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Asymptotic stability of stochastic differential equations driven by G-Lévy process with delay feedback control

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Abstract. Given an unstable stochastic differential equations, the stabilisation by delay feedback controls for such equations under Lipschitz conditions or highly nonlinear conditions have been discussed by several authors. However, there is few works on the stabilisation by delay feedback controls under the sub-linear expectation associated with a G-Lévy process. The aim of this paper is to design delay feedback controls in the drift part and obtain the asymptotical stability in mean square and quasi-surely asymptotical stability for the stochastic differential equations driven by G-Lévy process with the polynomial growth condition. Lastly, we give an example to verify the obtained theory.

1. Introduction

Non-additive expectations and non-additive probabilities are important tools for studying uncertainties in statistics, measures of risk, and non-linear stochastic calculus (see, for example, Denis and Martini [4], Marinacci [22]). Recently, Peng [29] introduce a notion of the sub-linear expectation which is generated by one dimensional fully nonlinear heat equation, called G-heat equation. Under the sub-linear expectation, a new type of G-Brownian motion and the related calculus of Itôs type were introduced ([29–31]). G-Brownian motion has a very rich and interesting new structure which non-trivially generalizes the classical Brownian motion, there have been some interesting works (see, for example, Denis, Hu and Peng [5], Gao [9], Gao and Jiang [10], Soner, Touzi and Zhang [38], Li and Peng [16], Bai and Lin [2], Zhang [42], Zhu and Huang [43] and the references therein).

On the other hand, one feels that G-Brownian motion is not sufficient to model the financial world, as both G-Brownian motion and the standard Brownian motion share the same property, which makes them often unsuitable for modelling, namely the continuity of paths. Therefore, the natural generalization of G-Brownian motion is to consider a jump processes and the uncertainty associated with the drift, the volatility and the jump component. Hu and Peng [14] introduced the process with jumps, which they called G-Lévy process and studied the distribution property, i.e., Lévy-Khintchine formula, of a Lévy process under sub-linear expectations. However, in contrast to the extensive studies on G-Brownian motion, there has been little systematic development on G-Lévy process in the literature. The main reason is the complexity of dependence structures for G-Lévy process. To the best of our knowledge, we only

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find that Ren [34] considered the representation of a sub-linear expectation associated with G-Lévy process. Paczka [25] considered the integration theory for G-Lévy processes with finite activity. Paczka [26] studied the properties of the Poisson random measure and the Poisson integral associated with a G-Lévy process. Wang and Yuan [39] obtained the existence of solution for stochastic differential equations driven by G-Lévy process with discontinuous coefficients. Qiao and Wu [33] proved that a kind of additive functionals of stochastic differential equations (SDEs) driven by G-Lévy processes has path independence under some assumptions. Faizullah et al. [6] obtained the exponential estimate for solutions of stochastic functional differential equations driven by G-Lévy process.

In the past decades, SDEs have come to play an important role in many branches of science and industry, such as biology, physics, economics, engineering and financial market (Gikhman and Skorokhod [11]). One of the important issues in the study of SDEs is the analysis of stability (see Mao [20]). Wang and Gao [40] considered the SDEs driven by G-Lévy Process and proposed the sufficient conditions for the mean exponential stability to the following SDEs:

$$dx(t) = f(t, x(t))dt + h(t, x(t))d\langle B\rangle(t) + \sigma(t, x(t))dB(t) + \int_{\mathbb{R}_0^d} K(t, x(t), z)L(dt, dz), t \geq 0. \tag{1.1}$$

Shen, Wu and Yin [37] gave sufficient conditions for the mean square exponential instability of the solution for the SDEs (1.1). Hence, a meaningful question is whether we can design a feedback control $u(t, x(t))$ based on the current state $x(t)$, so that the controlled system

$$dx(t) = [f(t, x(t)) + u(t, x(t)]dt + h(t, x(t))d\langle B\rangle(t) + \sigma(t, x(t))dB(t) + \int_{\mathbb{R}_0^d} K(t, x(t), z)L(dt, dz), \qquad t \geq 0,$$

becomes stable? However, taking into account a time lag $\tau(> 0)$ between the time when the observation of the state is made and the time when the feedback control reaches the system, it is more realistic that the control depends on a past state $x(t - \tau)$. Hence, the stabilisation problem becomes to design a delay feedback control $u(t, x(t - \tau))$ such that the controlled system (G-SDDEs, in short)

$$dx(t) = [f(t, x(t)) + u(t, x(t - \tau)]dt + h(t, x(t))d\langle B\rangle(t) + \sigma(t, x(t))dB(t) + \int_{\mathbb{R}_0^d} K(t, x(t), z)L(dt, dz), \quad t \geq 0, \tag{1.2}$$

is stable. Suppose that the underlying G-SDDEs with the initial data

$$\{x(t) : -\tau \leq t \leq 0\} = \xi \in C([-\tau, 0]; \mathbb{R}^n), \tag{1.3}$$

with $\widehat{\mathbb{E}}|\xi|^2 < \infty$. $B(\cdot)$ is d-dimensional G-Brownian motion, $\langle B\rangle(\cdot)$ is the quadratic variation process of the G-Brownian motion, $L(\cdot, \cdot)$ is a Poisson random measure associated with the G-Lévy process. The coefficients f, h, σ are in the space $M_G^2([0, T]; \mathbb{R}^n)$, $K \in H_G^2([0, T] \times \mathbb{R}_0^d; \mathbb{R}^n)$ for any $x \in \mathbb{R}^n$ (the precise definition are given in Section 2).

The main contributions of this paper are presented as follows:

(i) A class of unstable stochastic differential equations driven by G-Lévy process are stabilised via delay feedback controls in the drift part based on continuous observation.

(ii) The sufficient conditions on stabilisation criteria are obtained based on constructing appropriate G-Lyapunov function.

(iii) The stochastic calculus on G-Lévy process is applied to solve the stability of the systems.

Note that the ordinary differential equations with delay feedback controls have been well developed (see, for example, Ahlborn and Parlitz [1], Cao, Li and Ho [3], Pyragas [32]). On the other hand, there has been increasing interest and demanding for investigating stochastic differential equations with feedback control. Mao, Lam and Huang [21] were the first to study stabilisation problem for a given unstable hybrid stochastic differential equations with the delay feedback control, since then, there have been some interesting works (see, for example, Mao [19], You, Liu, Lu, Mao and Qiu [41], Lu, Hu and Mao [18], Li and Mao [17], Fei et al. [7], Mei et al. [23], Ren et al. [35], Shao [36], Li et al.[15], Hu et al. [12], just mention a few.)

The rest of this paper is organized as follows. In Section 2 we introduce preliminary results in the G-framework. In Section 3, we use the method of Lyapunov functionals to investigate the asymptotic stability of the solutions for the controlled system driven by G-Lévy process. Finally, in Section 4, we give an example to verify the obtained theory.

2. Preliminaties

In this section, we introduce briefly some notations about the G-framework, for the detail we can see Peng [29], Neufeld and Nutz [24], Paczka [27] and the references therein.

- Ω denotes the space of all continuous functions on $\mathbb{R}_+$.
- $C_{b,lip}(\mathbb{R}_d)$ is the space of all bounded real-valued Lipschitz continuous functions.
- $\mathcal{H}$ donotes a linear space of real-valued functions defined on Ω such that if $X_i \in \mathcal{H}, i = 1, 2, ..., d$, then $\varphi(X_1, ..., X_d) \in \mathcal{H}$ for all $\varphi \in C_{b,lip}(\mathbb{R}_d)$.
- Denote $\Omega_T := \{\omega_{\cdot\wedge T} : \omega \in \Omega\}$. Let

  $$Lip(\Omega_T) := \{\xi \in L^0(\Omega) : \xi = \phi(X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}})\},$$

  where $\phi \in C_{b,lip}(\mathbb{R}_d), 0 \le t_1 < \cdots < t_n \le T$. $L^p_G(\Omega_T)$ is the completion of $Lip(\Omega_T)$ under the norm $\|\cdot\|_p := \widehat{\mathbb{E}}[|\cdot|^p]^{1/p}, p \ge 1$.
- Consider the type of simple process: for a given partition $\pi_T = \{t_0, t_1, ..., t_N\}$ of [0,T], let

  $$\eta_t(\omega) = \sum_{k=0}^{N-1} \eta_k(\omega) I_{(t_k, t_{k+1}]}(t),$$

  where $\eta_k \in L^p_G(\Omega_{t_k}), k = 0, 1, ..., N-1$ are given. The collection of these processes is denoted by $M^{p,0}_G(0,T)$. Let $M^p_G(0,T)$ denotes the completion of $M^{p,0}_G(0,T)$ under the norm

  $$\|\eta\|_{M^p_G(0,T)} = \Big[\int_0^T \widehat{\mathbb{E}}[|\eta(t)|^p]dt\Big]^{\frac{1}{p}}.$$

- C stand for a positive constant and its value may be different in different appearances, and this assumption is also adaptable to C_p depending only on the subscripts.

Definition 2.1. *A sublinear expectation $\widehat{\mathbb{E}}$ is a functional $\widehat{\mathbb{E}}$: $\mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*

(i) Monotonicity $\widehat{\mathbb{E}}(X) \ge \widehat{\mathbb{E}}(Y)$ if $X \ge Y$.

(ii) Constant preserving $\widehat{\mathbb{E}}(C) = C$ for $C \in \mathbb{R}$.

(iii) Sub-additivity $\widehat{\mathbb{E}}(X + Y) \le \widehat{\mathbb{E}}(X) + \widehat{\mathbb{E}}(Y)$.

(iv) Positive homogeneity $\widehat{\mathbb{E}}(\lambda X) = \lambda\widehat{\mathbb{E}}(X)$ for $\lambda \ge 0$.

The tripe $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sub-linear expectation space (compared with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$), $\widehat{\mathbb{E}}$ is called a linear expectation if (iii) and (iv) is replaced by $\widehat{\mathbb{E}}[X + \alpha Y] = \widehat{\mathbb{E}}(X) + \alpha\widehat{\mathbb{E}}(Y)$ for $\alpha \in \mathbb{R}$. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. We also denote lower expectation $\widetilde{\mathbb{E}}[x] := -\widehat{\mathbb{E}}[-x]$ for each $X \in \mathcal{H}$.

Definition 2.2. *(G-Lévy process [14]) A d-dimensional càdlàg process $X = (X_t)_{t\geq 0}$ defined on a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a Lévy process if the following properties are satisfied:*

(i) $X_0 = 0$.

(ii) For each $s, t \geq 0$, the increment $X_{t+s} - X_t$ is independent of $(X_{t_1}, ..., X_{t_n})$ for every $n \in N$ and every partition $0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq t$.

(iii) The distribution of the increment $X_{t+s} - X_s, s, t \geq 0$ does not depend on t.

Moreover, a Lévy process X is a G-Lévy process if it satisfies the following conditions:

(iv) There exists a 2d-dimensional Lévy process $(X_t^c, X_t^d)_{t\geq 0}$ such that $X_t = X_t^c + X_t^d$, for each $t \geq 0$.

(v) The processes X_t^c and X_t^d satisfying the following assumption:

$$\lim_{t\downarrow 0} \widehat{\mathbb{E}}[|X_t^c|^3]t^{-1} = 0; \quad \widehat{\mathbb{E}}[|X_t^d|] \leq Ct, \quad for \ all \ t \geq 0,$$

where C is a positive constant.

Note that the condition (v) implies that X_t^c is a generalized G-Brownian motion, the jump part X_t^d is of finite variation (see Hu and Peng [14] for details).

Lemma 2.3. *(Lévy-Khintchine representation [14]) Let X be a G-Lévy process in $\mathbb{R}^d$. Defined nonlocal operator*

$$G_X[f(\cdot)] := \lim_{\delta\downarrow 0} \widehat{\mathbb{E}}[f(X_\delta)]\delta^{-1}, \ for \ f \in C_b^3(\mathbb{R}^d) \ with \ f(0) = 0.$$

Then, G_X has the following Lévy-Khintchine representation

$$G_X[f(\cdot)] = \sup_{(v,p,Q)\in\mathcal{U}} \left\{ \int_{\mathbb{R}_0^d} f(z)v(dz) + \langle D(f(0), p\rangle + \frac{1}{2}tr[D^2 f(0)QQ^T] \right\}$$

where $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$, $\mathcal{U}$ is a subset $\mathcal{U} \subset \mathcal{V} \times \mathbb{R}^d \times \mathcal{Q}$, and $\mathcal{V}$ is a set of all Borel measures on $(\mathbb{R}_0^d, \mathcal{B}(\mathbb{R}_0^d))$. $\mathcal{Q}$ is a set of all d-dimensional positive definite symmetric matrices in S^d (S^d is the space of all $d \times d$-dimensional symmetric matrices) such that

$$\sup_{(v,p,Q)\in\mathcal{U}} \left\{ \int_{\mathbb{R}_0^d} |z|v(dz) + |p| + tr\Big[QQ^T\Big] \right\} < \infty. \tag{2.1}$$

Definition 2.4. *For the sub-linear expectation $\widehat{\mathbb{E}}$, we introduce the capacity c and $\widetilde{c}$ related to $\widehat{\mathbb{E}}$ and $\widetilde{\mathbb{E}}$ as, respectively*

$$c(A) := \sup_{\mathbb{P}\in\mathfrak{B}} \mathbb{P}(A), A \in \mathcal{B}(\Omega),$$

$$\widetilde{c}(A) := \inf_{\mathbb{P}\in\mathfrak{B}} \mathbb{P}(A), A \in \mathcal{B}(\Omega),$$

where $\mathfrak{B}$ is a relatively compact family of probability measures.

We will say that a set $A \in \mathcal{B}(\Omega)$ is polar if $c(A) = 0$. We say that a property holds quasi-surely ($q.s.$) if it holds outside a polar set.

Lemma 2.5. *Let $X \in L_G^1(\Omega_T)$ and for some $p > 0$, $\widehat{\mathbb{E}}[|X|^p] < \infty$. Then, for each $M > 0$,*

$$c(|X| > M) \leq \frac{\widehat{\mathbb{E}}[|X|^p]}{M^p}.$$

In this paper, we assume that G-Lévy process X has finite activity, i.e.,

$$\lambda := \sup_{v \in \mathcal{V}} v(\mathbb{R}_0^d) < \infty.$$

Without loss of generality we will also assume that $\lambda = 1$ and that $\mu(\mathbb{R}_0^d) = 1$. Let X_{u-} denotes the left limit of X at point u, $\Delta X_u = X_u - X_{u-}$. We define a Poisson random measure $L(ds, dz)$ associated with the G-Lévy process X by considering

$$L((s,t], A) = \sum_{s<u\leq t} \mathbb{I}_A(\Delta X_u), \quad q.s.,$$

for any $0 < s < t < \infty$ and $A \in \mathcal{B}(\mathbb{R}_0^d)$. The random measure is well-defined and may be used to define the pathwise integral.

Let $H_G^S([0,T] \times \mathbb{R}_0^d)$ be a space of all the elementary random fields on $[0,T] \times \mathbb{R}_0^d$ of the form

$$K(r,z)(w) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} F_{k,l}(w) \mathbb{I}_{(t_k, t_{k+1}]}(r) \psi_l(z), n, m \in \mathbb{N},$$

where $0 \leq t_1 < ... < t_n \leq T$ is a partition of $[0,T]$, $\{\psi_l\}_{l=1}^m \subset C_{b,lip}(\mathbb{R}^d)$ are functions with disjoint supports such that $\psi_l(0) = 0$ and $F_{k,l} = \phi_{k,l}(X_{t_1}, ..., X_{t_k} - X_{t_{k-1}})$, $\phi_{k,l} \in C_{b,lip}(\mathbb{R}^{d\times k})$. We introduce the norm on this space

$$\|K\|^p_{H_G^p([0,T]\times\mathbb{R}_0^d)} := \widehat{\mathbb{E}}\Big[\int_0^T \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} |K(r,z)|^p v(dz) dr\Big], \ \ p = 1, 2.$$

Definition 2.6. *Let $0 \leq s < t \leq T$. The Itô integral of $K \in H_G^S([0,T] \times \mathbb{R}_0^d)$ with respect to the jump measure L is defined as*

$$\int_s^t \int_{\mathbb{R}_0^d} K(r,z) L(dr, dz) := \sum_{s<r\leq t} K(r, \Delta X_r), \ \ q.s..$$

It is worth note that for every $K \in H_G^S([0,T] \times \mathbb{R}_0^d)$, $\int_0^T \int_{\mathbb{R}_0^d} K(r,z) L(dr,dz)$ is an element of $L_G^1(\Omega_T)$ and $L_G^2(\Omega_T)$.

Let $H_G^p([0,T]\times\mathbb{R}_0^d)$ denote the topological completion of $H_G^S([0,T]\times\mathbb{R}_0^d)$ under the norm $\|\cdot\|_{H_G^p([0,T]\times\mathbb{R}_0^d)}$, $p = 1, 2$. Then Itô integral can be continuously extended to the whole space $H_G^p([0,T] \times \mathbb{R}_0^d)$, $p = 1, 2$. Moreover, the extended integral takes values in $L_G^p(\Omega_T)$, $p = 1, 2$. The formula from Definition 2.6 still holds for all $K \in H_G^p([0,T] \times \mathbb{R}_0^d)$.

For $K(r,z) \in H_G^2([0,T] \times \mathbb{R}_0^d)$, we know that

$$M(t) := \int_0^T \int_{\mathbb{R}_0^d} K(r,z) L(dr,dz) - \int_0^T \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} K(r,z) v(dz) dr$$

is a G-martingale, hence we have

$$\widehat{\mathbb{E}}\Big[\sup_{0\leq t\leq T} |M(t)|^2\Big] \leq C\widehat{\mathbb{E}}\Big[\int_0^T \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} K^2(r,z) v(dz) dr\Big],$$

where C is a positive constant.

Throughout this paper, let $C([-\tau,\infty)\times\mathbb{R}^n;\mathbb{R}_+)$ denote the family of all continuous functions from $[-\tau,\infty)\times\mathbb{R}^n$ to $\mathbb{R}_+$, $C^{1,2}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R}_+)$ denote the family of all continuous non-negative functions $W(t,x)$ defined on $\mathbb{R}_+\times\mathbb{R}^n$, they are continuously once differentiable in t and twice in x. Given $W\in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R}_+)$, we define the function $\mathfrak{L}W:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}$ by

$$\begin{aligned}\mathfrak{L}W &:= W_t(t,x)+\langle W_x(t,x), f(t,x))\rangle+\sup_{Q\in\mathcal{Q}} tr[(\langle W_x(t,x),h(t,x)\rangle\\&+\frac{1}{2}\langle W_{xx}(t,x)\sigma(t,x),\sigma(t,x)\rangle)QQ^T]+\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}(W(t,x+K(t,x,z))-W(t,x))v(dz),\end{aligned}$$

where

$$W_t(t,x)=\frac{\partial W(t,x)}{\partial t}, W_x(t,x)=(\frac{\partial W(t,x)}{\partial x_1},\frac{\partial W(t,x)}{\partial x_2},\cdots,\frac{\partial W(t,x)}{\partial x_n}),$$

and

$$W_{xx}(t,x)=(\frac{\partial^2 W(t,x)}{\partial x_i\partial x_j})_{n\times n}.$$

3. Asymptotic Stability

In this section, we will use the method of Lyapunov functionals to investigate the asymptotic stability of controlled G-SDDEs (1.2). For the stability of this paper, we suppose the

$$f(t,0)=u(t,0)=h(t,0)=\sigma(t,0)=K(t,0,z)=0,\ \ t\geq 0, \tag{3.1}$$

which implies that $x(t)\equiv 0$ is the trivial solution of the G-SDDEs (1.2). In order to obtain the existence and uniqueness of the global solution of equation (1.2), we need the following conditions.

Assumption 3.1. *Assume that for any real number $m>0$, $x_1,x_2\in\mathbb{R}^d$ and $|x_1|\vee|x_2|<m$, the functions $f(t,x)$, $h(t,x),\sigma(t,x),K(t,x,z)$ satisfy*

$$\begin{aligned}&|f(t,x_1)-f(t,x_2)|+|h(t,x_1)-h(t,x_2)|+|\sigma(t,x_1)-\sigma(t,x_2)|\\&+\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}|K(t,x_1,z)-K(t,x_2,z)|v(dz)\leq L_m|x_1-x_2|,\end{aligned} \tag{3.2}$$

where L_m is a positive constant.
And for $x_1,x_2\in\mathbb{R}^d$, there exists a positive constant L such that

$$|u(t,x_1)-u(t,x_2)|\leq L|x_1-x_2|. \tag{3.3}$$

Moreover, there exist positive constants C and $p_i, i=1,2,3,4$ such that

$$\begin{aligned}&|f(t,x)|\leq C(1+|x|^{p_1}),\\&|h(t,x)|\leq C(1+|x|^{p_2}),\\&|\sigma(t,x)|\leq C(1+|x|^{p_3}),\\&\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}|K(t,x,z)|v(dz)\leq C(1+|x|^{p_4}).\end{aligned} \tag{3.4}$$

The condition (3.4) is referred as the polynomial growth condition which means one of the $p_i, i=1,2,3,4$ more than 1. When $p_i=1, i=1,2,3,4$, the condition (3.4) is the familiar linear growth condition.

It is well-known that (3.2), (3.3) and linear growth condition can guarantee the existence and uniqueness of the global solution of (1.2). Without the linear growth condition, the solutions of the G-SDDEs (1.2) may explode to infinity at a finite time. To avoid such a possible explosion, we propose our alternative condition as follows, it's weaker than the linear growth condition.

Assumption 3.2. *Assume that there exist functions $\bar{W} \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ and $W_1 \in C^{1,2}([-\tau, \infty) \times \mathbb{R}^n; \mathbb{R}_+)$, as well as nonnegative constants $p \geq 2p_i$, $i = 1, 2, 3, 4$ and q_j $(j = 1, 2, 3)$ with $q_2 > q_3$, such that*

$$|x|^p \leq \bar{W}(t, x) \leq W_1(t, x), \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{3.5}$$

and

$$\mathfrak{L}\bar{W}(t, x) + \bar{W}_x(t, x)u(t, y) \leq q_1 - q_2 W_1(t, x) + q_3 W_1(t - \tau, y), \tag{3.6}$$

for all $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$.

Theorem 3.1. *Under Assumptions 3.1 and 3.2, for any given initial data* (1.3) *there exists a unique global solution $x(t)$ to the G-SDDEs* (1.2) *on $t \in [-\tau, \infty)$ and the solution has the property:*

$$\sup_{-\tau \leq t < \infty} \widehat{\mathbb{E}}|x(t)|^p < \infty,$$

where $\gamma > 0$ is the unique root to the equation $q_2 = \gamma + e^{\gamma\tau} q_3$.

Proof. The existence and uniqueness of the global solution can be proved by the standard technique of Picard iterations, we omit the details (we can see, for example, Wang and Gao [40], Hu et al. [13] for the details). Next, we will prove $\sup_{-\tau \leq t \leq \infty} \widehat{\mathbb{E}}|x(t)|^p < \infty$.

In fact, applying the G-Itô formula to $e^{\gamma t}\bar{W}(t, x(t)), t \geq 0$, we have

$$\begin{aligned}
&e^{\gamma t}\bar{W}(t, x(t)) - \bar{W}(0, x(0)) \\
&= \int_0^t e^{\gamma r}[\gamma \bar{W}(r, x(r)) + \mathfrak{L}\bar{W}(r, x(r)) + \bar{W}_x(t, x)u(t, y)]dr + \int_0^t e^{\gamma r}\langle \bar{W}_x(r, x(r)), \sigma(r, x(r))\rangle dB(r) + M_t^0 + P_t^0,
\end{aligned}$$

where

$$\begin{aligned}
M_t^s :=& \int_s^t e^{\gamma r}[\langle \bar{W}_x(r, x(r)), h(r, x(r))\rangle + \frac{1}{2}\langle \bar{W}_{xx}(r, x(r))\sigma(r, x(r)), \sigma(r, x(r))\rangle]d\langle B\rangle(r) \\
&- \int_s^t e^{\gamma r} \sup_{Q \in \boldsymbol{Q}} tr[\langle \bar{W}_x(r, x(r)), h(r, x(r))\rangle + \frac{1}{2}\langle \bar{W}_{xx}(r, x(r))\sigma(r, x(r)), \sigma(r, x(r))\rangle QQ^T]dr, \\
P_t^s :=& \int_s^t \int_{\mathbb{R}_0^d} e^{\gamma r}[\bar{W}(r, x(r^-) + K(r, x(r), z)) - \bar{W}(r, x(r^-))]L(dr, dz) \\
&- \int_s^t \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_0^d} e^{\gamma r}[\bar{W}(r, x(r^-) + K(r, x(r), z)) - \bar{W}(r, x(r^-))]v(dz)dr.
\end{aligned}$$

By conditions (3.5) and (3.6), then we have

$$\begin{aligned}
&e^{\gamma t}|x|^p - W_1(0, x(0)) \\
&\le \int_0^t e^{\gamma r}[q_1 - (q_2 - \gamma)W_1(r, x) + q_3 W_1(r - \tau, x(r - \tau))]dr \\
&\quad + \int_0^t e^{\gamma r}\langle \bar{W}_x(r, x(r)), \sigma(r, x(r))\rangle dB(r) + M_t^0 + P_t^0 \\
&\le \frac{q_1}{\gamma}e^{\gamma t} - (q_2 - \gamma)\int_0^t e^{\gamma r}W_1(r, x)dr + \int_0^t e^{\gamma r}q_3 W_1(r - \tau, x(r - \tau))dr \\
&\quad + \int_0^t e^{\gamma r}\langle \bar{W}_x(r, x(r)), \sigma(r, x(r))\rangle dB(r) + M_t^0 + P_t^0 \\
&\le \frac{q_1}{\gamma}e^{\gamma t} - (q_2 - \gamma)\int_0^t e^{\gamma r}W_1(r, x)dr + e^{\gamma\tau}\int_{-\tau}^0 q_3 W_1(r, x(r))dr \\
&\quad + e^{\gamma\tau}\int_0^t e^{\gamma r}q_3 W_1(r, x(r))dr + \int_0^t e^{\gamma r}\langle \bar{W}_x(r, x(r)), \sigma(r, x(r))\rangle dB(r) + M_t^0 + P_t^0.
\end{aligned}$$

This implies

$$\begin{aligned}
&e^{\gamma t}|x|^p \\
&\le W_1(0, x(0)) + \frac{q_1}{\gamma}e^{\gamma t} - (q_2 - \gamma - e^{\gamma\tau}q_3)\int_0^t e^{\gamma r}W_1(r, x)dr + e^{\gamma\tau}\int_{-\tau}^0 q_3 W_1(r, x(r))dr \\
&\quad + \int_0^t e^{\gamma r}\langle \bar{W}_x(r, x(r)), \sigma(r, x(r))\rangle dB(r) + M_t^0 + P_t^0.
\end{aligned}$$

Note that $\{M_t^s\}, \{P_t^s\}$ are G-martingale (Peng [28] and Paczka [27]). Then taking the expectation on both sides, we have

$$\widehat{\mathbb{E}}(e^{\gamma t}|x|^p) \le K + \frac{q_1}{\gamma}e^{\gamma t},$$

where

$$K = W_1(0, x(0)) + e^{\gamma\tau}\int_{-\tau}^0 q_3 W_1(r, x(r))dr.$$

Which means

$$\sup_{-\tau\le t<\infty}\widehat{\mathbb{E}}|x(t)|^p < \infty.$$

This completes the proof. □

In order to use Lyapunov functional to study the asymptotic stability of controlled G-SDDEs (1.2), we define $x_t := \{x(t+s) : -2\tau \le s \le 0\}$ for $t \ge 0$, x_t is well defined for $0 \le t < 2\tau$. Let $x(s) = \xi(-\tau)$ for $s \in [-2\tau, -\tau)$. In this paper, we will use the following Lyapunov functional

$$\begin{aligned}
\tilde{V}(t, x_t) = {}& W(t, x(t)) + \rho\int_{-\tau}^0\int_{t+s}^t [\tau|f(r, x(r)) + u(r, x(r - \tau))|^2 \\
&+ C_2^{'}\tau|h(r, x(r))|^2 + C_2|\sigma(r, x(r))|^2 + C_\tau\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}K^2(r, x(r), z)v(dz)]drds,
\end{aligned} \tag{3.7}$$

for $t \ge 0$, where $W \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$, and ρ is a determined positive constant and let

$$f(r, x) = f(0, x), u(r, x) = u(0, x), h(r, x) = h(0, x),$$

$$\sigma(r,x)=\sigma(0,x),\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}K(r,x,z)v(dz)=\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}K(0,x,z)v(dz),$$

for $(r,x)\in[-2\tau,0)\times\mathbb{R}^n$.
Applying G-Itô formula to $W(t,x(t))$, we get

$$\begin{aligned}dW(t,x(t))&=[\mathfrak{L}W(t,x(t))+W_x(t,x(t))u(t,x(t-\tau))]dt\\&\quad+\langle W_x(t,x(t)),\sigma(t,x(t))\rangle dB(t)+dN_t^0+dD_t^0,\quad t\geq 0,\end{aligned}\tag{3.8}$$

where

$$\begin{aligned}N_t^s&:=\int_s^t[\langle W_x(r,x(r)),h(r,x(r))\rangle+\frac{1}{2}\langle W_{xx}(r,x(r))\sigma(r,x(r)),\sigma(r,x(r))\rangle]d\langle B\rangle(r)\\&-\int_s^t\sup_{Q\in\mathcal{Q}}tr[\langle W_x(r,x(r)),h(r,x(r))\rangle+\frac{1}{2}\langle W_{xx}(r,x(r))\sigma(r,x(r)),\sigma(r,x(r))\rangle QQ^T]dr,\\D_t^s&:=\int_s^t\int_{\mathbb{R}_0^d}[W(r,x(r^-)+K(r,x(r),z))-W(r,x(r^-))]L(dr,dz)\\&\qquad-\int_s^t\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}[W(r,x(r^-)+K(r,x(r),z))-W(r,x(r^-))]v(dz)dr.\end{aligned}$$

Note that $\{N_t^s\},\{D_t^s\}$ are G-martingale (Peng [28] and Paczka [27]). It is easy to obtain that

$$\begin{aligned}&d\Big(\rho\int_{-\tau}^0\int_{t+s}^t\Big[\tau|f(r,x(r))+u(r,x(r-\tau))|^2+C_2'\tau|h(r,x(r))|^2\\&\quad+C_2|\sigma(r,x(r))|^2+C_\tau\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}K^2(r,x(r),z)v(dz)\Big]drds\Big)\\&\leq\Big(\rho\tau\Big[\tau|f(t,x(t))+u(t,x(t-\tau))|^2+C_2'\tau|h(t,x(t))|^2+C_2|\sigma(t,x(t))|^2\\&\quad+C_\tau\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}K^2(t,x(t),z)v(dz)\Big]-\rho(\int_{t-\tau}^t[\tau|f(r,x(r))+u(r,x(r-\tau))|^2\\&\quad+C_2'\tau|h(r,x(r))|^2+C_2|\sigma(r,x(r))|^2+C_\tau\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}K^2(r,x(r),z)v(dz)]dr)\Big)dt.\end{aligned}\tag{3.9}$$

Thus we have

$$d\tilde{V}(t,x_t)=L\tilde{V}(t,x_t)dt+\langle W_x(t,x(t)),\sigma(t,x(t))\rangle dB(t)+dN_t^0+dD_t^0,\tag{3.10}$$

where

$$\begin{aligned}L\tilde{V}(t,x_t)&=\mathfrak{L}W(t,x(t))+W_x(t,x(t))u(t,x(t-\tau))\\&\quad+\rho\tau\Big[\tau|f(t,x(t))+u(t,x(t-\tau))|^2+C_2'\tau|h(t,x(t))|^2+C_2|\sigma(t,x(t))|^2\\&\quad+C_\tau\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}K^2(t,x(t),z)v(dz)\Big]-\rho(\int_{t-\tau}^t[\tau|f(r,x(r))+u(r,x(r-\tau))|^2\\&\quad+C_2'\tau|h(r,x(r))|^2+C_2|\sigma(r,x(r))|^2+C_\tau\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}K^2(r,x(r),z)v(dz)dr).\end{aligned}\tag{3.11}$$

In order to study the asymptotic stability of the controlled G-SDDEs (1.2), we need to impose the following assumption.

Assumption 3.3. *Assume that there is a function $W \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ such that*

$$\begin{aligned} &\mathfrak{L}W(t, x(t)) + W_x(t, x(t))u(t, x(t)) + \alpha_1|W_x(t, x(t))|^2 + \alpha_2|f(t, x(t))|^2 \\ &\quad + \alpha_3|h(t, x(t))|^2 + \alpha_4|\sigma(t, x(t))|^2 + \alpha_5 \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} K^2(t, x(t), z)v(dz) \\ &\leq -\alpha_0|x(t)|^2, \end{aligned} \tag{3.12}$$

for all $(t, x(t)) \in (\mathbb{R}_+ \times \mathbb{R}^n)$, where $\alpha_i, i = 0, \cdots, 5$ are positive constants.

Theorem 3.2. *Under Assumptions 3.1, 3.2 and 3.3, let*

$$\tau < \sqrt{\frac{\alpha_1\alpha_2}{2L^2}} \wedge \sqrt{\frac{\alpha_1\alpha_3}{L^2C_2'}} \wedge \frac{\alpha_1\alpha_4}{L^2C_2} \wedge \frac{\alpha_1\alpha_5}{L^2C_\tau}, \quad and \quad \tau < \sqrt{\frac{\alpha_0\alpha_1}{2L^4}}. \tag{3.13}$$

Then for any given initial data (1.3)*, the solution of G-SDDEs* (1.2) *has the following property*

$$\widetilde{\mathbb{E}}(\int_0^\infty |x(s)|^2 ds) < \infty. \tag{3.14}$$

Proof. For any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n)$, let $k_0 > 0$ be a large enough integer such that $\|\xi\| < k_0$. For any integer $k \geq k_0$, define the following stopping time

$$\zeta_k = \inf\{t \geq 0 : |x(t)| \geq k\}.$$

It comes from Theorem 3.1, we know that as $k \to \infty$, the ζ_k is increasing to infinity q.s.. By G-Itô formula to $\tilde{V}(t, x_t)$ and (3.10), we have, for all $t \geq 0$ and $k \geq k_0$,

$$\widehat{\mathbb{E}}\tilde{V}(t \wedge \zeta_k, x_{t\wedge\zeta_k}) = \tilde{V}(t_0, x_0) + \widehat{\mathbb{E}} \int_0^{t\wedge\zeta_k} L\tilde{V}(s, x_s)ds. \tag{3.15}$$

By Assumption 3.1, we have

$$W_x(t, x(t))[u(t, x(t - \tau)) - u(t, x(t))] \leq \alpha_1|W_x(t, x(t))|^2 + \frac{L^2}{4\alpha_1}|x(t) - x(t - \tau)|^2. \tag{3.16}$$

By (3.11), (3.16) and Assumption 3.2, we obtain

$$\begin{aligned} &L\tilde{V}(t, x_t) \\ &\leq \mathfrak{L}W(t, x(t)) + W_x(t, x(t))u(t, x(t - \tau)) \\ &\quad + \rho\tau\Big[\tau|f(t, x(t)) + u(t, x(t - \tau))|^2 + C_2'\tau|h(t, x(t))|^2 + C_2|\sigma(t, x(t))|^2 \\ &\quad + C_\tau \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} K^2(t, x(t), z)v(dz)\Big] - \rho(\int_{t-\tau}^t [\tau|f(r, x(r)) + u(r, x(r - \tau))|^2 \\ &\quad + C_2'\tau|h(r, x(r))|^2 + C_2|\sigma(r, x(r))|^2 + C_\tau \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} K^2(r, x(r), z)v(dz)]dr) \\ &\leq \mathfrak{L}W(t, x(t)) + W_x(t, x(t))u(t, x(t)) + \alpha_1|W_x(t, x(t))|^2 + \frac{L^2}{4\alpha_1}|x(t) - x(t - \tau)|^2 \\ &\quad + 2\rho\tau^2|f(t, x(t))|^2 + 2\rho\tau^2|u(t, x(t - \tau))|^2 + \rho C_2'\tau^2|h(t, x(t))|^2 + \rho C_2\tau|\sigma(t, x(t))|^2 \\ &\quad + \rho C_\tau\tau \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} K^2(t, x(t), z)v(dz)\Big] - \rho(\int_{t-\tau}^t [\tau|f(r, x(r)) + u(r, x(r - \tau))|^2 \\ &\quad + C_2'\tau|h(r, x(r))|^2 + C_2|\sigma(r, x(r))|^2 + C_\tau \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} K^2(r, x(r), z)v(dz)]dr). \end{aligned} \tag{3.17}$$

Let $\rho = \frac{L^2}{\alpha_1}$, where ρ is the free parameter in the definition of the Lyapunov functional (see (3.7)). Let $\tau < \sqrt{\frac{\alpha_1\alpha_2}{2L^2}} \wedge \sqrt{\frac{\alpha_1\alpha_3}{L^2C_2'}} \wedge \frac{\alpha_1\alpha_4}{L^2C_2} \wedge \frac{\alpha_1\alpha_5}{L^2C_\tau}$, by the Assumption 3.3, we can obtain

$$\begin{aligned} L\tilde{V}(t,x_t) \leq & -\alpha_0|x(t)|^2 + 2\rho\tau^2L^2|x(t-\tau)|^2 + \frac{L^2}{4\alpha_1}|x(t)-x(t-\tau)|^2 \\ & -\frac{L^2}{\alpha_1}(\int_{t-\tau}^{t}[\tau|f(r,x(r))+u(r,x(r-\tau))|^2 + C_2'\tau|h(r,x(r))|^2 \\ & + C_2|\sigma(r,x(r))|^2 + C_\tau \sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}K^2(r,x(r),z)v(dz)]dr). \end{aligned} \tag{3.18}$$

It comes from (3.10), we have

$$\begin{aligned} 0 \leq & \tilde{V}(t,x_t) \\ = & \tilde{V}(0,x_0) + \int_0^{t\wedge\zeta_k} L\tilde{V}(s,x_s)ds + \int_0^{t\wedge\zeta_k}\langle W_x(s,x(s)),\sigma(s,x(s))\rangle dB(s) + N_t^0 + D_t^0 \\ \leq & \tilde{V}(0,x_0) + \int_0^{t\wedge\zeta_k}[-\alpha_0|x(s)|^2 + 2\rho\tau^2L^2|x(s-\tau)|^2]ds + \int_0^{t\wedge\zeta_k}[\frac{L^2}{4\alpha_1}|x(s)-x(s-\tau)|^2]ds \\ & - \frac{L^2}{\alpha_1}\int_0^{t\wedge\zeta_k}\int_{s-\tau}^{s}[\tau|f(r,x(r))+u(r,x(r-\tau))|^2 + C_2'\tau|h(r,x(r))|^2 + C_2|\sigma(r,x(r))|^2 \\ & + C_\tau\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}K^2(r,x(r),z)v(dz)]drds + \int_0^{t\wedge\zeta_k}\langle W_x(s,x(s)),\sigma(s,x(s))\rangle dB(s) + N_t^0 + D_t^0. \end{aligned} \tag{3.19}$$

For convenience, we let

$$\begin{aligned} \phi_1 := & \int_0^{t\wedge\zeta_k}[-\alpha_0|x(s)|^2 + 2\rho\tau^2L^2|x(s-\tau)|^2]ds, \\ \phi_2 := & \frac{L^2}{4\alpha_1}\int_0^{t\wedge\zeta_k}|x(s)-x(s-\tau)|^2ds, \\ \phi_3 := & \frac{L^2}{\alpha_1}\int_0^{t\wedge\zeta_k}\int_{s-\tau}^{s}[\tau|f(r,x(r))+u(r,x(r-\tau))|^2 + C_2'\tau|h(r,x(r))|^2 + C_2|\sigma(r,x(r))|^2 \\ & + C_\tau\sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d}K^2(r,x(r),z)v(dz)]drds. \end{aligned} \tag{3.20}$$

For ϕ_1, we know

$$\int_0^{t\wedge\zeta_k}|x(s-\tau)|^2ds \leq \int_{-\tau}^{t\wedge\zeta_k}|x(s)|^2ds.$$

Hence, we have

$$\begin{aligned} \phi_1 \leq & 2\rho\tau^2L^2\int_{-\tau}^{0}|x(s)|^2ds - (\alpha_0 - 2\rho\tau^2L^2)\int_0^{t\wedge\zeta_k}|x(s)|^2ds \\ \leq & 2\rho\tau^3L^2\|\xi\|^2 - (\alpha_0 - 2\rho\tau^2L^2)\int_0^{t\wedge\zeta_k}|x(s)|^2ds. \end{aligned} \tag{3.21}$$

Substituting (3.21) and (3.20) into (3.19) and let $k \to \infty$, we can get

$$\begin{aligned} \bar{\phi_3} \leq & C_1 - (\alpha_0 - 2\rho\tau^2L^2)\int_0^t|x(s)|^2ds \\ & + \bar{\phi_2} + \int_0^t\langle W_x(s,x(s)),\sigma(s,x(s))\rangle dB(s) + N_t^0 + D_t^0, \end{aligned} \tag{3.22}$$

with $C_1 = \tilde{V}(0, x_0) + 2\rho\tau^3 L^2\|\xi\|^2$, where

$$\begin{aligned}\bar{\phi_2} &:= \frac{L^2}{4\alpha_1}\int_0^t |x(s) - x(s-\tau)|^2 ds,\\ \bar{\phi_3} &:= \frac{L^2}{\alpha_1}\int_0^t\int_{s-\tau}^s [\tau|f(r, x(r)) + u(r, x(r-\tau))|^2 + C_2'\tau|h(r, x(r))|^2 + C_2|\sigma(r, x(r))|^2\\ &\quad + C_\tau \sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d} K^2(r, x(r), z)v(dz)]drds.\end{aligned} \tag{3.23}$$

For $t \in [0, \tau]$, we obtain

$$\begin{aligned}\bar{\phi_2} &\le \frac{L^2}{4\alpha_1}\int_0^\tau \Big(2|x(s)|^2 + 2|x(s-\tau)|^2\Big)ds\\ &\le \frac{L^2}{2\alpha_1}\int_0^\tau \Big(|x(s)|^2 + |x(s-\tau)|^2\Big)ds \le \frac{L^2\tau}{\alpha_1}\Big(\sup_{-\tau\le s\le\tau}|x(s)|^2\Big) := C_3.\end{aligned} \tag{3.24}$$

For $t > \tau$, we have

$$\bar{\phi_2} \le C_3 + \frac{L^2}{4\alpha_1}\int_\tau^t \Big(|x(s) - x(s-\tau)|^2\Big)ds. \tag{3.25}$$

Substituting (3.25) into (3.22), we have

$$\begin{aligned}\bar{\phi_3} \le& C_1 + C_3 - (\alpha_0 - 2\rho\tau^2 L^2)\int_0^t |x(s)|^2 ds + \frac{L^2}{4\alpha_1}\int_\tau^t \Big(|x(s) - x(s-\tau)|^2\Big)ds\\ &+ \int_0^t \langle W_x(s, x(s)), \sigma(s, x(s))\rangle dB(s) + N_t^0 + D_t^0.\end{aligned} \tag{3.26}$$

Taking the expectation on both sides we obtain

$$\begin{aligned}\widehat{\mathbb{E}}(\bar{\phi_3}) \le& C_1 + C_3 + \widehat{\mathbb{E}}\Big[-(\alpha_0 - 2\rho\tau^2 L^2)\int_0^t |x(s)|^2 ds\Big]\\ &+ \widehat{\mathbb{E}}\Big(\frac{L^2}{4\alpha_1}\int_\tau^t |x(s) - x(s-\tau)|^2 ds\Big).\end{aligned} \tag{3.27}$$

On the other hand, by the Burkholder-Davis-Gundy-type inequalities (Gao [9], Wang and Gao [40]), we have

$$\begin{aligned}&\widehat{\mathbb{E}}\Big(|x(j) - x(j-\tau)|^2\Big)\\ &\le \widehat{\mathbb{E}}\Big(\int_{j-\tau}^j [f(s, x(s)) + u(s, x(s-\tau))]ds + \int_{j-\tau}^j h(s, x(s))d\langle B\rangle(s)\\ &\quad + \int_{j-\tau}^j \sigma(s, x(s))dB(s) + \int_{j-\tau}^j\int_{\mathbb{R}_0^d} K(s, x(s), z)L(ds, dz))|^2\Big)\\ &\le 4\Big(\widehat{\mathbb{E}}\big(\sup_{0\le r\le j}\Big[|\int_{r-\tau}^r [f(s, x(s)) + u(s, x(s-\tau))]ds|^2 + |\int_{r-\tau}^r h(s, x(s))d\langle B\rangle(s)|^2\\ &\quad + |\int_{r-\tau}^r \sigma(s, x(s))dB(s)|^2 + |\int_{r-\tau}^r\int_{\mathbb{R}_0^d} K(s, x(s), z)L(ds, dz))|^2\Big]\big)\Big)\\ &\le 4\widehat{\mathbb{E}}\int_{j-\tau}^j (\tau|f(s, x(s)) + u(s, x(s-\tau))|^2)ds + 4C_2'\tau\int_{j-\tau}^j \widehat{\mathbb{E}}|h(s, x(s))|^2 ds\\ &\quad + 4C_2\int_{j-\tau}^j \widehat{\mathbb{E}}|\sigma(s, x(s))|^2 ds + 4C_\tau\widehat{\mathbb{E}}[\int_{j-\tau}^j \sup_{v\in\mathcal{V}}\int_{\mathbb{R}_0^d} K^2(s, x(s), z)v(dz)ds].\end{aligned} \tag{3.28}$$

Hence,

$$
\begin{aligned}
&-\widehat{\mathbb{E}}\Big[-(\alpha_0-2\rho\tau^2L^2)\int_0^t |x(s)|^2ds\Big]\\
&\leq C_1+C_3+\widehat{\mathbb{E}}\Big(\frac{L^2}{4\alpha_1}\int_\tau^t\Big[|x(s)-x(s-\tau)|^2\Big]ds\Big)-\widehat{\mathbb{E}}(\bar{\phi}_3)\\
&\leq C_1+C_3.
\end{aligned}
\tag{3.29}
$$

Let $\tau<\sqrt{\frac{\alpha_0\alpha_1}{2L^4}}$, by $\rho=\frac{L^2}{\alpha_1}$, then $\alpha_0-2\rho\tau^2L^2>0$. Let $t\to\infty$, imply that

$$
\widetilde{\mathbb{E}}(\int_0^\infty |x(s)|^2ds)=-\widehat{\mathbb{E}}\int_0^\infty -|x(s)|^2ds<\infty. \tag{3.30}
$$

This completes the proof. □

Theorem 3.3. *Under the same assumptions of Theorem 3.2. Then for any given initial data* (1.3)*, the solution of the G-SDDEs* (1.2) *is asymptotically stable in mean square, that is, the solution has the property that*

$$
\lim_{t\to\infty}\widetilde{\mathbb{E}}(|x(t)|^2)=0. \tag{3.31}
$$

Proof. Applying the G-Itô formula, we have

$$
\begin{aligned}
&|\widetilde{\mathbb{E}}|x(t_2)|^2-\widetilde{\mathbb{E}}|x(t_1)|^2|\\
&\leq\widetilde{\mathbb{E}}\int_{t_1}^{t_2}\Big(2|x(t)|[f(t,x(t))+u(t,x(t-\tau))]+2|x(t)|h(t,x(t))\\
&\quad+|\sigma(t,x(t))|^2+\int_{\mathbb{R}_0^d}\Big[2|x(t)|K(t,x,z)+K^2(t,x,z)\Big]v(dz)\Big)dt.
\end{aligned}
\tag{3.32}
$$

It follows from (3.4) that

$$
\begin{aligned}
&\Big|\widetilde{\mathbb{E}}|x(t_2)|^2-\widetilde{\mathbb{E}}|x(t_1)|^2\Big|\\
&\leq\widehat{\mathbb{E}}\int_{t_1}^{t_2}\Big(2|x(t)|[f(t,x(t))+u(t,x(t-\tau))]+2|x(t)|h(t,x(t))\\
&\quad+|\sigma(t,x(t))|^2+\int_{\mathbb{R}_0^d}\Big[2|x(t)|K(t,x,z)+K^2(t,x,z)\Big]v(dz)\Big)dt\\
&\leq\int_{t_1}^{t_2}\Big(c_1+c_2(1+\widehat{\mathbb{E}}|x(t)|^p)\Big)dt.
\end{aligned}
\tag{3.33}
$$

Hence, by Theorem 3.1, we can obtain

$$
|\widetilde{\mathbb{E}}|x(t_2)|^2-\widetilde{\mathbb{E}}|x(t_2)|^2|\leq(t_2-t_1)(c_1+c_2+c_2c_3),
$$

where $c_i, i=1,2,3$ are constants and $c_1=5C^2, c_2=9C+6C^2+2L, c_3=\sup_{-\tau\leq t<\infty}\widehat{\mathbb{E}}|x(t)|^p<\infty$. That is, $\widetilde{\mathbb{E}}|x(t)|^2$ is uniformly continuous in t, combining with the (3.14), we have $\lim_{t\to\infty}\widetilde{\mathbb{E}}(|x(t)|^2)=0$. This completes the proof. □

Theorem 3.4. *Under the same assumptions of Theorem 3.2, then the solution of the controlled G-SDDEs satisfies*

$$
\lim_{t\to\infty}x(t)=0,\qquad q.s.. \tag{3.34}
$$

That is, the controlled system is quasi-surely asymptotically stable.

Proof. Assume that the assertion (3.34) is not true, which means we can find a positive and small enough constant $\varepsilon \in (0, \frac{1}{4})$ sufficiently small such that

$$\widetilde{c}\Big(\Omega_1\Big) \geq 4\varepsilon, \tag{3.35}$$

where $\Omega_1 = \{\limsup_{t\to\infty} |x(t)|^2 > 2\varepsilon\}$. For any $k \geq \|\xi\|$, let ζ_k be the same stopping time as defined in the proof of Theorem 3.2 and from Theorem 3.2 we have

$$\widetilde{\mathbb{E}}|x(t \wedge \zeta_k)|^2 \leq C,$$

this means

$$k^2\widetilde{c}(\zeta_k \leq t) \leq C, \quad \forall t \geq 0,$$

this, letting $t \to \infty$, implies there is a positive integer k_1 large enough such that

$$\limsup_{k\to\infty} k^2\widetilde{c}(\zeta_k \leq \infty) \leq C + 1, \quad \forall k \geq k_1. \tag{3.36}$$

Hence we can then choose a sufficiently large fixed $k_2 \geq k_1$ such that $\frac{C+1}{k_2^2} \leq \varepsilon$ to get $\widetilde{c}(\zeta_{k_2} < \infty) \leq \varepsilon$. This means that

$$\widetilde{c}(\Omega_2) \geq 1 - \varepsilon, \tag{3.37}$$

where $\Omega_2 = \{|x(t)| < k_2, \forall t \geq -\tau\}$. Combining (3.35) with (3.37), we have

$$\widetilde{c}(\Omega_1 \cap \Omega_2) \geq 3\varepsilon. \tag{3.38}$$

Define the stopped process $\bar{x}(t) = x(t \wedge \zeta_{k_2})$. It is obvious that $\bar{x}(t)$ satisfies the form

$$d\bar{x}(t) = \bar{f}(t)dt + \bar{h}(t)d\langle B\rangle(t) + \bar{\sigma}(t)dB(t) + \int_{\mathbb{R}_0^d} \bar{K}(t,z)L(dt,dz), \tag{3.39}$$

where

$$\bar{f}(t) = [f(t,x(t)) + u(t,x(t-\tau))]I_{[0,\zeta_{k_2})}(t),$$
$$\bar{h}(t) = h(t,x(t))I_{[0,\zeta_{k_2})}(t),$$
$$\bar{\sigma}(t) = \sigma(t,x(t))I_{[0,\zeta_{k_2})}(t),$$
$$\sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} \bar{K}(t,z)v(dz) = \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} K(t,x(t),z)I_{[0,\zeta_{k_2})}(t)v(dz),$$

It is easy to see that $\bar{f}(t)$, $\bar{h}(t)$, $\bar{\sigma}(t)$ and $\sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} \bar{K}(t,z)v(dz)$ are bounded processes, that is,

$$|\bar{f}(t)| \vee |\bar{h}(t)| \vee |\bar{\sigma}(t)| \vee |\sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} \bar{K}(t,z)v(dz)| \leq C_5, \quad q.s., \tag{3.40}$$

for all $t \geq 0$. Next, we will define a sequence of stopping times

$$\beta_1 = \inf\{t \geq 0 : |\bar{x}(t)|^2 \geq 2\varepsilon\},$$
$$\beta_{2l} = \inf\{t \geq \beta_{2l-1} : |\bar{x}(t)|^2 \leq \varepsilon\}, l = 1, 2, \cdots,$$
$$\beta_{2l+1} = \inf\{t \geq \beta_{2l} : |\bar{x}(t)|^2 \geq 2\varepsilon\}, l = 1, 2, \cdots.$$

It comes from Theorem 3.2 that

$$C_4 := \widetilde{\mathbb{E}} \int_0^\infty (|x(t)|^2)dt < \infty. \tag{3.41}$$

This means that

$$\liminf_{t\to\infty}|x(t)|^2 = 0, \quad q.s.. \tag{3.42}$$

By the definition of Ω_1 and Ω_2 and (3.42) we have

$$\Omega_1 \cap \Omega_2 \subset \{\beta_l < \infty\} \cap \{\zeta_{k_2} = \infty\}, l = 1, 2, \cdots . \tag{3.43}$$

Choose a small positive number θ and a large positive integer ϖ such that

$$4\theta C_5^2(\theta + \theta C_2^{'} + C_2 + C_\theta) \le \varepsilon\delta^2 \quad and \quad C_4 < \varepsilon^2\theta\varpi. \tag{3.44}$$

By (3.38) and (3.43), we can further choose a sufficiently large number N such that

$$\widetilde{c}(\beta_{2\varpi} \le N) \ge 2\varepsilon. \tag{3.45}$$

In particular, if $\beta_{2\varpi} \le N$, then $|\bar{x}(\beta_{2\varpi})| = \varepsilon$. So, by the definition of $\bar{x}(t)$, we have $\beta_{2\varpi} \le \zeta_{k_2}$. This implies

$$\bar{x}(t, w) = x(t, w) \; for \; all \; 0 \le t \le \beta_{2\varpi} \; and \; w \in \{\beta_{2\varpi} \le N\}. \tag{3.46}$$

By (3.39) and the Burkholder-Davis-Gundy-type inequalities, for $1 \le l \le \varpi$,

$$\begin{aligned}
&\widehat{\mathbb{E}}(\sup_{0\le t\le\theta} ||\bar{x}(\beta_{2l-1} \wedge N + t)| - |\bar{x}(\beta_{2l-1} \wedge N)||^2) \\
\le&\widehat{\mathbb{E}}(\sup_{0\le t\le\theta} |\bar{x}(\beta_{2l-1} \wedge N + t) - \bar{x}(\beta_{2l-1} \wedge N)|^2) \\
=&\widehat{\mathbb{E}}(\sup_{0\le t\le\theta} |\int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+t} \bar{f}(s, x(s))ds + \int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+t} \bar{h}(s, x(s))d\langle B\rangle(s) \\
&+ \int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+t} \bar{\sigma}(s, x(s))dB(s) + \int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+t} \int_{\mathbb{R}_0^d} \bar{K}(s, x(s), z)L(ds, dz)|^2) \\
\le&4\widehat{\mathbb{E}}(\sup_{0\le t\le\theta} |\int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+t} \bar{f}(s, x(s))ds|^2) + 4\widehat{\mathbb{E}}(\sup_{0\le t\le\theta} |\int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+t} \bar{h}(s, x(s))d\langle B\rangle(s)|^2) \\
&+ 4\widehat{\mathbb{E}}(\sup_{0\le t\le\theta} |\int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+t} \bar{\sigma}(s, x(s))dB(s)|^2) \\
&+ 4\widehat{\mathbb{E}}(\sup_{0\le t\le\theta} |\int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+t} \int_{\mathbb{R}_0^d} \bar{K}(s, x(s), z)L(ds, dz)|^2) \\
\le&4\theta \int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+\theta} \widehat{\mathbb{E}}(|\bar{f}(s, x(s))|^2)ds + 4\theta C_2^{'} \int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+\theta} \widehat{\mathbb{E}}(|\bar{h}(s, x(s))|^2)ds \\
&+ 4C_2 \int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+\theta} \widehat{\mathbb{E}}(|\bar{\sigma}(s, x(s))|^2)ds + 4C_\theta\widehat{\mathbb{E}}[\int_{\beta_{2l-1}\wedge N}^{\beta_{2l-1}\wedge N+\theta} \sup_{v\in V} \int_{\mathbb{R}_0^d} \bar{K}^2(s, x(s), z)(ds)dz] \\
\le&4\theta C_5^2(\theta + \theta C_2^{'} + C_2 + C_\theta).
\end{aligned}$$

This, together with (3.44) and Chebyshev inequality, we can get

$$\widetilde{c}(\sup_{0\le t\le\theta} |\bar{x}(\beta_{2l-1} \wedge N + t) - \bar{x}(\beta_{2l-1} \wedge N)| \ge \delta) \le \varepsilon.$$

By (3.45) and the above inequality, we can obtain

$$\begin{aligned}
&\widetilde{c}(\{\beta_{2\varpi} \le N\} \cap \{\sup_{0\le t\le v} ||\bar{x}(\beta_{2l-1} + t)| - |\bar{x}(\beta_{2l-1})|| < \delta\}) \\
=&\widetilde{c}(\beta_{2\varpi} \le N) - \widetilde{c}(\{\beta_{2\varpi} \le N\} \cap \{\sup_{0\le t\le v} ||\bar{x}(\beta_{2l-1} + t)| - |\bar{x}(\beta_{2l-1})|| \ge \delta\}) \\
\ge&\widetilde{c}(\beta_{2\varpi} \le N) - \widetilde{c}(\sup_{0\le t\le\theta} ||\bar{x}(\beta_{2l-1} \wedge N + t)| - |\bar{x}(\beta_{2l-1} \wedge N)|| \ge \delta) \ge \varepsilon.
\end{aligned}$$

Hence,

$$\widetilde{c}(\{\beta_{2\varpi} \le N\}) \cap \{\beta_{2l} - \beta_{2l-1} \ge \theta\}) \ge \varepsilon. \tag{3.47}$$

At lastly, by (3.41), (3.46) and (3.47), we have

$$\begin{aligned} C_4 &= \widetilde{\mathbb{E}} \int_0^\infty |x(t)|^2 dt \\ &\ge \sum_{i=1}^{\varpi} \widetilde{\mathbb{E}}(I_{\{\beta_{2\varpi} \le N\}} \int_{\beta_{2l-1}}^{\beta_{2l}} |\bar{x}(t)|^2 dt) \ge \varepsilon \sum_{i=1}^{\varpi} \widetilde{\mathbb{E}}(I_{\{\beta_{2\varpi} \le N\}}(\beta_{2l} - \beta_{2l-1})) \\ &\ge \varepsilon\theta \sum_{i=1}^{\varpi} \widetilde{c}(\{\beta_{2\varpi} \le N\} \cap \{\beta_{2l} - \beta_{2l-1} \ge \theta\}) \ge \varepsilon^2 \theta \varpi. \end{aligned}$$

However, this contradicts with the inequality in (3.44). Therefore the required assertion (3.34) holds. This completes the proof. □

4. Example

In order to illustrate the obtained theory, in this section, given a stochastic differential equation driven by G-Lévy process, we design delay feedback controls $u(t, x(t-\tau))$ in the drift part such that the control systems have asymptotical stability in mean square and quasi-surely asymptotical stability.

Example 4.1. *Consider the following stochastic differential equation driven by the G-Lévy process:*

$$dx(t) = f(t, x(t))dt + h(t, x(t))d\langle B\rangle(t) + \sigma(t, x(t))dB(t) + \int_{\mathbb{R}_0^d} K(t, x(t), z)L(dt, dz), t \ge 0. \tag{4.1}$$

Assume

$$f(t, x) = x - x^3, h(t, x) = x, \sigma(t, x) = \sqrt{2}x, K(t, x, z) = 2xR(z).$$

Moreover, we assume that the function R(z) satisfies

$$\begin{aligned} \frac{1}{2} - \sup_{Q\in\mathcal{Q}} tr[IQQ^T] &< \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} \Big(|R(z)|^2 + R(z)\Big)v(dz) \\ &< \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} \Big((1 + 2R(z))^6 - 1\Big)v(dz) < 1 - \sup_{Q\in\mathcal{Q}} tr[IQQ^T], \end{aligned}$$

where I is a $d \times d$ dimensional matrix whose elements is 1.

Now, we define the delay feedback control $u(t, x(t-\tau)) = -10x(t-\tau)$. Consider the corresponding control systems

$$\begin{aligned} dx(t) = &[f(t, x(t)) + u(t, x(t-\tau)]dt + h(t, x(t))d\langle B\rangle(t) + \sigma(t, x(t))dB(t) \\ &+ \int_{\mathbb{R}_0^d} K(t, x(t), z)L(dt, dz), \qquad t \ge 0. \end{aligned} \tag{4.2}$$

Thus the Assumption 3.1 holds obviously. Next, our aim is to verify Assumptions 3.2—3.3 and get the bound of the time delay τ.

To verify the Assumption 3.2, define $\bar{W}(t,x(t)) = x^6$, then we obtain

$$\begin{aligned}
&\mathfrak{L}\bar{W}(t,x) + \bar{W}_x(t,x)u(t,y)\\
&= \bar{W}_t(t,x) + \langle \bar{W}_x(t,x), f(t,x)\rangle + \bar{W}_x(t,x)u(t,y) + \sup_{Q\in\mathcal{Q}} tr[(\langle \bar{W}_x(t,x), h(t,x)\rangle\\
&\quad + \frac{1}{2}\langle \bar{W}_{xx}(t,x)\sigma(t,x), \sigma(t,x)\rangle)QQ^T] + \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} (\bar{W}(t,x+K(t,x,z)) - \bar{W}(t,x))v(dz)\\
&= 6x^6 - 6x^8 + [36 \sup_{Q\in\mathcal{Q}} tr[IQQ^T] + \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} [(1+2R(z))^6 - 1]v(dz)]x^6 - 60x^5 y\\
&\le 42x^6 - 6x^8 - 60x^5 y.
\end{aligned}$$

Noting that $60x^5y \le 30x^6 + 30y^6$, which means

$$\mathfrak{L}\bar{W}(t,x) + \bar{W}_x(t,x)u(t,y) \le 105x^6 - 6x^8 - 33x^6 + 30y^6.$$

Let $W_1(t,x) = 3x^6$, $q_1 = \sup_{x\in\mathbb{R}}(105x^6 - 6x^8) < \infty$, $q_2 = 11$ and $q_3 = 10$, the Assumption 3.2 is fulfilled.
To verify the Assumption 3.3, define $W(t,x(t)) = x^2 + 0.5x^4$, and let $\alpha_1 = \alpha_2 = \alpha_3 = 0.1$, $\alpha_4 = \alpha_5 = 0.6$, then

$$\begin{aligned}
&\mathfrak{L}W(t,x(t)) + W_x(t,x(t))u(t,x(t)) + \alpha_1|W_x(t,x(t))|^2 + \alpha_2|f(t,x(t))|^2\\
&\quad + \alpha_3|h(t,x(t))|^2 + \alpha_4|\sigma(t,x(t))|^2 + \alpha_5 \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} K^2(v,x(v),z)v(dz)\\
&\le 2x^2 - 2x^6 - 20x^2 - 20x^4 + \sup_{Q\in\mathcal{Q}} tr[(2x^2 + 2x^4 + 2x^2 + 6x^4)QQ^T]\\
&\quad + 4\sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} (|R(z)|^2 + R(z))v(dz)x^2 + 0.5\sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} \Big((1+2R(z))^4 - 1\Big)v(dz)x^4\\
&\quad + \alpha_1(4x^2 + 8x^4 + 4x^6) + \alpha_2(x^2 - 2x^4 + x^6) + \alpha_3 x^2 + 2\alpha_4 x^2\\
&\quad + 4\alpha_5 x^2 \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} |R(z)|^2 v(dz)\\
&\le \Big(2 - 20 + 4\sup_{Q\in\mathcal{Q}} tr[IQQ^T] + 4\sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} (|R(z)|^2 + R(z))v(dz) + 4\alpha_1 + \alpha_2\\
&\quad + \alpha_3 + 2\alpha_4 + 4\alpha_5 \sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} |R(z)|^2 v(dz)\Big)x^2 + \Big(-20 + 16\sup_{Q\in\mathcal{Q}} tr[IQQ^T]\\
&\quad + 0.5\sup_{v\in\mathcal{V}} \int_{\mathbb{R}_0^d} [(1+2R(z))^4 - 1]v(dz) + 8\alpha_1 - 2\alpha_2\Big)x^4 + \Big(-2 + 4\alpha_1 + \alpha_2\Big)x^6\\
&\le -9.8x^2 - 3.4x^4 - 1.5x^6.
\end{aligned}$$

Thus, the Assumption 3.3 are satisfied. Moreover, we could let $\alpha_0 = 9.8$ and $C_2' \vee C_2 \vee C_\tau < 50$. From the (3.13), we have $\tau < 0.0012$. Hence the solution of G-SDDEs (4.2) is asymptotically stable in mean square and quasi-surely asymptotically stable.

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