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Hybrid inertial accelerated extragradient algorithms for split pseudomonotone equilibrium problems and fixed point problems of demicontractive mappings

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Abstract. In this paper, we present a new hybrid extragradient algorithm for finding a common element of the fixed point problem for a demicontractive mapping and the split equilibrium problem for a pseudomonotone and Lipschitz-type continuous bifunction. By using a new technique of choosing the step size of the proposed method, our algorithms do not need any prior information of the operator norm. In fact, we propose an inertial type algorithm in order to accelerate its convergence rate and then prove strong convergence theorem of our proposed method under some control conditions. Moreover, we give some numerical experiments to support our main results.

1. Introduction

The *equilibrium problem* provides a unified approach to address a variety of mathematical problems arising in disciplines such as physics, transportation, game theory, economics and network (see[12, 19]).

Let H_1 and H_2 be real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C_1 and C_2 be nonempty closed convex subset of H_1 and H_2, respectively. Let $T : C_1 \to C_1$ be a mapping. We denoted $Fix(T)$ by the set of all fixed points of T, i.e., $Fix(T) = \{x \in C_1 : Tx = x\}$. Let $f_1 : C_1 \times C_1 \to \mathbb{R}$ be a bifunction. The *equilibrium problem* (shortly, **(EP)**) is as follows:

$$\text{Find a point } \bar{x} \in C_1 \text{ such that } f_1(\bar{x}, y) \geq 0 \text{ for all } y \in C_1. \tag{1}$$

The set of all solutions of the problem **(EP)** is denoted by $EP(f_1)$. The equilibrium problem is a generalization of the variational inequality problem, the optimization problem, the Nash equilibrium problem

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and some others (see [4, 6, 7, 11, 20]). Recently, some nonlinear problems to find a common point of the solution set of the equilibrium problem and the set of fixed points of a nonexpansive mapping becomes an attractive field for many researchers (see [1, 8–10, 17, 18, 22, 25, 27–29]).

Let $f_1 : C_1 \times C_1 \to \mathbb{R}$ and $f_2 : C_2 \times C_2 \to \mathbb{R}$ be two bifunctions. Let $A : H_1 \to H_2$ be a bounded linear operator. The *split equilibrium problem* (shortly, **(SEP)**) [21] is as follows:

$$\text{Find a point } \bar{x} \in C_1 \text{ such that } f_1(\bar{x}, y) \geq 0 \text{ for all } y \in C_1 \tag{2}$$

and such that

$$\bar{y} = A\bar{x} \in C_2 \text{ solves } f_2(\bar{y}, z) \geq 0 \text{ for all } z \in C_2. \tag{3}$$

The solution set of the problem **(SEP)** is denoted by

$$\Omega = \{z \in EP(f_1) : Az \in EP(f_2)\}.$$

The split equilibrium problem is said to be *monotone* if bifunctions f_1 and f_2 are monotone.

Obviously, if $f_2 = 0$ and $C_2 = H_2$ in the problem **(SEP)**, then the split equilibrium problem becomes the equilibrium problem.

In 2012, He [21] proposed a new algorithm for solving the split monotone equilibrium problem and investigated the convergence behaviour in several ways including the strong convergence and he also generated the sequence $\{x_n\}$ iteratively as follows:

$$\begin{cases} x_1 \in C_1 = C, \\ f_1(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ f_2(w_n, z) + \dfrac{1}{r_n}\langle z - w_n, w_n - Au_n\rangle \geq 0, \quad \forall z \in D, \\ y_n = P_C[u_n - \gamma A^*(w_n - Au_n)], \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|u_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad \forall n \geq 1, \end{cases} \tag{4}$$

where C and D are nonempty closed convex subsets of H_1 and H_2, respectively, A^* is the adjoint operator of A, $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ and $\{r_n\}$ is a sequence in $[r, \infty) \subset (0, \infty)$ with some conditions.

To find a solution of a system of equilibrium problems for pseudomonotone monotone and Lipschitz-type continuous bifunctions in $\mathbb{R}^m$, in [32], Tran et al. introduced the following extragradient method $\{x_n\}$:

$$\begin{cases} x_0 \in C_1, \\ y_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|y - x_n\|^2 + \lambda_n f_1(x_n, y) : y \in C_1\Big\}, \\ x_{n+1} = \operatorname{argmin}\Big\{\dfrac{1}{2}\|y - x_n\|^2 + \lambda_n f_1(y_n, y) : y \in C_1\Big\}, \quad \forall n \geq 0, \end{cases} \tag{5}$$

where $\lambda_n \in (0, 1]$. They proved that the sequence $\{x_n\}$ converges to a solution of the equilibrium problem.

Recently, Anh [3] presented a hybrid extragradient iteration method $\{x_n\}$ for finding a common element of the set of fixed points of a nonexpansive self-mapping and the set of solutions of the equilibrium problem for a pseudomonotone and Lipschitz-type continuous bifunction as follows:

$$\begin{cases} x_0 \in C_1, \\ y_n = \text{argmin}\Big\{\frac{1}{2}\|y - x_n\|^2 + \lambda_n f_1(x_n, y) : y \in C_1\Big\}, \\ t_n = \text{argmin}\Big\{\frac{1}{2}\|t - x_n\|^2 + \lambda_n f_1(y_n, t) : t \in C_1\Big\}, \\ z_n = \alpha_n x_n + (1 - \alpha_n)T(t_n), \\ D_n = \{z \in C_1 : \|z_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C_1 : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{D_n \cap Q_n}, \quad \forall n \ge 0. \end{cases} \tag{6}$$

Also, he showed that, under certain appropriate conditions imposed on λ_n and α_n, the sequences $\{x_n\}$ strongly converges to a common solution of the solution sets of the fixed point problem and the equilibrium problem. Further, some more iterative algorithms for finding a common element of the set of fixed points of a nonlinear mapping and the set of solutions of the equilibrium problem for pseudomonotone bifunctions in real Hilbert spaces have been studied by some authors (see[2, 13, 23, 31, 33]).

Very recently, Dong et al. [14–17], Hieu et al. [22] and some others have studied some kinds of inertial algorithms to converge strongly and weakly to some fixed points of nonlinear mappings and some solutions of some variational inequality problems, equilibrium problems and split feasibility problems in Hilbert spaces.

In this paper, motivated and inspired by the results [3, 21], first we apply the inertial term, that is, inertial extrapolation, to some algorithms and then our control conditions on the step sizes do not require any prior knowledge of the operator norm. Second, we prove some strong convergence theorems of the proposed algorithms for approximating a common solution of the set of solutions of the split pseudomonotone equilibrium problem and the set of fixed points of a demicontractive mapping in real Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let the symbols $\to$ ($\rightharpoonup$) be denoted the strong and weak convergence, respectively, and let $\omega_w(x_n)$ denote the set of cluster points of the sequence $\{x_n\}$ in the weak topology, that is, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x$. Let $f : H \to \mathbb{R}$ be a function. Define the set of minimizers of the function f by

$$\underset{y \in C \subseteq H}{\text{argmin}} f(y) = \{y \in C : f(y) \le f(z),\ \forall z \in C\}.$$

It is known that $\text{argmin}\{f(y) + a : y \in C\} = \text{argmin}\{f(y) : y \in C\}$ for all $a \in \mathbb{R}$. A mapping P_C is called the *metric projection* of H onto C if, for any $x \in H$, there exists a unique nearest point in C denoted by $P_C(x)$, i.e.,

$$P_C(x) = \text{argmin}\{\|y - x\| : y \in C\}.$$

It is known that P_C is a firmly nonexpansive mapping and, moreover, P_C is characterized by the following property:

$$\langle x - P_C x, y - P_C x\rangle \le 0, \quad \forall x \in H, \ \ y \in C.$$

Now, we recall the following definition:

Definition 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be:*

(1) firmly nonexpansive if

$$\|Tu - Tv\|^2 \le \langle Tu - Tv, u - v\rangle, \quad \forall u, v \in C;$$

(2) nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \ \ \forall u, v \in C;$$

(3) quasi-nonexpansive if $Fix(T) \neq \emptyset$ *and*

$$\|Tu - v\| \leq \|u - v\|, \ \ \forall u \in C, \ v \in Fix(T);$$

(4) k-demicontractive if $Fix(T) \neq \emptyset$ *and there exists* $k \in [0, 1)$ *such that*

$$\|Tu - v\|^2 \leq \|u - v\|^2 + k\|u - Tu\|^2, \ \ \forall u \in C, \ v \in Fix(T).$$

Noted the following:
(1) Every firmly nonexpansive mapping is nonexpansive.
(2) Every nonexpansive mapping is quai-nonexpansive.
(3) Every quasi-nonexpansive mapping is demicontractive.
(4) If T is a demicontractive mapping with $Fix(T) \neq \emptyset$, then $Fix(T)$ is closed convex.

Definition 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let* $T : C \to C$ *be a mapping and I be the identity mapping on C. The mapping* $T - I$ *is said to be demiclosed at zero if, for any sequence* $\{x_n\}$ *in C which* $x_n \rightharpoonup x$ *and* $Tx_n - x_n \to 0$*, we have* $x \in Fix(T)$.

Next, we list some well-known definitions for the next section.

Definition 2.3. *The bifunction* $f : C \times C \to \mathbb{R}$ *is said to be:*
(1) strongly monotone on C if there exists a constant $\gamma > 0$ *such that* $f(x, y) + f(y, x) \leq -\gamma\|x - y\|^2, \ \ \forall x, y \in C;$
(2) monotone on C if $f(x, y) + f(y, x) \leq 0, \ \ \forall x, y \in C;$
(3) pseudomonotone if $f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \ \ \forall x, y \in C;$
(4) Lipschitz-type continuous on C if there exist two positive constants c_1, c_2 *such that*

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \ \ \forall x, y, z \in C.$$

From the definitions above, it is clear that (1) $\implies$ (2) $\implies$ (3).
Now, we assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:
(b1) $f(x, x) = 0$ for all $x \in C$ and f is pseudomonotone on C;
(b2) f is Lipschitz-type continuous;
(b3) for each $x \in C$, $y \mapsto f(x, y)$ is convex and subdifferentiable;
(b4) $f(x, y)$ is weakly continuous on $C \times C$, that is, if $\{x_n\}, \{y_n\} \subseteq C$ weakly converges to $x, y \in C$, respectively, then $f(x_n, y_n) \to f(x, y)$.
Note that, if f satisfies the condition (b1) and $EP(f) \neq \emptyset$, then $EP(f)$ is convex (see [7]). By the condition (b4), we can show that $EP(f)$ is closed.

Lemma 2.4 ([30]). *Let H be a real Hilbert space. Then the following results hold:*
(1) *for all* $t \in [0, 1]$ *and* $u, v \in H$,

$$\|tu + (1 - t)v\|^2 = t\|u\|^2 + (1 - t)\|v\|^2 - t(1 - t)\|u - v\|^2.$$

(2) $\|u \pm v\|^2 = \|u\|^2 \pm 2\langle u, v\rangle + \|v\|^2$ *for all* $u, v \in H$.

Lemma 2.5 ([24]). *Let C be a closed and convex subsets of a real Hilbert space H. Then, for any* $x, y, z \in H$ *and* $a \in \mathbb{R}$*, the set*

$$D := \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle w, v\rangle + a\} \tag{7}$$

is closed and convex.

Lemma 2.6 ([24]). *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H, $u \in H$ and let $q = P_C u$. Suppose that the sequence $\{x_n\}$ in H satisfies the following conditions:*

$$\omega_w(x_n) \subseteq C, \ \ \|x_n - u\| \leq \|u - q\|, \ \ \forall n \geq 1.$$

Then $x_n \to q$.

Lemma 2.7 ([2, 3]). *Let C be a nonempty closed convex subset of a real Hilbert spaces H and $f : C \times C \to \mathbb{R}$ be a psedumonotone and Lipschitz-type continuous bifunction with constants $c_1, c_2 > 0$. For each $x \in C$, let $f(x, \cdot)$ be convex and subdifferentiable on C. Let $\{v_n\}$, $\{z_n\}$ and $\{w_n\}$ be the sequences generated by*

$$\begin{cases} v_0 \in C, \\ z_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|z - v_n\|^2 + \lambda_n f(v_n, z) : z \in C\Big\}, \\ w_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|w - v_n\|^2 + \lambda_n f(z_n, w) : w \in C\Big\}, \ \ \forall n \geq 0, \end{cases} \tag{8}$$

where $\lambda_n > 0$ for all $n \geq 0$. Then, for each $x^* \in EP(f)$,

$$\lambda_n[f(v_n, z) - f(v_n, z_n)] \geq \langle z_n - v_n, z_n - z\rangle, \ \ \forall z \in C, \tag{9}$$

and

$$\|w_n - x^*\|^2 \leq \|v_n - x^*\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2 - (1 - 2\lambda_n c_1)\|v_n - z_n\|^2, \ \ \forall n \geq 0. \tag{10}$$

3. Main Results

Throughout this section, let H_1 and H_2 be real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C_1 and C_2 be nonempty closed convex subsets of H_1 and H_2, respectively. and let I be the identity mapping on H_1. We assume that

- $T : H_1 \to H_1$ is a k-demicontractive mapping such that $T - I$ demiclosed at zero;
- $A : H_1 \to H_2$ is a bounded linear operator with its adjoint operator A^*;
- $f_1 : C_1 \times C_1 \to \mathbb{R}$ is the bifunction satisfies the conditions (b1)-(b4) with the Lipschitz constants $c_1, c_2 > 0$;
- $f_2 : C_2 \times C_2 \to \mathbb{R}$ is the bifunction satisfies the conditions (b1)-(b4) with the Lipschitz constants $b_1, b_2 > 0$;
- $Fix(T) \cap \Omega \neq \emptyset$.

For our main results, that is, some strong convergence theorems, we start with the following important lemmas:

Lemma 3.1. *Let $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ be the sequences generated by*

$$\begin{cases} x_0 \in H_1, \\ y_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|y - P_{C_2}Ax_n\|^2 + \beta_n f_2(P_{C_2}Ax_n, y) : y \in C_2\Big\}, \\ t_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|t - P_{C_2}Ax_n\|^2 + \beta_n f_2(y_n, t) : t \in C_2\Big\}, \ \ \forall n \geq 0, \end{cases} \tag{11}$$

where $0 < \beta_n < \min\left\{\frac{1}{2b_1}, \frac{1}{2b_2}\right\}$ for all $n \geq 0$. Then we have

$$\|Ax_n - t_n\|^2 \leq 2\langle Ax_n - Ax^*, Ax_n - t_n\rangle \tag{12}$$

and

$$\|x_n - \gamma_n A^*(Ax_n - t_n) - x^*\|^2 \leq \|x_n - x^*\|^2 - \gamma_n\left[\|Ax_n - t_n\|^2 - \gamma_n\|A^*(Ax_n - t_n)\|^2\right] \tag{13}$$

for all $n \geq 0$ and $x^* \in H_1$ such that $Ax^* \in EP(f_2)$.

Proof. *Let $n \geq 0$ and $x^* \in H_1$ be such that $Ax^* \in EP(f_2)$. By Lemma 2.7, we have*

$$\begin{aligned}\|Ax_n - t_n\|^2 &\leq \|Ax_n - Ax^*\|^2 - 2\langle Ax_n - Ax^*, t_n - Ax^*\rangle + \|t_n - Ax^*\|^2 \\ &\leq \|Ax_n - Ax^*\|^2 - 2\langle Ax_n - Ax^*, t_n - Ax^*\rangle + \|P_{C_2}Ax_n - Ax^*\|^2 \\ &\quad - (1 - 2\beta_n b_2)\|t_n - y_n\|^2 - (1 - 2\beta_n b_1)\|P_{C_2}Ax_n - y_n\|^2.\end{aligned}$$

Since $2\beta_n b_1, 2\beta_n b_2 < 1$ and P_{C_2} is a firmly nonexpansive mapping, we obtain

$$\begin{aligned}\|Ax_n - t_n\|^2 &\leq \|Ax_n - Ax^*\|^2 - 2\langle Ax_n - Ax^*, t_n - Ax^*\rangle + \|P_{C_2}Ax_n - Ax^*\|^2 \\ &\leq \|Ax_n - Ax^*\|^2 - 2\langle Ax_n - Ax^*, t_n - Ax^*\rangle + \|Ax_n - Ax^*\|^2 \\ &= 2\langle Ax_n - Ax^*, Ax_n - t_n\rangle. \end{aligned} \tag{14}$$

From (14)*, it follows that*

$$\begin{aligned}&\|x_n - \gamma_n A^*(Ax_n - t_n) - x^*\|^2 \\ &\quad = \|x_n - x^*\|^2 - 2\gamma_n\langle x_n - x^*, A^*(Ax_n - t_n)\rangle + \gamma_n^2\|A^*(Ax_n - t_n)\|^2 \\ &\quad = \|x_n - x^*\|^2 - 2\gamma_n\langle Ax_n - Ax^*, Ax_n - t_n\rangle + \gamma_n^2\|A^*(Ax_n - t_n)\|^2 \\ &\quad \leq \|x_n - x^*\|^2 - \gamma_n\left[\|Ax_n - t_n\|^2 - \gamma_n\|A^*(Ax_n - t_n)\|^2\right].\end{aligned} \tag{15}$$

This completes the proof. □

Remark 3.2. *Let $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ be the sequences generated by* (11) *and let $A^{-1}(EP(f_2)) \neq \emptyset$. Then, by* (12)*, we have*

$$Ax_n - t_n = 0 \Longleftrightarrow A^*(Ax_n - t_n) = 0, \quad \forall n \geq 0. \tag{16}$$

Lemma 3.3. *Let $\{u_n\}$ be the sequence generated by*

$$\begin{cases} s_0 \in H_1, \\ u_n = (1 - \alpha_n)s_n + \alpha_n T s_n, \quad \forall n \geq 0, \end{cases} \tag{17}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$. Then we have

$$\|u_n - x^*\|^2 \leq \|s_n - x^*\|^2 - \alpha_n(1 - k - \alpha_n)\|(T - I)s_n\|^2, \quad \forall n \geq 0,\ x^* \in F(T).$$

Proof. Let $x^* \in F(T)$. Since T is a k-demicontractive mapping, by Lemma 2.4 (1), we have

$$\begin{aligned}\|u_n - x^*\|^2 &= \|(1 - \alpha_n)(s_n - x^*) + \alpha_n(Ts_n - x^*)\|^2 \\ &= (1 - \alpha_n)\|s_n - x^*\|^2 + \alpha_n\|Ts_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|(T - I)s_n\|^2 \\ &\leq (1 - \alpha_n)\|s_n - x^*\|^2 + \alpha_n\|s_n - x^*\|^2 + \alpha_n k\|Ts_n - s_n\|^2 \\ &\quad - \alpha_n(1 - \alpha_n)\|(T - I)s_n\|^2 \\ &= \|s_n - x^*\|^2 - \alpha_n(1 - k - \alpha_n)\|(T - I)s_n\|^2.\end{aligned} \tag{18}$$

This completes the proof. □

Now, we introduce the hybrid extragradient algorithm for solving the split pseudomonotone equilibrium problem and the fixed point problem of a k-demicontractive mapping.

Algorithm 3.1. Initialization. Choose $\{\lambda_n\}$, $\{\beta_n\} \subseteq (0,\infty)$, $\{\alpha_n\} \subseteq (0,1)$, $\{\theta_n\} \subseteq [0,\infty)$. Take $x_1 = w_0 \in H_1$ and for $n \geq 1$.

Step 1. Solve the strongly convex problem:

$$\begin{cases} y_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|y - P_{C_2}Ax_n\|^2 + \beta_n f_2(P_{C_2}Ax_n, y) : y \in C_2\Big\}, \\ t_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|t - P_{C_2}Ax_n\|^2 + \beta_n f_2(y_n, t) : t \in C_2\Big\}. \end{cases} \tag{19}$$

Step 2. Compute v_n using

$$v_n = P_{C_1}[x_n - \gamma_n A^*(Ax_n - t_n)], \tag{20}$$

where γ_n is chosen such that $\{\gamma_n\}$ is bounded and there exists $\varepsilon > 0$ such that

$$\gamma_n \in \left[\varepsilon, \frac{\|Ax_n - t_n\|^2}{2\|A^*(Ax_n - t_n)\|^2}\right], \ \ n \in \Gamma = \{k : Ax_k - t_k \neq 0\}. \tag{21}$$

Otherwise, $\gamma_n = \gamma$, where γ is a nonnegative real number.

Step 3. Solve the strongly convex problem:

$$\begin{cases} z_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|z - v_n\|^2 + \lambda_n f_1(v_n, z) : z \in C_1\Big\}, \\ w_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|w - v_n\|^2 + \lambda_n f_1(z_n, w) : w \in C_1\Big\}. \end{cases} \tag{22}$$

Step 4. If $x_n = Tx_n$, $y_n = Ax_n$ and $z_n = x_n$, then $x_n \in Fix(T) \cap \Omega$ and **stop**. Otherwise, go to Step 5.

Step 5. Compute s_n, u_n and x_{n+1} using

$$\begin{cases} s_n = w_n + \theta_n(w_n - w_{n-1}), \\ u_n = (1 - \alpha_n)s_n + \alpha_n Ts_n, \\ x_{n+1} = P_{D_n \cap Q_n}(x_1), \end{cases} \tag{23}$$

where

$$D_n = \{p \in H_1 : \|u_n - p\|^2 \leq \|x_n - p\|^2 + 2\theta_n\langle w_n - p, w_n - w_{n-1}\rangle + \theta_n^2\|w_n - w_{n-1}\|^2\} \tag{24}$$

and

$$Q_n = \{p \in H_1 : \langle x_n - p, x_1 - x_n\rangle \geq 0\}. \tag{25}$$

Then update $n := n + 1$ and go to Step 1.

Lemma 3.4. *If $x_n = Tx_n$, $y_n = Ax_n$ and $z_n = x_n$ in Algorithm 3.1, then $x_n \in Fix(T) \cap \Omega$.*

Proof. Since $x_n = Tx_n$, we get $x_n \in Fix(T)$. By (9), we see that

$$\lambda_n f_2(Ax_n, y) = \lambda_n[f_2(P_{C_2}Ax_n, y) - f_2(P_{C_2}Ax_n, y_n)] \geq \langle y_n - P_{C_2}Ax_n, y_n - y\rangle = 0, \ \ \forall y \in C_2. \tag{26}$$

Since $\lambda_n > 0$ for all $n \geq 0$, we have $Ax_n \in EP(f_2)$. Since $y_n = Ax_n$ and $z_n = x_n$, we get $t_n = Ax_n$ and $z_n = x_n = v_n$. Similarly, we can prove that $x_n \in EP(f_1)$. Therefore $x_n \in Fix(T) \cap \Omega$. This completes the proof. □

Lemma 3.5. *Let $\{x_n\}$ be a sequence in Algorithm 3.1 satisfying the following conditions:*

(a) $0 < \alpha_n < 1 - k$ *for all* $n \geq 1$;

(b) $0 < \lambda_n < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ *for all* $n \geq 0$;

(c) $0 < \beta_n < \min\left\{\frac{1}{2b_1}, \frac{1}{2b_2}\right\}$ *for all* $n \geq 0$.

Then $\{x_n\}$ is well defined and $Fix(T) \cap \Omega \subseteq D_n \cap Q_n$ for all $n \geq 1$.

Proof. It is easy to see that Q_n is closed and convex. By Lemma 2.5, it follows that D_n is closed and convex. So, we have $D_n \cap Q_n$ is closed and convex for all $n \geq 1$.

Let $x^* \in Fix(T) \cap \Omega$. By Lemma 3.3 and the condition on α_n, we have

$$\begin{aligned} \|u_n - x^*\|^2 &\leq \|s_n - x^*\|^2 - \alpha_n(1 - k - \alpha_n)\|(T - I)s_n\|^2 \\ &= \|w_n + \theta_n(w_n - w_{n-1}) - x^*\|^2 - \alpha_n(1 - k - \alpha_n)\|(T - I)s_n\|^2 \\ &\leq \|w_n - x^*\|^2 + 2\theta_n\langle w_n - x^*, w_n - w_{n-1}\rangle + \theta_n^2\|w_n - w_{n-1}\|^2 \\ &\quad - \alpha_n(1 - k - \alpha_n)\|(T - I)s_n\|^2 \\ &\leq \|w_n - x^*\|^2 + 2\theta_n\langle w_n - x^*, w_n - w_{n-1}\rangle + \theta_n^2\|w_n - w_{n-1}\|^2. \end{aligned} \tag{27}$$

By (10) and the condition on λ_n, we have

$$\|w_n - x^*\|^2 \leq \|v_n - x^*\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2 - (1 - 2\lambda_n c_1)\|v_n - z_n\|^2 \leq \|v_n - x^*\|^2. \tag{28}$$

By (13) and the condition on γ_n, we have

$$\begin{aligned} \|v_n - x^*\|^2 &= \|P_{C_1}[x_n - \gamma_n A^*(Ax_n - t_n)] - P_{C_1}x^*\|^2 \\ &\leq \|x_n - \gamma_n A^*(Ax_n - t_n) - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \gamma_n\left[\|Ax_n - t_n\|^2 - \gamma_n\|A^*(Ax_n - t_n)\|^2\right] \\ &\leq \|x_n - x^*\|^2. \end{aligned} \tag{29}$$

From (27), (28) and (29), it follows that

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 + 2\theta_n\langle w_n - x^*, w_n - w_{n-1}\rangle + \theta_n^2\|w_n - w_{n-1}\|^2, \quad \forall n \geq 1, \tag{30}$$

that is, $x^* \in D_n$ for all $n \geq 1$. So, we have $Fix(T) \cap \Omega \subseteq D_n, \ \forall n \geq 1$.

Next, we show, by induction, that $\{x_n\}$ is well defined and $Fix(T) \cap \Omega \subseteq D_n \cap Q_n$ for all $n \geq 1$. For $n = 1$, we have $Q_1 = H_1$ and hence $Fix(T) \cap \Omega \subseteq D_1 \cap Q_1$. Suppose that $Fix(T) \cap \Omega \subseteq D_k \cap Q_k$ for some $k \geq 1$. There exists a unique element $x_{k+1} \subseteq D_k \cap Q_k$ such that $x_{k+1} = P_{D_k \cap Q_k}(x_1)$ is equivalent to

$$\langle x_{k+1} - x, x_1 - x_{k+1}\rangle \geq 0, \quad \forall x \in D_k \cap Q_k. \tag{31}$$

Since $Fix(T) \cap \Omega \subseteq D_k \cap Q_k$, we get $\langle x_{k+1} - x, x_1 - x_{k+1}\rangle \geq 0, \ \forall x \in Fix(T) \cap \Omega$ and hence $Fix(T) \cap \Omega \subseteq Q_{k+1}$. Therefore, by induction, we have $Fix(T) \cap \Omega \subseteq D_{k+1} \cap Q_{k+1}$. This completes the proof. □

Theorem 3.6. *If the sequences $\{\beta_n\}$, $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\alpha_n\}$ satisfy the following conditions: for some positive real numbers a_i for each $i = 1, \cdots, 6$,*

(C1) $\{\beta_n\} \subseteq [a_1, a_2] \subseteq \left(0, \min\left\{\frac{1}{2b_1}, \frac{1}{2b_2}\right\}\right)$;

(C2) $\{\lambda_n\} \subseteq [a_3, a_4] \subseteq \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right)$;

(C3) $\{\alpha_n\} \subseteq [a_5, a_6] \subseteq (0, 1 - k)$;

(C4) $\{\theta_n\} \subseteq [0, \infty)$ *and* $\lim_{n\to\infty} \theta_n = 0$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $P_{Fix(T)\cap\Omega}(x_1)$.

Proof. By Lemma 3.4, we assume that the stop criterion at Step 4 can not be satisfied for all $n \geq 1$. Since $\Omega \cap Fix(T)$ is a nonempty closed convex subset of H_1, there exists a unique element $z_1 \in \Omega \cap Fix(T)$ such that

$$z_1 = P_{\Omega \cap Fix(T)}(x_1). \tag{32}$$

From $x_{n+1} = P_{D_n \cap Q_n}(x_1)$, we have

$$\|x_{n+1} - x_1\| \leq \|p - x_1\|, \ \ \forall p \in D_n \cap Q_n. \tag{33}$$

Since $z_1 \in \Omega \cap Fix(T) \subseteq D_n \cap Q_n$, we have

$$\|x_{n+1} - x_1\| \leq \|z_1 - x_1\|, \ \ \forall n \geq 1. \tag{34}$$

This implies that $\{x_n\}$ is bounded. Otherwise, for each $p \in Q_n$, we have

$$\langle x_n - p, x_1 - x_n \rangle \geq 0, \ \ \forall n \geq 1, \tag{35}$$

and hence $x_n = P_{Q_n}(x_1)$. Since $x_{n+1} \in D_n \cap Q_n \subseteq Q_n$, we have

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \ \ \forall n \geq 1. \tag{36}$$

So, the sequence $\{\|x_n - x_1\|\}$ is bounded and non-decreasing and so
$\lim_{n\to\infty} \|x_n - x_1\|$ exists. Since $x_{n+1} \in Q_n$, we have $\langle x_n - x_{n+1}, x_1 - x_n \rangle \geq 0$ and so

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_1\|^2 + \|x_{n+1} - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_1 \rangle \\ &= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_n \rangle \\ &\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2. \end{aligned} \tag{37}$$

This implies that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{38}$$

From $x_{n+1} = P_{D_n \cap Q_n}(x_1)$, it follows that $x_{n+1} \in D_n$, i.e.,

$$\|u_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + 2\theta_n \langle w_n - x_{n+1}, w_n - w_{n-1} \rangle + \theta_n^2 \|w_n - w_{n-1}\|^2. \tag{39}$$

Since $\{x_n\}$ is bounded, we also have $\{u_n\}$, $\{w_n\}$ and $\{v_n\}$ are bounded. By (38) and $\lim_{n\to\infty} \theta_n = 0$, we get

$$\lim_{n\to\infty} \|u_n - x_{n+1}\| = 0. \tag{40}$$

Hence we have

$$\|u_n - x_n\| \leq \|u_n - x_{n+1}\| + \|x_n - x_{n+1}\| \to 0 \text{ as } n \to \infty. \tag{41}$$

Let $x^* \in Fix(T) \cap \Omega$. By (27), (28) and (29), we obtain

$$\begin{aligned} \|u_n - x^*\|^2 \leq{}& \|x_n - x^*\|^2 + 2\theta_n \langle w_n - x^*, w_n - w_{n-1} \rangle + \theta_n^2 \|w_n - w_{n-1}\|^2 - \alpha_n(1 - k - \alpha_n)\|(T - I)s_n\|^2 \\ & - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2 - (1 - 2\lambda_n c_1)\|v_n - z_n\|^2 - \gamma_n \left[\|Ax_n - t_n\|^2 - \gamma_n \|A^*(Ax_n - t_n)\|^2 \right]. \end{aligned} \tag{42}$$

From (42), it follows that

$$\begin{aligned} \alpha_n(1 - k - \alpha_n)\|(T - I)s_n\|^2 &\leq \|x_n - x^*\|^2 - \|u_n - x^*\|^2 + 2\theta_n \langle w_n - x^*, w_n - w_{n-1} \rangle + \theta_n^2 \|w_n - w_{n-1}\|^2 \\ &= (\|x_n - x^*\| + \|u_n - x^*\|)\|x_n - u_n\| \\ &\quad + 2\theta_n \langle w_n - x^*, w_n - w_{n-1} \rangle + \theta_n^2 \|w_n - w_{n-1}\|^2. \end{aligned} \tag{43}$$

By (41) and the conditions on α_n, θ_n, we get

$$\lim_{n\to\infty} \|(T-I)s_n\| = 0 \tag{44}$$

and we have

$$\lim_{n\to\infty} \|u_n - s_n\| = 0. \tag{45}$$

Similarly, we have

$$\lim_{n\to\infty} \|w_n - z_n\| = \lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{46}$$

By (41) and (45), we get

$$\|x_n - s_n\| \le \|x_n - u_n\| + \|u_n - s_n\| \to 0 \text{ as } n \to \infty. \tag{47}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x} \in H_1$ as $i \to \infty$. By (47), we also have $s_{n_i} \rightharpoonup \bar{x} \in H_1$ as $i \to \infty$. Using (44) and the demiclosedness of $T - I$, we have $\bar{x} \in Fix(T)$.

If Γ is finite, then $Ax_n - t_n = 0$ for all $n \in \mathbb{N}\backslash\Gamma$. It follows from Remark 3.2 that

$$\lim_{n\to\infty} \|Ax_n - t_n\| = \lim_{n\to\infty} \|A^*(Ax_n - t_n)\| = 0.$$

Suppose that Γ is infinite. It is noted that, if $n \notin \Gamma$, then we have

$$\lim_{n\to\infty} \|Ax_n - t_n\| = \lim_{n\to\infty} \|A^*(Ax_n - t_n)\| = 0.$$

For each $n \in \Gamma$, again, from (42) and the condition of γ_n, it follows that

$$\begin{aligned}
\frac{\varepsilon}{2}\|Ax_n - t_n\|^2 &\le \frac{\gamma_n}{2}\|Ax_n - t_n\|^2 \\
&\le \gamma_n \left[\|Ax_n - t_n\|^2 - \gamma_n\|A^*(Ax_n - t_n)\|^2\right] \\
&\le (\|x_n - x^*\| + \|u_n - x^*\|)\|x_n - u_n\| + 2\theta_n\langle w_n - x^*, w_n - w_{n-1}\rangle + \theta_n^2\|w_n - w_{n-1}\|^2.
\end{aligned} \tag{48}$$

By (41) and the conditions on θ_n, we get

$$\lim_{n\to\infty} \|Ax_n - t_n\| = 0 \tag{49}$$

and then

$$\lim_{n\to\infty} \|A^*(Ax_n - t_n)\| = 0. \tag{50}$$

Since P_{C_1} is firmly nonexpansive, it follows from (13) and (29) that

$$\begin{aligned}
\|v_n - x^*\|^2 &= \|P_{C_1}[x_n - \gamma_n A^*(Ax_n - t_n)] - x^*\|^2 \\
&\le \langle v_n - x^*, x_n - \gamma_n A^*(Ax_n - t_n) - x^*\rangle \\
&= \frac{1}{2}\left[\|v_n - x^*\|^2 + \|x_n - \gamma_n A^*(Ax_n - t_n) - x^*\|^2\right] - \frac{1}{2}\|(v_n - x_n) + \gamma_n A^*(Ax_n - t_n)\|^2 \\
&\le \|x_n - x^*\|^2 - \frac{1}{2}\|v_n - x_n\|^2 - \langle v_n - x_n, \gamma_n A^*(Ax_n - t_n)\rangle - \frac{\gamma_n^2}{2}\|A^*(Ax_n - t_n)\|^2.
\end{aligned} \tag{51}$$

By (27), (28) and (51), we have

$$\begin{aligned}
\|u_n - x^*\|^2 &\le \|v_n - x^*\|^2 + 2\theta_n\langle w_n - x^*, w_n - w_{n-1}\rangle + \theta_n^2\|w_n - w_{n-1}\|^2 \\
&\le \|x_n - x^*\|^2 + 2\theta_n\langle w_n - x^*, w_n - w_{n-1}\rangle + \theta_n^2\|w_n - w_{n-1}\|^2 - \frac{\gamma_n^2}{2}\|A^*(Ax_n - t_n)\|^2 \\
&\quad - \frac{1}{2}\|v_n - x_n\|^2 - \langle v_n - x_n, \gamma_n A^*(Ax_n - t_n)\rangle.
\end{aligned} \tag{52}$$

This implies that

$$\frac{1}{2}\|v_n - x_n\|^2 \leq (\|x_n - x^*\| + \|u_n - x^*\|)\|x_n - u_n\| + 2\theta_n\langle w_n - x^*, w_n - w_{n-1}\rangle \\ + \theta_n^2\|w_n - w_{n-1}\|^2 - \langle v_n - x_n, \gamma_n A^*(Ax_n - t_n)\rangle. \quad (53)$$

By (41), (50) and the condition on θ_n, we have

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \quad (54)$$

By (46) and (54), we get

$$\|x_n - z_n\| \leq \|x_n - v_n\| + \|v_n - z_n\| \to 0 \text{ as } n \to \infty, \quad (55)$$

and

$$\|x_n - w_n\| \leq \|x_n - z_n\| + \|z_n - w_n\| \to 0 \text{ as } n \to \infty. \quad (56)$$

By (54), (55) and (56), we have $v_{n_i} \rightharpoonup \bar{x} \in C_1$, $z_{n_i} \rightharpoonup \bar{x} \in C_1$, $w_{n_i} \rightharpoonup \bar{x} \in C_1$, respectively. Now, we show that $\bar{x} \in \Omega$. By (9), we have

$$\lambda_{n_i}[f_1(v_{n_i}, z) - f_1(v_{n_i}, z_{n_i})] \geq \langle z_{n_i} - v_{n_i}, z_{n_i} - z\rangle, \ \ \forall z \in C_1. \quad (57)$$

Taking $i \to \infty$ in (57), from $(b1), (b4)$, (46) and the condition on λ_n, it follows that

$$f_1(\bar{x}, z) \geq 0, \ \ \forall z \in C, \quad (58)$$

that is, $\bar{x} \in EP(f_1)$. Using (49), we get

$$\|P_{C_2}Ax_n - t_n\| = \|P_{C_2}Ax_n - P_{C_2}t_n\| \leq \|Ax_n - t_n\| \to 0 \text{ as } n \to \infty. \quad (59)$$

By (10), we have

$$\|t_n - Ax^*\|^2 \leq \|P_{C_2}Ax_n - Ax^*\|^2 - (1 - 2\beta_n b_2)\|t_n - y_n\|^2 - (1 - 2\beta_n b_1)\|P_{C_2}Ax_n - y_n\|^2. \quad (60)$$

Hence we have

$$(1 - 2\beta_n b_1)\|P_{C_2}Ax_n - y_n\|^2 \leq \|P_{C_2}Ax_n - Ax^*\|^2 - \|t_n - Ax^*\|^2 \\ \leq (\|P_{C_2}Ax_n - Ax^*\| + \|t_n - Ax^*\|)\|\|P_{C_2}Ax_n - t_n\|. \quad (61)$$

By (59) and the condition on β_n, we get

$$\lim_{n\to\infty} \|P_{C_2}Ax_n - y_n\| = 0. \quad (62)$$

Similarly, we have

$$\lim_{n\to\infty} \|t_n - y_n\| = 0. \quad (63)$$

By (49) and (63), we have

$$\|y_n - Ax_n\| \leq \|y_n - t_n\| + \|t_n - Ax_n\| \to 0 \text{ as } n \to \infty. \quad (64)$$

Since A is a bounded linear and $x_{n_i} \rightharpoonup \bar{x} \in H_1$, we have $Ax_{n_i} \rightharpoonup A\bar{x} \in H_2$. Since $\{y_n\} \subseteq C_2$ and (64), we have $y_{n_i} \rightharpoonup A\bar{x} \in C_2$. Using (62), we get $P_{C_2}Ax_n \rightharpoonup A\bar{x} \in C_2$. By (9), we have

$$\beta_{n_i}[f_2(P_{C_2}Ax_{n_i}, z) - f_2(P_{C_2}Ax_{n_i}, y_{n_i})] \geq \langle y_{n_i} - P_{C_2}Ax_{n_i}, P_{C_2}Ax_{n_i} - y\rangle, \ \ \forall y \in C_2. \quad (65)$$

Taking $i \to \infty$ in (65), it follows from (b1), (b4), (62) and the condition on β_n that

$$f_2(A\bar{x}, y) \geq 0, \ \ \forall y \in C_2, \quad (66)$$

that is, $A\bar{x} \in EP(f_2)$. Therefore, we have $\bar{x} \in Fix(T) \cap \Omega$, i.e., $\omega_w(x_n) \subseteq Fix(T) \cap \Omega$. Therefore, it follows the inequality (34) and Lemma 2.6 that $\{x_n\} \to P_{Fix(T)\cap\Omega}(x_1)$ as $n \to \infty$. This completes the proof. □

If we set $\theta_n = 0$ for all $n \geq 1$ in Algorithm 3.1, then we obtain the following result for the split pseudomonotone equilibrium problem and the fixed point problem of a demicontractive mapping:

Corollary 3.7. *Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in H_1 \\ y_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|y - P_{C_2}Ax_n\|^2 + \beta_n f_2(P_{C_2}Ax_n, y) : y \in C_2\Big\}, \\ t_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|t - P_{C_2}Ax_n\|^2 + \beta_n f_2(y_n, t) : t \in C_2\Big\}, \\ v_n = P_{C_1}[x_n - \gamma_n A^*(Ax_n - t_n)], \\ z_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|z - v_n\|^2 + \lambda_n f_1(v_n, z) : z \in C_1\Big\}, \\ w_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|w - v_n\|^2 + \lambda_n f_1(z_n, w) : w \in C_1\Big\}, \\ u_n = (1 - \alpha_n)w_n + \alpha_n T w_n, \\ D_n = \{p \in H_1 : \|u_n - p\| \leq \|x_n - p\|\}, \\ Q_n = \{p \in H_1 : \langle x_n - p, x_1 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{D_n \cap Q_n}(x_1), \quad n \geq 1, \end{cases} \tag{67}$$

where $\{\gamma_n\}$ is bounded and satisfies the condition (21)*. If $\{\beta_n\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the following conditions: for some positive real number a_i for each $i = 1, \cdots, 6$,*

(C1) $\{\beta_n\} \subseteq [a_1, a_2] \subseteq \left(0, \min\left\{\frac{1}{2b_1}, \frac{1}{2b_2}\right\}\right)$;

(C2) $\{\lambda_n\} \subseteq [a_3, a_4] \subseteq \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right)$;

(C3) $\{\alpha_n\} \subseteq [a_5, a_6] \subseteq (0, 1 - k)$,

Then the sequence $\{x_n\}$ generated by (67) *converges strongly to $P_{Fix(T)\cap\Omega}(x_1)$.*

If we set $T = I$ in Algorithm 3.1, then we obtain the following result for the split pseudomonotone equilibrium:

Corollary 3.8. *Suppose that $\Omega \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in H_1 \\ y_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|y - P_{C_2}Ax_n\|^2 + \beta_n f_2(P_{C_2}Ax_n, y) : y \in C_2\Big\}, \\ t_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|t - P_{C_2}Ax_n\|^2 + \beta_n f_2(y_n, t) : t \in C_2\Big\}, \\ v_n = P_{C_1}[x_n - \gamma_n A^*(Ax_n - t_n)], \\ z_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|z - v_n\|^2 + \lambda_n f_1(v_n, z) : z \in C_1\Big\}, \\ w_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|w - v_n\|^2 + \lambda_n f_1(z_n, w) : w \in C_1\Big\}, \\ u_n = w_n + \theta_n(w_n - w_{n-1}), \\ x_{n+1} = P_{D_n \cap Q_n}(x_1), \quad n \geq 1, \end{cases} \tag{68}$$

where $D_n = \{p \in H_1 : \|u_n - p\|^2 \leq \|x_n - p\|^2 + 2\theta_n\langle w_n - p, w_n - w_{n-1}\rangle\theta_n^2\|w_n - w_{n-1}\|^2\}$, $Q_n = \{p \in H_1 : \langle x_n - p, x_1 - x_n\rangle \geq 0\}$ and $\{\gamma_n\}$ is bounded and satisfies the condition (21)*. If $\{\beta_n\}$, $\{\lambda_n\}$ and $\{\theta_n\}$ satisfy the following conditions: for some positive real number a_i for each $i = 1, \cdots, 4$,*

(C1) $\{\beta_n\} \subseteq [a_1, a_2] \subseteq \left(0, \min\left\{\frac{1}{2b_1}, \frac{1}{2b_2}\right\}\right)$;
(C2) $\{\lambda_n\} \subseteq [a_3, a_4] \subseteq \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right)$;
(C3) $\{\theta_n\} \subseteq (-\infty, \infty)$ *and* $\lim_{n\to\infty} \theta_n = 0$.

Then the sequence $\{x_n\}$ *generated by* (68) *converges strongly to* $P_\Omega(x_1)$.

If we set $f_2 = 0$ and $C_2 = H_2$ in Algorithm 3.1, So, we obtain the following result for the pseudomonotone equilibrium and the fixed point problem of a demicontractive mapping:

Corollary 3.9. *Suppose that* $Fix(T) \cap EP(f_1) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 = w_0 \in H_1, \\ z_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|z - P_{C_1}x_n\|^2 + \lambda_n f_1(P_{C_1}x_n, z) : z \in C_1\Big\}, \\ w_n = \operatorname{argmin}\Big\{\dfrac{1}{2}\|w - P_{C_1}x_n\|^2 + \lambda_n f_1(z_n, w) : w \in C_1\Big\}, \\ s_n = w_n + \theta_n(w_n - w_{n-1}), \\ u_n = (1-\alpha_n)s_n + \alpha_n T s_n, \\ x_{n+1} = P_{D_n \cap Q_n}(x_1), \ \ n \geq 1, \end{cases} \tag{69}$$

where $D_n = \{p \in H_1 : \|u_n - p\|^2 \leq \|x_n - p\|^2 + 2\theta_n\langle w_n - p, w_n - w_{n-1}\rangle + \theta_n^2\|w_n - w_{n-1}\|^2\}, Q_n = \{p \in H_1 : \langle x_n - p, x_1 - x_n\rangle \geq 0\}$. *If* $\{\lambda_n\}$, $\{\alpha_n\}$ *and* $\{\theta_n\}$ *satisfy the following conditions: for some positive real number* a_i *for each* $i = 1, \cdots, 4$,

(C1) $\{\lambda_n\} \subseteq [a_1, a_2] \subseteq \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right)$;
(C2) $\{\alpha_n\} \subseteq [a_3, a_4] \subseteq (0, 1-k)$;
(C3) $\{\theta_n\} \subseteq (-\infty, \infty)$ *and* $\lim_{n\to\infty} \theta_n = 0$.

Then the sequence $\{x_n\}$ *generated by* (69) *converges strongly to* $P_{Fix(T)\cap EP(f_1)}(x_1)$.

4. Numerical Experiments

Now, we present a numerical experiment for supporting our main theorems, where all codes were written in Matlab and run on laptop Intel core i5, 4.00 GB RAM, windows 8 (64-bit).

Example 4.1. *Let* $H_1 = \mathbb{R}^5$, $H_2 = \mathbb{R}$ *and*

$$C_1 = \begin{cases} x = (x_1, x_2, \cdots, x_5)^T \in \mathbb{R}^5_+ := \{x \in \mathbb{R}^5_+ : x_i \geq 0, \forall i = 1, 2, \cdots, 5\}, \\ x_1 + x_2 + x_3 + 2x_4 + x_5 \leq 10, \\ 2x_1 + x_2 - x_3 + x_4 + 3x_5 \leq 15, \\ x_1 + x_2 + x_3 + x_4 + 0.5x_5 \geq 4. \end{cases}$$

Define a bifunction $f_1 : C_1 \times C_1 \to \mathbb{R}$ *by* $f_1(x, y) = \langle Bx + \chi^5(y + x) + \mu - \alpha, y - x\rangle, \ \ \forall x, y \in C_1$, *where*

$$B = \begin{bmatrix} 0 & \chi & \chi & \chi & \chi \\ \chi & 0 & \chi & \chi & \chi \\ \chi & \chi & 0 & \chi & \chi \\ \chi & \chi & \chi & 0 & \chi \\ \chi & \chi & \chi & \chi & 0 \end{bmatrix}, \ \ \chi = 3, \ \ \alpha = (2, 2, 2, 2, 2)^T, \ \ \mu = (3, 4, 5, 7, 6)^T.$$

Then we have f_1 *is a pseudomonotone on* C_1, *but it is not monotone on* C_1 *(see [5]). It is known that* f_1 *is Lipschitz-type continuous on* C_1 *with the constants* $c_1 = c_2 = \frac{\|B\|_2}{2} = 6$. *Let* $C_2 = [0, 1]$. *Define a bifunction* $f_2 : C_2 \times C_2 \to \mathbb{R}$ *by*

$$f_2(x, y) = H(x)(y - x), \ \ \forall x, y \in C_2,$$

where

$$H(x) = \begin{cases} 0, & \text{if } 0 \le x \le \frac{1}{2}, \\ e^{(x-\frac{1}{2})} + \sin(x - \frac{1}{2}) - 1, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Then we have f_2 is a monotone on C_2 and Lipschitz-type continuous on C_2 with the constants $b_1 = b_2 = 2$ (see [23]). The linear operator $A : \mathbb{R}^5 \to \mathbb{R}$ is defined by $A(x) = \langle a, x\rangle$, where a is a vector in $\mathbb{R}^5$ whose elements are randomly generated in $[1,5]$. Thus $A^*(y) = y \cdot a$ for all $y \in \mathbb{R}$. Define the mapping $T : \mathbb{R}^5 \to \mathbb{R}^5$ by

$$T(x) = \begin{cases} x, & \text{if } x \in (-\infty, 0], \\ -2x, & \text{if } x \in [0, \infty), \end{cases}$$

for all $x = (x_1, x_2, \cdots, x_5)^T \in \mathbb{R}^5$. Then T is $\frac{1}{3}$-demicontractive mapping, but it is not quasi-nonexpansive mapping. By Algorithm 3.1, we have the following:

Step 1. *Solve the strong convex problem:*

$$y_n = \operatorname{argmin}\Big\{\frac{1}{2}(y - P_{[0,1]}Ax_n)^2 + \beta_n H(P_{[0,1]}Ax_n)(y - P_{[0,1]}Ax_n) : y \in [0,1]\Big\}, \tag{70}$$

where $\beta_n = \frac{n}{100n-1}$ for all $n \ge 1$. A simple computation shows that (70) is equivalent to the following:

$$y_n = P_{[0,1]}Ax_n - \beta_n H(P_{[0,1]}Ax_n), \ \ \forall n \ge 1.$$

Similarly, we get $t_n = P_{[0,1]}Ax_n - \beta_n H(y_n), \ \forall n \ge 1$.
Step 2. *Compute v_n using*

$$v_n = P_{C_1}[x_n - \gamma_n(Ax_n - t_n) \cdot a], \ \ \forall n \ge 1,$$

where $a = (1,1,1,1,1)^T \in \mathbb{R}^5$ and $\gamma_n = \frac{1}{100\|a\|_2^2}, \ \forall n \ge 1$.
Step 3. *Solve the strong convex problem:*

$$z_n = \operatorname{argmin}\Big\{\frac{1}{2}\|z - v_n\|_2^2 + \lambda_n\langle Bv_n + \chi^5(z + v_n) + \mu - \alpha, z - v_n\rangle : z \in C_1\Big\}$$

and

$$w_n = \operatorname{argmin}\Big\{\frac{1}{2}\|z - v_n\|_2^2 + \lambda_n\langle Bz_n + \chi^5(z + z_n) + \mu - \alpha, z - z_n\rangle : z \in C_1\Big\},$$

where $\lambda_n = \frac{n}{100n-1}$ for all $n \ge 1$.
Step 4. *Compute s_n, u_n and x_{n+1} where $\theta_n = \frac{1}{100n}$ and $\alpha_n = \frac{n}{3n-1}$ for all $n \ge 1$.*

In the experiment, we choose the stopping criterion is $E_n =: \|x_n\|_2 < 10^{-10}$, Time (s) is the average of execution times and Iter. := Number of iterations. So, the numerical result and the graph of error are shown in the Table 1 and Figure 1.

Table 1: Numerical result of Algorithm 3.1 with start $x_1 = w_0 = (0.5, 1, 0.5, 3, 2)^T$.

Time (s)	Iter.	Approximate solution	E_n
0.4493	1	$(0.3040, 0.7648, 1.2256, 0.7710, 0.3893)^T$	1.7104
	2	$(0.1525, 0.3083, 0.4640, 0.2985, 0.1472)^T$	0.6666
	3	$(2.6\times10^{-7}, 0.0081, 0.2664, 0.0576, 0.0465)^T$	0.2766
	4	$(2.8\times10^{-6}, 0.2445, 4.6\times10^{-7}, 2.4\times10^{-6}, 2.6\times10^{-6})^T$	0.2445
	5	$(0.0001, 0.0352, 0.0897, 0.0389, 0.0204)^T$	0.1059
	6	$(7.8\times10^{-11}, 4.2\times10^{-11}, 2.7\times10^{-11}, 4.2\times10^{-11}, 8.3\times10^{-11})^T$	1.3×10^{-10}
	7	$(3.9\times10^{-11}, 4.1\times10^{-11}, 4.1\times10^{-11}, 4.2\times10^{-11}, 8.3\times10^{-11})^T$	1.2×10^{-10}
	8	$(3.6\times10^{-11}, 3.9\times10^{-11}, 4.3\times10^{-11}, 3.9\times10^{-11}, 8.1\times10^{-11})^T$	1.1×10^{-10}
	9	$(5.1\times10^{-11}, 4.7\times10^{-11}, 4.3\times10^{-11}, 4.8\times10^{-11}, 9.9\times10^{-11})^T$	1.3×10^{-10}
	10	$(2.6\times10^{-11}, 3.3\times10^{-13}, 2.2\times10^{-11}, 2.6\times10^{-11}, 5.9\times10^{-11})^T$	7.3×10^{-11}

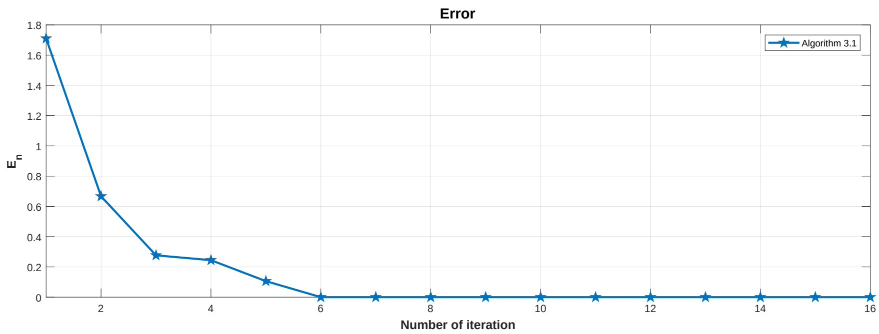

Figure 1: Graph of error for Example 4.1

Example 4.2. *Let $H = \mathbb{R}^2$ and $C = \{(x_1, x_2) : x_i \geq 0\ \forall i = 1, 2\}$. Define a bifunction $f : C \times C \to \mathbb{R}$ by $f(x, y) = 2(y_2 - x_2)\|x\|_2$, for all $x = (x_1, x_2), y = (y_1, y_2) \in C$. Define the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = -0.9x$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. The stopping criterion is given by $E_n =: \|x_n\|_2 < 10^{-4}$. Choose $\theta_n = \frac{1}{10n}, \alpha_n = 0.6$ and $\lambda_n = \frac{n}{100n-1}$. So, the comparison of numerical results between Anh Algorithm [3] and Corollary 3.9 are shown in the Table 3.9 and Figure* **??**.

Table 2: Comparison of numerical results between Anh Algorithm and Corollary 3.9.

		Anh Algorithm [3]		**Corollary 3.9**	
Case	**Starting points**	**Iter.**	**Tims (s)**	**Iter.**	**Tims (s)**
1	$x_1 = w_0 = (2, 0.8)$	774	0.2322	27	0.3431
2	$x_1 = w_0 = (1, 2)$	1708	0.2203	56	0.3057
3	$x_1 = w_0 = (1.5, 0.7)$	1233	0.2280	37	0.3478
4	$x_1 = w_0 = (1, 5)$	867	0.2335	53	0.3389

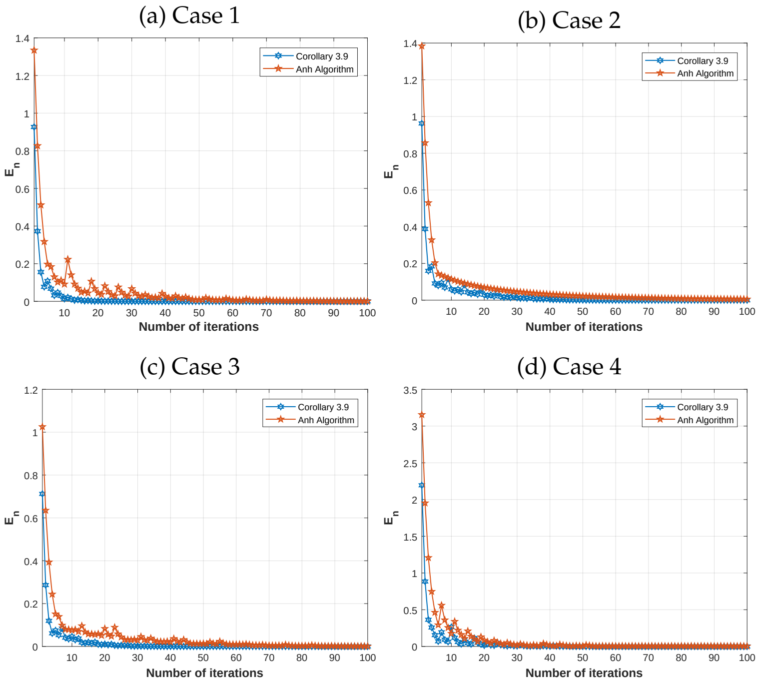

Figure 2: Plot of error by Anh Algorithm and Corollary 3.9.

5. Conclusion

In this paper, we proposed a new hybrid extragradient method for solving a common solutions of the fixed point problem of a demicontractive mapping and the split equilibrium problem for a pseudomonotone and Lipschitz-type continuous bifunction and proved some strong convergence results of the proposed method under some control conditions. Moreover, we gave some numerical experiments to support our main results. The novelty of this paper is as follows:

(1) We introduced a new method for solving a common solutions of the fixed point problem of a demicontractive mapping and the split equilibrium problem for a pseudomonotone and Lipschitz-type continuous bifunction;

(2) We obtained some strong convergence results of our proposed algorithm which is more desirable than the methods of Tran et al. [32] and Anh [3];

(3) Finally, we gave some examples to illustrate our main results and the comparison of the methods of Anh [3] with our method.

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