, we have*

$$(\lambda S - A)^{-1} = \sum_{n \geq 0} (\lambda - \lambda_0)^n S^n (\lambda_0 S - A)^{-(n+1)}.$$ ◊

Definition 2.2. (*i*) *Let $A \in \mathcal{L}(H)$. The resolvent set and the spectrum set of A are define, respectively, by:*

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ is invertible}\}$$

and $\sigma(A) = \mathbb{C} \backslash \rho(A)$.

(*ii*) *Let $A \in \mathcal{L}(H)$. The spectral radius of A is defined by:*

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

(iii) Let $S \in \mathcal{L}(H)$ such that $S \neq 0$. For $A \in \mathcal{L}(H)$, we define the S-resolvent set of A by:

$$\rho_S(A) = \{\lambda \in \mathbb{C} : \lambda S - A \text{ has a bounded inverse}\},$$

and the S-spectrum of A by: $\sigma_S(A) = \mathbb{C} \backslash \rho_S(A)$. ◊

Remark 2.1. [6, Proposition 3.3.1] *Let $A \in \mathcal{L}(H)$, $S \in \mathcal{L}(H)$ such that $S \neq 0$. Then, the S-resolvent set $\rho_S(A)$ is open.* ◊

Lemma 2.2. [6, Remark 3.3.1] *If $A \in \mathcal{L}(H)$ and S is an invertible bounded operator, then*

$$\sigma_S(A) = \sigma(S^{-1}A) \bigcap \sigma(AS^{-1}).$$ ◊

Remark 2.2. *Let $A \in \mathcal{L}(H)$. Let S be a non-null bounded operator such that $S \neq A$.*

$$\rho_S(A) = \overline{\rho_{S'}(A')}.$$ ◊

Indeed, it follows from Proposition 2.1 and Lemma 2.1 that

$$\begin{aligned} \rho_S(A) &= \{\lambda \in \mathbb{C} : \lambda S - A \text{ has a bounded inverse}\} \\ &= \{\lambda \in \mathbb{C} : (\lambda S - A)' \text{ has a bounded inverse}\} \\ &= \{\lambda \in \mathbb{C} : \overline{\lambda} S' - A' \text{ has a bounded inverse}\} \\ &= \overline{\rho_{S'}(A')}. \end{aligned}$$

3. Main results

The goal of this section is to study some proprieties of S-pseudospectra of linear operator in a Hilbert space and to find a relationship between S-spectra and S-pseudospectra.

Definition 3.1. *Let $A \in \mathcal{L}(H)$ and $\varepsilon > 0$. Let S be a non-null bounded operator such that $S \neq A$. We define the S-pseudospectra of A by:*

$$\Sigma_{S,\varepsilon}(A) = \sigma_S(A) \bigcup \left\{\lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}\right\},$$

by convention $\|(\lambda S - A)^{-1}\| = +\infty$ if, and only if, $\lambda \in \sigma_S(A)$. ◊

Lemma 3.1. *Let $A \in \mathcal{L}(H)$ and $\varepsilon > 0$. Let S be a non-null bounded operator such that $S \neq A$. Then, $\Sigma_{S,\varepsilon}(A)$ is closed.* ◊

Proof. We consider the following function

$$\begin{array}{rcl} \varphi : \rho_S(A) & \longrightarrow & \mathbb{R}_+ \\ \lambda & \longmapsto & \|(\lambda S - A)^{-1}\|. \end{array}$$

It is clear that φ is continuous and

$$\left\{\lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| < \frac{1}{\varepsilon}\right\} = \varphi^{-1}\left(\left]-\infty, \frac{1}{\varepsilon}\right[\right).$$

So, we can deduce that $\left\{\lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| < \frac{1}{\varepsilon}\right\}$ is open. Finally, the use of Remark 2.1 allows us to conclude that $\rho_{S,\varepsilon}(A)$ is open. This is equivalent to saying that $\Sigma_{S,\varepsilon}(A)$ is closed. □

Proposition 3.1. *Let $A \in \mathcal{L}(H)$ and $\varepsilon > 0$. Let S be a non-null bounded operator such that $S \neq A$. Then,*

$$\Sigma_{S',\varepsilon}(A') = \overline{\Sigma_{S,\varepsilon}(A)}.$$ ◊

Proof. By using Lemma 2.1 and proposition 2.1, we obtain

$$\begin{array}{rcl}\|(\lambda S-A)^{-1}\| & = & \|((\lambda S-A)^{-1})'\| \\ & = & \|((\lambda S-A)')^{-1}\| \\ & = & \|(\overline{\lambda}S'-A')^{-1}\|.\end{array}$$

Finally, the use of Remark 2.2 allows us to conclude that $\Sigma_{S',\varepsilon}(A')=\overline{\Sigma_{S,\varepsilon}(A)}$. □

Theorem 3.1. *Let A be a bounded invertible operator on H, $S=A^{-1}$ and $\varepsilon>0$. If A is self-adjoint, then we have*

(*i*) $\Sigma_{S,\varepsilon}(A)\subseteq\sigma(S^{-1}A)\bigcup\Big\{\lambda\in\mathbb{C}:\inf\limits_{\mu\in\sigma(S^{-1}A)}|\lambda-\mu|\le\|S^{-1}\|\,\varepsilon\Big\}$.

(*ii*) $\sigma(S^{-1}A)\bigcup\Big\{\lambda\in\mathbb{C}:\inf\limits_{\mu\in\sigma(S^{-1}A)}|\lambda-\mu|\le\|S\|^{-1}\varepsilon\Big\}\subseteq\Sigma_{S,\varepsilon}(A)$.

(*iii*) *Moreover, if $\|A\|=\|A^{-1}\|=1$, then*

$$\Sigma_{S,\varepsilon}(A)=\sigma(S^{-1}A)\bigcup\Big\{\lambda\in\mathbb{C}:\inf_{\mu\in\sigma(S^{-1}A)}|\lambda-\mu|\le\varepsilon\Big\}.$$ ◊

Proof. Since $S=A^{-1}$, then S is invertible, $S^{-1}=A$ and $S^{-1}A=AS^{-1}$. It follows from Lemma 2.2 that

$$\sigma_S(A)=\sigma(S^{-1}A)=\sigma(AS^{-1}).$$

Consequently,

$$\Sigma_{S,\varepsilon}(A)=\sigma(S^{-1}A)\bigcup\Big\{\lambda\in\mathbb{C}:\|(\lambda S-A)^{-1}\|\ge\frac{1}{\varepsilon}\Big\}. \tag{3.1}$$

(*i*) For $\lambda\in\mathbb{C}$, we can write

$$\begin{array}{rcl}\|(\lambda S-A)^{-1}\| & = & \|(S(\lambda-S^{-1}A))^{-1}\| \\ & = & \|(\lambda-S^{-1}A)^{-1}S^{-1}\| \\ & \le & \|(\lambda-S^{-1}A)^{-1}\|\,\|S^{-1}\|.\end{array}$$

Therefore,

$$\|(\lambda S-A)^{-1}\|\,\|S^{-1}\|^{-1}\le\|(\lambda-S^{-1}A)^{-1}\|. \tag{3.2}$$

Let $\lambda\in\Sigma_{S,\varepsilon}(A)$. Then, by (3.1), we have

$$\lambda\in\sigma(S^{-1}A)\bigcup\Big\{\lambda\in\mathbb{C}:\|(\lambda S-A)^{-1}\|\ge\frac{1}{\varepsilon}\Big\}.$$

It is clear that

$$\sigma(S^{-1}A)\subset\sigma(S^{-1}A)\bigcup\Big\{\lambda\in\mathbb{C}:\inf_{\mu\in\sigma(S^{-1}A)}|\lambda-\mu|\le\|S^{-1}\|\,\varepsilon\Big\}.$$

Then, it is sufficient to show that

$$\Sigma_{S,\varepsilon}(A)\backslash\sigma(S^{-1}A)\subset\sigma(S^{-1}A)\bigcup\Big\{\lambda\in\mathbb{C}:\inf_{\mu\in\sigma(S^{-1}A)}|\lambda-\mu|\le\|S^{-1}\|\,\varepsilon\Big\}.$$

Let $\lambda\in\Big\{\lambda\in\mathbb{C}:\|(\lambda S-A)^{-1}\|\ge\frac{1}{\varepsilon}\Big\}$. Then, using (3.2), we obtain

$$\|(\lambda-S^{-1}A)^{-1}\|\ge\frac{1}{\varepsilon\,\|S^{-1}\|}. \tag{3.3}$$

Now, combining the fact that $S = A^{-1}$ and (*iii*) of Lemma 2.1, we infer that

$$\begin{aligned}(S^{-1}A)' &= A'(S^{-1})' \\ &= AA' \\ &= S^{-1}A,\end{aligned}$$

which yields $S^{-1}A$ is self-adjoint. By referring to (1.1), we have

$$\|(\lambda - S^{-1}A)^{-1}\| = \frac{1}{d(\lambda, \sigma(S^{-1}A))} = \frac{1}{\inf\limits_{\mu\in\sigma(S^{-1}A)} |\lambda - \mu|}. \tag{3.4}$$

Hence, by (3.3), we conclude that $\inf\limits_{\mu\in\sigma(S^{-1}A)} |\lambda - \mu| \leq \|S^{-1}\|\, \varepsilon$. This shows that

$$\Sigma_{S,\varepsilon}(A) \subset \sigma(S^{-1}A) \bigcup \Big\{\lambda \in \mathbb{C} : \inf_{\mu\in\sigma(S^{-1}A)} |\lambda - \mu| \leq \|S^{-1}\|\, \varepsilon\Big\}.$$

(*ii*) For $\lambda \in \mathbb{C}$, we can write

$$\begin{aligned}\|(\lambda - S^{-1}A)^{-1}\| &= \|(S^{-1}(\lambda S - A))^{-1}\| \\ &\leq \|(\lambda S - A)^{-1}\|\, \|S\|.\end{aligned}$$

Therefore,

$$\|(\lambda S - A)^{-1}\|\| \geq \|(\lambda - S^{-1}A)^{-1}\|\, \|S\|^{-1}. \tag{3.5}$$

Let us assume that $\lambda \in \Big\{\lambda \in \mathbb{C} : \inf\limits_{\mu\in\sigma(S^{-1}A)} |\lambda - \mu| \leq \|S\|^{-1}\, \varepsilon\Big\}$, then by (3.4), we infer that

$$\|(\lambda - S^{-1}A)^{-1}\| \geq \frac{\|S\|}{\varepsilon}.$$

By referring to (3.5), we have

$$\|(\lambda S - A)^{-1}\|\| \geq \frac{1}{\varepsilon}.$$

The use of (3.1) makes us conclude that

$$\sigma(S^{-1}A) \bigcup \Big\{\lambda \in \mathbb{C} : \inf_{\mu\in\sigma(S^{-1}A)} |\lambda - \mu| \leq \|S\|^{-1}\varepsilon\Big\} \subseteq \Sigma_{S,\varepsilon}(A).$$

(*iii*) Using the fact that $S = A^{-1}$ and $\|A\| = \|A^{-1}\| = 1$, then

$$\|S^{-1}\| = \|A\| = \|A^{-1}\| = \|S\| = 1. \tag{3.6}$$

Finally, the use of (*i*), (*ii*) of Theorem 3.1 and (3.6) allows us to conclude that

$$\Sigma_{S,\varepsilon}(A) = \sigma(S^{-1}A) \bigcup \Big\{\lambda \in \mathbb{C} : \inf_{\mu\in\sigma(S^{-1}A)} |\lambda - \mu| \leq \varepsilon\Big\}.$$ □

Remark 3.1. *From Theorem 3.1, it follows immediately that*

$$\sigma_{\varepsilon\|A\|^{-1}}(A^2) \subseteq \Sigma_{A,\varepsilon}(A) \subseteq \Sigma_{\varepsilon\|A\|}(A^2) \tag{3.7}$$

and that equality holds in (3.7), *if* $\|A\| = \|A^{-1}\| = 1$. ◊

Theorem 3.2. *Let $A \in \mathcal{L}(H)$ and $\varepsilon > 0$. Let $S \in \mathcal{L}(H)$ such that $S \neq 0$ and $S \neq A + D$, for all $D \in \mathcal{L}(H)$ with $\|D\| < \varepsilon$. Then,*

$$\bigcup_{\|D\|<\varepsilon} \sigma_S(A+D) \subset \Sigma_{S,\varepsilon}(A). \qquad \diamond$$

Proof. Let us assume that $\lambda \in \bigcup_{\|D\|<\varepsilon} \sigma_S(A+D)$. Then, there exists $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $\lambda \in \sigma_S(A+D)$. We derive a contradiction from the assumption that $\lambda \in \rho_S(A)$ and $\|(\lambda S - A)^{-1}\| < \frac{1}{\varepsilon}$. For $\lambda \in \rho_S(A)$, we can write

$$\lambda S - A - D = (\lambda S - A)\Big(I - (\lambda S - A)^{-1} D\Big). \tag{3.8}$$

Since

$$\begin{array}{rcl} \|(\lambda S - A)^{-1} D\| & \leq & \|(\lambda S - A)^{-1}\| \, \|D\| \\ & < & \dfrac{\varepsilon}{\varepsilon} \\ & < & 1, \end{array}$$

then by using (*i*) of Proposition 2.2, we infer that $I - (\lambda S - A)^{-1} D$ is invertible. By referring to (3.8), we conclude that $\lambda S - A - D$ is invertible. This is equivalent to say that $\lambda \in \rho_S(A+D)$. $\square$

As an immediate consequence of Lemma 3.1 and Theorem 3.2, we have

Corollary 3.1. *Let $A \in \mathcal{L}(H)$ and $\varepsilon > 0$. Let $S \in \mathcal{L}(H)$ such that $S \neq 0$ and $S \neq A + D$, for all $D \in \mathcal{L}(H)$ with $\|D\| \leq \varepsilon$, then we have*

$$clos\left(\bigcup_{\|D\|<\varepsilon} \sigma_S(A+D) \right) \subset \Sigma_{S,\varepsilon}(A),$$

where $clos(\cdot)$: denotes the closure. $\diamond$

Proposition 3.2. *Let $A, S \in \mathcal{L}(H)$ such that S is invertible, $S \neq A$ and $SA = AS$. Suppose that $\lambda S - A$ is invertible for all λ in some open subset $U \subset \mathbb{C}$ and $\|(\lambda S - A)^{-1}\| \leq M$, for all $\lambda \in U$. Then,*

$$\|(\lambda S - A)^{-1}\| < M, \text{ for all } \lambda \in U.$$

Proof. A little thought reveals that what we must show is the following: if U is an open subset of $\mathbb{C}$ containing 0 and $\|(\lambda S - A)^{-1}\| \leq M$, then

$$\|(\lambda S - A)^{-1}\| < M, \text{ for all } \lambda \in U.$$

To prove this assume the contrary

$$\|(\lambda S - A)^{-1}\| = M, \text{ for all } \lambda \in U.$$

If $\lambda = 0$, then

$$\|A^{-1}\| = M. \tag{3.9}$$

Using the fact that $SA = AS$, then by using (*ii*) of Proposition 2.2, we have

$$(\lambda S - A)^{-1} = \sum_{n \geq 0} \lambda^n S^n A^{-(n+1)}, \text{ for all } |\lambda| < \|A^{-1} S\|^{-1}. \tag{3.10}$$

Let $x \in H$ and $|\lambda| < \|A^{-1}S\|^{-1}$. Hence, by (3.10), we infer that

$$\begin{aligned}\|(\lambda S - A)^{-1}x\|^2 &= \langle(\lambda S - A)^{-1}x, (\lambda S - A)^{-1}x\rangle \\ &= \Big\langle \sum_{k\geq 0} \lambda^k S^k A^{-(k+1)}x, \sum_{j\geq 0} \lambda^j S^j A^{-(j+1)}x\Big\rangle \\ &= \sum_{k,j\geq 0} \lambda^k \overline{\lambda^j} \Big\langle S^k A^{-(k+1)}x, \sum_{j\geq 0} S^j A^{-(j+1)}x\Big\rangle.\end{aligned}$$

Let $r \leq \|A^{-1}S\|^{-1}$.Therefore, for all $x \in H$ and $|\lambda| \leq r$

$$\|(\lambda S - A)^{-1}x\|^2 = \sum_{k,j\geq 0} \lambda^k \overline{\lambda^j} \Big\langle S^k A^{-(k+1)}x, S^j A^{-(j+1)}x\Big\rangle. \tag{3.11}$$

Integrating (3.11) along the circle $|\lambda| = r$, we obtain

$$\int_0^1 \|(re^{2it\pi}S - A)^{-1}x\|^2 dt = \sum_{k\geq 0} r^{2k} \Big\langle S^k A^{-(k+1)}x, S^k A^{-(k+1)}x\Big\rangle = \sum_{k\geq 0} r^{2k} \|S^k A^{-(k+1)}x\|^2. \tag{3.12}$$

Using (3.12) and the hypothesis $\|(re^{2it\pi}S - A)^{-1}x\| \leq M\|x\|$, then we arrive at

$$\|A^{-1}x\|^2 + \|SA^{-2}x\|^2 \leq M^2\|x\|^2. \tag{3.13}$$

Now pick an arbitrary $\varepsilon > 0$. It follows from (3.9) that there is an $x_0 \in H$ such that $\|x_0\| = 1$ and

$$\|A^{-1}x_0\|^2 > M^2 - \varepsilon. \tag{3.14}$$

In view of (3.13) and (3.14) implies that

$$\|SA^{-2}x_0\|^2 < \varepsilon r^{-2}. \tag{3.15}$$

Consequently, by referring to (3.15), we have

$$1 = \|x_0\|^2 \leq \|(SA^{-2})^{-1}\|\,\|SA^{-2}x_0\|^2 < \|(SA^{-2})^{-1}\|\varepsilon r^{-2},$$

which is impossible if $\varepsilon > 0$ is sufficiently small. This contradiction shows that $\|(\lambda S - A)^{-1}\| < M$, for all $\lambda \in U$. □

Remark 3.2. (*i*) *In Proposition 3.2, we proved that the S-resolvent of a bounded operator acting in Hilbert space cannot have constant norm on any open set.*
(*ii*) *Proposition 3.2 is a generalization of* [3, Proposition 6.1]. ◇

Theorem 3.3. *Let $\varepsilon > 0$ and $A, S \in \mathcal{L}(H)$ such that S is invertible, $SA = AS$ and $S \neq A + D$ for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$. Then,*

$$\Sigma_{S,\varepsilon}(A) \subseteq clos\left(\bigcup_{\|D\|<\varepsilon} \sigma_S(A + D)\right). \quad ◇$$

Proof. Let $\lambda \in \Sigma_{S,\varepsilon}(A) = \sigma_S(A) \bigcup \Big\{\lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}\Big\}$.
First case. If $\lambda \in \sigma_S(A)$, we may put $D = 0$.
Second case. If $\lambda \in \Big\{\lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}\Big\}\backslash\sigma_S(A)$, then

$$\|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon} \text{ and } \lambda \in \rho_S(A).$$

This leads to $\|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}$, for $\lambda \in \rho_S(A)$. Therefore, by Remarks 2.1 and 3.2 (*i*), we obtain

$$\|(\lambda S - A)^{-1}\| > \frac{1}{\varepsilon} \text{ for all } \lambda \in \rho_S(A).$$

This implies that there exists y_0 such that $\|y_0\| = 1$ and $\|(\lambda S - A)^{-1} y_0\| > \frac{1}{\varepsilon}$. Putting

$$x_0 = \|(\lambda S - A)^{-1} y_0\|^{-1} (\lambda S - A)^{-1} y_0.$$

Therefore, $x_0 \in H$, $\|x_0\| = 1$ and

$$\begin{aligned} \|(\lambda S - A)x_0\| &= \|(\lambda S - A)^{-1} y_0\|^{-1} \\ &< \varepsilon. \end{aligned}$$

Consequently, there exists $x_0 \in H$ such that $\|x_0\| = 1$ and $\|(\lambda S - A)x_0\| < \varepsilon$. By the Hahn-Banach theorem, there exists $x' \in X'$ such that $\|x'\| = 1$ and $x'(x_0) = 1$. We consider the following linear operator

$$D(x) := x'(x)\ (\lambda S - A)x.$$

Let us observe that

$$\begin{aligned} \|D(x)\| &\leq \|x'\|\ \|x\|\ \|(\lambda S - A)x\| \\ &< \varepsilon \|x\|, \end{aligned}$$

then we have $\|D\| < \varepsilon$ and D is everywhere defined. Therefore, D is bounded. Moreover, we have

$$(\lambda S - A - D)x_0 = 0, \text{ for } \|x_0\| = 1.$$

Hence, $\lambda \in \sigma_S(A + D)$ and we can deduce that $\lambda \in clos\left(\bigcup_{\|D\|<\varepsilon} \sigma_S(A + D)\right)$. □

As a direct consequence of Corollary 3.1 and Theorem 3.3, we infer the following result

Corollary 3.2. *Let $\varepsilon > 0$ and $A, S \in \mathcal{L}(H)$ such that S is invertible, $SA = AS$ and $S \neq A + D$ for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$. Then,*

$$\Sigma_{S,\varepsilon}(A) = clos\left(\bigcup_{\|D\|<\varepsilon} \sigma_S(A + D)\right). \qquad \diamond$$

Theorem 3.4. *Let $\varepsilon > 0$ and $A, S \in \mathcal{L}(H)$. Then,*

$$\Sigma_{S,\varepsilon}(A) = \bigcup_{\|D\|\leq\varepsilon} \sigma_S(A + D). \qquad \diamond$$

Proof. Let us assume that $\lambda \in \bigcup_{\|D\|\leq\varepsilon} \sigma_S(A + D)$. Then, there exists $D \in \mathcal{L}(H)$ such that $\|D\| \leq \varepsilon$ and $\lambda S - A - D$ is not invertible. If $\lambda \in \sigma_S(A)$, then $\lambda \in \Sigma_{S,\varepsilon}(A)$. So we can suppose that $\lambda S - A$ is invertible. Therefore, we can write

$$\lambda S - A - D = (\lambda S - A)(I - (\lambda S - A)^{-1} D).$$

Consequently, $I - (\lambda S - A)^{-1} D$ is not invertible which yields $\|(\lambda S - A)^{-1} D\| \geq 1$. This implies that

$$\begin{aligned} 1 &\leq \|(\lambda S - A)^{-1} D\| \\ &\leq \|(\lambda S - A)^{-1}\|\ \|D\| \\ &\leq \varepsilon\ \|(\lambda S - A)^{-1}\|. \end{aligned}$$

Hence, $\|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}$. This enables us to conclude that

$$\bigcup_{\|D\|\leq\varepsilon} \sigma_S(A+D) \subset \Sigma_{S,\varepsilon}(A).$$

Conversely, we suppose for contrary that there exists a $\lambda \in \Sigma_{S,\varepsilon}(A)$ such that $\lambda S - A - D$ is invertible for all $D \in \mathcal{L}(H)$ with $\|D\| \leq \varepsilon$. Setting $D = 0$, we get the invertibility of $\lambda S - A$. It follows from Remark 2.2 that $\overline{\lambda} S' - A'$ is invertible. Setting $D = \mu(\overline{\lambda} S' - A')^{-1}$ where μ is arbitrary complex number satisfying

$$0 < |\mu| \leq \frac{\varepsilon}{\|(\overline{\lambda} S' - A')^{-1}\|}. \tag{3.16}$$

For μ satisfying (3.16), we can write

$$\begin{aligned} \lambda S - A - D &= \lambda S - A - \mu(\overline{\lambda} S' - A')^{-1} \\ &= \mu(\lambda S - A)\left(\frac{1}{\mu} - (\lambda S - A)^{-1}(\overline{\lambda} S' - A')^{-1}\right). \end{aligned}$$

Consequently, $\frac{1}{\mu} - (\lambda S - A)^{-1}(\overline{\lambda} S' - A')^{-1}$ is invertible for μ satisfying (3.16) which yields

$$r\Big((\lambda S - A)^{-1}(\overline{\lambda} S' - A')^{-1}\Big) < \frac{\|(\overline{\lambda} S' - A')^{-1}\|}{\varepsilon}.$$

Using the fact that $(\lambda S - A)^{-1}(\overline{\lambda} S' - A')^{-1}$ is self adjoint, then we have

$$\Big\|(\lambda S - A)^{-1}(\overline{\lambda} S' - A')^{-1}\Big\| = r\Big((\lambda S - A)^{-1}(\overline{\lambda} S' - A')^{-1}\Big) < \frac{\|(\overline{\lambda} S' - A')^{-1}\|}{\varepsilon}.$$

Hence,

$$\|(\overline{\lambda} S' - A')^{-1}\|^2 < \frac{\|(\overline{\lambda} S' - A')^{-1}\|}{\varepsilon}.$$

Finally, the use of Proposition 2.1 (*ii*) allows us to conclude that

$$\|(\overline{\lambda} S' - A')^{-1}\| = \|(\lambda S - A)^{-1}\| < \frac{1}{\varepsilon},$$

which is a contradiction. □

Remark 3.3. *Theorem 3.4 is a generalization of T. Finck and T. Ehrhardt's result* [9]. ◇

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