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Some applications of p-(DPL) sets

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Abstract. In this paper, we introduce a new class of subsets of class bounded linear operators between Banach spaces which is called p-(DPL) sets. Then, the relationship between these sets with equicompact sets is investigated. Moreover, we define p-version of Right sequentially continuous differentiable mappings and get some characterizations of these mappings. Finally, we prove that a mapping $f : X \to Y$ between real Banach spaces is Fréchet differentiable and f' takes bounded sets into p-(DPL) sets if and only if f may be written in the form $f = g \circ S$ where the intermediate space is normed, S is a Dunford-Pettis p-convergent operator, and g is a Gáteaux differentiable mapping with some additional properties.

1. Introduction

The study localized properties in the geometry of Banach spaces, e.g., p-(V) sets and p-Right sets show how these notions can be used to study more global structure properties. For instance, it is well known [14], that a bounded linear operator $T \in L(X, Y)$ between Banach spaces is Dunford-Pettis p-convergent iff it's adjoint $T^* \in L(Y^*, X^*)$ takes bounded subsets of Y^* into p-Right subsets of X^*. Motivated by this work and the research works of Cilia et al. [9–11], we give similar results for differentiable mappings. In this paper, we introduce the notions p-(DPL) sets and p-Right sequentially continuous differentiable mappings. Then, we answer to the following interesting questions:

- For given a differentiable mapping $f : U \to Y$ whose it's derivative $f' : U \to L(X, Y)$ is uniformly continuous on the U-bounded subsets of U, under which conditions does f' takes U-bounded Dunford-Pettis weakly p-precompact subsets of U into p-(DPL) subsets of $L(X, Y)$?
- If $f : X \to Y$ is a differentiable mapping between real Banach spaces, then under which conditions does f' takes bounded sets into p-(DPL) sets?

The present paper is organized as follows:
Section 2 of this article provides a wide range of definitions and concepts in Banach spaces. In Section 3, we introduce the concepts of p-(DPL) sets in $L(X, Y)$ and p-Right sequentially continuous differentiable mappings. In Section 4, we obtain a factorization result for differentiable mappings through Dunford-Pettis p-convergent operators.

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2. Notions and Definitions

Throughout this paper X, Y and Z will always denote real Banach spaces and U is an open convex subset of X. We denote the class of all bounded linear operators and weakly compact operators from X into Y by $L(X, Y)$ and $W(X, Y)$, respectively. The topological dual of X is denoted by X^* and the adjoint of an operator T is denoted by T^*. Also, we use $\langle x^*, x\rangle$ or $x^*(x)$ for the duality between $x \in X$ and $x^* \in X^*$. We denote the closed unit ball of X and the identity operator on X by B_X and id_X respectively. p^* will always denote the conjugate number of p for $1 \leq p < \infty$; if $p = 1$, ℓ_{p^*} plays the role of c_0. In this paper $1 \leq p \leq \infty$, except for the cases where we consider other assumptions.
A sequence $(x_n)_n$ in X is called weakly p-summable, if $(x^*(x_n))_n \in \ell_p$ for each $x^* \in X^*$. We denote the space of all weakly p-summable sequences in X by $\ell_p^w(X)$; see [12]. The weakly ∞-summable sequences are precisely the weakly null sequences. A sequence $(x_n)_n$ in X is called weakly p-convergent to $x \in X$ if $(x_n - x)_n \in \ell_p^w(X)$. A bounded subset K of X is said to be relatively weakly p-compact, if every sequence in K has a weakly p-convergent subsequence with limit in X; see [6]. A sequence $(x_n)_n$ in X is called weakly p-Cauchy, provided that $(x_{m_k} - x_{n_k})_k \in \ell_p^w(X)$ for any increasing sequences $(m_k)_k$ and $(n_k)_k$ of positive integers; see [8]. A subset K of X is said to be weakly p-precompact, provided that every sequence from K has a weakly p-Cauchy subsequence; see [8]. The weakly ∞-precompact sets are precisely the weakly precompact sets or Rosenthal sets. An operator $T \in L(X, Y)$ is said to be weakly p-precompact if $T(B_X)$ is weakly p-precompact. An operator $T \in L(X, Y)$ is called p-convergent if $\lim_{n\to\infty} \| T(x_n) \| = 0$ for all $(x_n)_n \in \ell_p^w(X)$; see [6]. We denote the space of all p-convergent operators from X into Y, by $C_p(X, Y)$. If the identity operator on X is p-convergent (in short, $id_X \in C_p$), we say that a Banach space X has the p-Schur property, which is equivalent to every weakly p-compact subset of X is norm compact. A Banach space X is said to have the Dunford-Pettis property of p (in short, $X \in (DPP_p)$), provided that for any Banach space Y, every weakly compact operator $T : X \to Y$ is p-convergent; see [6]. A bounded subset K of X is a p-(V^*) set if $\lim_{n\to\infty} \sup_{x\in K} |x_n^*(x)| = 0$, for every weakly p-summable sequence $(x_n^*)_n$ in X^*; see [17]. A bounded subset K of X is Dunford-Pettis, if every weakly null sequence $(x_n^*)_n$ in X^*, converges uniformly to zero on the set K [3]. For convenience, we apply the notions p-Right null and p-Right Cauchy sequences instead of weakly p-summable and weakly p-Cauchy sequences which are Dunford-Pettis sets, respectively. An operator $T \in L(X, Y)$ is said to be Dunford-Pettis p-convergent if it takes p-Right null sequences to norm null sequences; see [14]. The space of all Dunford-Pettis p-convergent operators from X into Y is denoted by $DPC_p(X, Y)$.
Given $x, y \in X$, the segment with bounds x and y denoted by $I(x, y)$. A subset B of U is U-bounded if it is bounded and the distance between B and the boundary of U is strictly positive; see [10]. The space of all differentiable mappings $f : U \to Y$ whose derivative $f' : U \to L(X, Y)$ is uniformly continuous on U-bounded subsets of U will be denoted by $C^{1u}(U, Y)$; see [9]. For given a mapping $f : U \to Y$ and a class $\mathcal{M}$ of subsets of U such that every singleton belongs to $\mathcal{M}$, the mapping f is $\mathcal{M}$-differentiable at $x \in U$ if there exists an operator $f'(x) \in L(X, Y)$ such that

$$\lim_{\varepsilon\to 0} \frac{f(x + \varepsilon y) - f(x) - f'(x)(\varepsilon y)}{\varepsilon} = 0$$

uniformly to y on each member of $\mathcal{M}$. In this case, we write $f \in D_{\mathcal{M}}(x, Y)$; see [16]. We say that a mapping f is Gâteaux differentiable at x if $f \in D_{\mathcal{M}}(x, Y)$ where $\mathcal{M}$ is the class of all single-point subsets of X. We also, say that f is Fréchet differentiable at x if $f \in D_{\mathcal{M}}(x, Y)$, where $\mathcal{M}$ is the class of all bounded subsets of X.

3. p-Right sequentially continuous differentiable mappings

In this Section, we find some equivalent conditions for all differentiable mappings $f : U \to Y$ whose derivative $f' : U \to L(X, Y)$ is uniformly continuous on U-bounded subsets of U such that f' takes U-bounded Dunford-Pettis and weakly p-precompact subsets of U into p-(DPL) subsets of $L(X, Y)$.

Definition 3.1. *Let $K \subset L(X, Y)$ and $1 \leq p \leq \infty$. We say that K is a p-(DPL) set if for every p-Right null sequence $(x_n)_n$ in X, it follows:*

$$\lim_n \sup_{T\in K} \|T(x_n)\| = 0.$$

Note that the definition of a p-(DPL) set in X^* coincides with the definition of a p-Right set introduced by Ghenciu [14]. Recall that a subset K of X^* is said to be a p-Right set provided that each p-Right null sequence $(x_n)_n$ in X tends to 0 uniformly on K.
The following Proposition gives some additional properties of p-(DPL) sets.

Proposition 3.2. (i) *If $K \subset DPC_p(X, Y)$ is a relatively compact set, then K is a p-(DPL) set in $L(X, Y)$;*
(ii) *Absolutely closed convex hull of a p-(DPL) set in $L(X, Y)$ is p-(DPL);*
(iii) *If $K \subset L(X, Y)$ is a p-(DPL) set, then every $T \in K$ is a Dunford-Pettis p-convergent operator;*
(iv) *If $K_1, \cdots, K_n$ are p-(DPL) sets in $L(X, Y)$, then $\bigcup_{i=1}^{n} K_i$ and $\sum_{i=1}^{n} K_i$ are p-(DPL) sets in $L(X, Y)$.*

Remark 3.3. (i) *It is clear that every q-(DPL) subset of $L(X, Y)$ is p-(DPL), whenever $1 \le p < q \le \infty$. Also, it is interesting to obtain conditions under which every p-(DPL) set in the space $L(X, Y)$ is q-(DPL). In my opinion, this is very interesting but, it's a difficult question. In particular if $K \subset X^*$, we answer to this question. Indeed, we obtain a characterization for those Banach spaces in which p-(DPL) sets are q-(DPL) (see Definition* 4.1 *and Theorem* 4.4 *in [2]).*
(ii) *Every relatively weakly compact subset of a topological dual Banach space is p-(DPL), while the converse of this implication is false. For instance, the unit ball of ℓ_∞ is a p-(DPL) set, but it is not weakly compact.*
(iii) *There is a relatively weakly compact set in $K(c_0, c_0)$ so that is not a p-(DPL) set. In fact, consider the operator $T : \ell_2 \to K(c_0, c_0)$ given by $T(\alpha)(x) = (\alpha_n x_n)$, $\quad \alpha = (\alpha_n) \in \ell_2$, $\quad x = (x_n) \in c_0$. It is clear that $T(B_{\ell_2})$ is relatively weakly compact, while it is not a p-(DPL) set in $K(c_0, c_0)$, since $T(e_n^2)(e_n) = e_n$.*

A subsets M of $K(X, Y)$ is said to be equicompact if for every bounded sequence $(x_n)_n$ in X, there exists a subsequence $(x_{k_n})_n$ such that $(Tx_{k_n})_n$ is uniformly convergent for $T \in M$; see [18].

Theorem 3.4. *Let X be a Banach space and $1 \le p \le \infty$. If there exists a non-zero Banach space Y so that every p-(DPL) subset of $K(X, Y)$ is equicompact, then $DPC_p(X, Y) = K(X, Y)$.*

Proof. Since the p-Right sets in X^* coincides with the p-(DPL) subsets of X^*, it is enough to show that every p-(DPL) subset M of X^* is relatively compact; see ([2, Theorem 3.15]). For this purpose, consider $y_0 \in S_Y$ and put $H = M \otimes \{y_0\}$. Obviously, H is a p-(DPL) subset of $K(X, Y)$. Hence, by the hypothesis, H is equicompact, which yields the equicompactness of M as a subset of $K(X, \mathbb{R})$. Hence, an application of Lemma 2.1 in [19] shows that, M is relatively compact. □

A subset M of $K(X, Y)$ is said to be collectively compact, if $\bigcup_{T \in M} T(B_X)$ is a relatively compact set. Recall that $M \subset K(X, Y)$ is equicompact if and only if $M^* = \{T^* : T \in M\}$ is collectively compact; see [18].

Proposition 3.5. *If $S : X \to Z$ is a weakly p-precompact operator, then for any Banach space Y and any $N \subset DPC_p(Z, Y)$ which is p-(DPL), the set $N \circ S := \{T \circ S : T \in N\}$ is equicompact.*

Proof. . We prove that $S^* \circ N^*$ is collectively compact. Consider a sequence $((S^* \circ T_n^*)y_n^*)_n$ in $\bigcup_{T \in N} S^* \circ T^*(B_Y)$ and put $A := \{T_n^* y_n^* : n \in \mathbb{N}\}$. It is easy to verify that, A is a p-(DPL) set in Z^*. Indeed, if $(z_n)_n$ is a p-Right null sequence in Z, we have

$$\lim_{n\to\infty} \sup_m |\langle z_n, T_m^*(y_m^*)\rangle| \le \lim_{n\to\infty} \sup_m \| T_m(z_n) \| = 0.$$

Let $(z_n^*)_n \subset A$ and let be a p-Right null sequence in Z, Consider an operator $S_1 : Z \to \ell_\infty$ defined by $S_1(z) := (z_n^*(z))$. Since A is a p-(DPL) set in Z^*, $\lim_n \| S_1(z_n) \| = \lim_n \sup_i |z_i^*(z_n)| = 0$, and so S_1 is Dunford-Pettis p-convergent. Hence, the operator $S_1 S : X \to \ell_\infty$ is compact, since $S : X \to Z$ is a weakly p-precompact operator. Thus $S^* \circ S_1^*$ is compact and so, $S^*(z_n^*)_n = (S^*(S_1^*(e_n^1))_n$ is relatively compact, where (e_n^1) is the unit basis of ℓ_1. Hence, $S^*(A)$ is a relatively compact set and so, $((S^* \circ T_n^*)y_n^*)_n$ has a convergent subsequence. □

Recall that, a subset K of $W(X, Y)$ is weakly equicompact if for every bounded sequence $(x_n)_n$vin X, there exists a subsequence $(x_{k_n})_n$ such that $(T(x_{k_n}))_n$ is weakly uniformly convergent for $T \in K$; see [19].

Proposition 3.6. *Let X be a Banach space and $1 \leq p \leq \infty$. If there exists a non-zero Banach space Y such that every p-(DPL) set of $W(X,Y)$ is weakly equicompact, then $DPC_p(X,Y) = K(X,Y)$.*

Proof. Let K be a p-(DPL) set in X^*. We show K is relatively compact. For this purpose, choose $y_0 \in Y$ and $y_0^* \in Y^*$ so that $\langle y_0^*, y_0\rangle = 1$. It is easy to verify that $M = K \bigotimes \{y_0\}$ is a p-(DPL) set in $W(X,Y)$ and so, by the hypothesis, M is weakly equicompact. Hence, by using Proposition 2.2 of [19], $K = \langle y_0^*, y_0\rangle K = M^*(y_0^*)$ is relatively compact. □

Definition 3.7. *Let $U \subset X$ be an open convex and $1 \leq p \leq \infty$. We say that the mapping $f : U \to Y$ is p-Right sequentially continuous or Right-sequentially continuous of order p, if it takes p-Right Cauchy U-bounded sequences of U into norm convergent sequences in Y. We denote the space of all such mappings by $C^p_{rsc}(U,Y)$.*

Note that, the mapping $f : U \to Y$ is ∞-Right sequentially continuous or Right-sequentially continuous, if it takes Right-Cauchy U-bounded sequences of U into norm convergent sequences in Y. It is easy to verify that if f is compact and takes U-bounded p-Right Cauchy sequences into weakly Cauchy sequences, then $f \in C^p_{rsc}(U,Y)$. Also, it is easy to verify that $C^q_{rsc}(U,Y) \subseteq C^p_{rsc}(U,Y)$ whenever $1 \leq p < q \leq \infty$. But, we do not have any example of a mapping $f \in C^{1u}(U,Y) \cap C^p_{rsc}(U,Y)$ which does not belong to $C^q_{rsc}(U,Y)$. Hence, it would be interesting to get conditions under which every p-Right sequentially continuous map is q-Right sequentially continuous. In my opinion, this is very interesting, but it is a difficult question?

Proposition 3.8. *Let U be an open convex subset of X and let $1 \leq p \leq \infty$. If $f \in C^{1u}(U,Y)$ so that $f' : U \to DPC_p(X,Y)$ is Right-sequentially continuous on U-bounded sets, then f' takes Dunford-Pettis U-bounded sets into p-(DPL) sets.*

Proof. Let K be a U-bounded and Dunford-Pettis set. It is well known that, K is a Rosenthal set (see,([13, Corollary 17])). So, by the hypothesis, $f'(K)$ is relatively norm compact in $DPC_p(X,Y)$. Hence, by the part (i) of Proposition 3.2, $f'(K)$ is a p-(DPL) set. □

Proposition 3.9. *If $f : U \to Y$ is a differentiable mapping such that $f' \in C^p_{rsc}(U, DPC_p(X,Y))$, then $f \in C^p_{rsc}(U,Y)$.*

Proof. Let $(x_n)_n$ be a U-bounded and p-Right Cauchy sequence. Therefore, for any increasing sequences $(m_k)_k$ and $(n_k)_k$ of positive integers the sequence $(x_{m_k} - x_{n_k})_k$ is weakly p-summable in X. By the Mean Value Theorem ([7, Theorem 6.4]), we have

$$\|f(x_{m_k}) - f(x_{n_k})\| \leq \|f'(c_k)(x_{m_k} - x_{n_k})\| \tag{1}$$

for some $c_k \in I(x_{n_k}, x_{m_k})$. Since the sequence (c_k) is U-bounded and p-Right Cauchy, the sequence $(f'(c_k))_k$ is norm convergent to some $T \in DPC_p(X,Y)$. So we have

$$\lim_{k\to\infty} \|f'(c_k)(x_{m_k} - x_{n_k})\| = \lim_{k\to\infty} \|f'(c_k)(x_{m_k} - x_{n_k}) - T(x_{m_k} - x_{n_k}) + T(x_{m_k} - x_{n_k})\|$$

$$\leq \lim_{k\to\infty} \|f'(c_k)(x_{m_k} - x_{n_k}) - T(x_{m_k} - x_{n_k})\| + \lim_{k\to\infty} \|T(x_{m_k} - x_{n_k})\|$$

$$\leq \lim_{k\to\infty} \|f'(c_k) - T\|\|x_{m_k} - x_{n_k}\| + \lim_{k\to\infty} \|T(x_{m_k} - x_{n_k})\| = 0.$$

As a consequence of inequality (1), we get $\lim_{k\to\infty} \|f(x_{m_k}) - f(x_{n_k})\| = 0$. Hence, the sequence $(f(x_n))_n$ is norm convergent. □

Theorem 3.10. *If $f : U \to Y$ is a differentiable mapping such that for every U-bounded and Dunford-Pettis set K, $f'(K)$ is a p-(DPL) set in $L(X,Y)$, then $f \in C^p_{rsc}(U,Y)$.*

Proof. Let $(x_n)_n$ be a U-bounded and weakly p-Right Cauchy sequence. So for any increasing sequences $(m_k)_k$ and $(n_k)_k$ of positive integers, $(x_{m_k} - x_{n_k})_k$ is a p-Right null sequence in X. Since U is convex, the segment $I(x_{n_k}, x_{m_k})$ is contained in U for all $k \in \mathbb{N}$. Applying the Mean Value Theorem ([7, Theorem 6.4]), there exists $c_k \in I(x_{n_k}, x_{m_k})$ so that

$$\|f(x_{m_k}) - f(x_{n_k})\| \leq \|f'(c_k)(x_{m_k} - x_{n_k})\| \leq \sup_{T \in f'(K)} \|T(x_{m_k} - x_{n_k})\| \tag{2}$$

in which $K := \{c_k : k \in \mathbb{N}\}$. Obviously, the set $K := \{c_k : k \in \mathbb{N}\}$ is contained in the convex hull of all x_n and then in U, since U is a convex set. Moreover K is still a U-bounded and Dunford-Pettis set. Therefore by the hypothesis, $f'(K)$ is a p-(DPL) set in $L(X, Y)$. Since $(x_{m_k} - x_{n_k})_k$ is a p-Right null sequence in X, it follows that $\lim_{k\to\infty} \sup_{T\in f'(K)} \|T(x_{m_k} - x_{n_k})\| = 0$. Hence, the inequality (2) implies that $\lim_{k\to\infty} \|f(x_{m_k}) - f(x_{n_k})\| = 0$. □

In the sequel, we denote the space of all real-valued k-times continuously differentiable functions on X, by $C^k(X)$.

Example 3.11. *Let $h \in C^1(\mathbb{R})$ and $1 < r < 2$. We define $f : \ell_{r^*} \to \mathbb{R}$ by $f((x_n)_n) = \sum_{n=1}^{\infty} \frac{h(x_n)}{2^n}$. The same argument as in the* ([11, Example 2.4])*, shows that f is differentiable such that $f'((x_n)_n) = (\frac{h'(x_n)}{2^n})_n \in \ell_r$. By Pitt's Theorem* ([1, Theorem 2.1.4])*, $f' : \ell_{r^*} \to \ell_r$ is compact and so, $f'(B_{\ell_{r^*}})$ is a relatively compact set in* $L(\ell_{r^*}, \mathbb{R}) = C_p(\ell_{r^*}, \mathbb{R})$. *Thus, the part* (i) *of Proposition* 3.2*, yields that $f'(B_{\ell_{r^*}})$ is a p-(DPL) set in* $L(\ell_{r^*}, \mathbb{R})$. *Hence, Theorem* 3.10 *implies that* $f \in C^p_{rsc}(U, Y)$.

In the following result, we find a method to get p-(DPL) subsets of $L(X, Y)$. Note that we adapt the proof of ([9, Theorem 2.1]).

Theorem 3.12. *Let $U \subseteq X$ be an open convex subset and $1 \leq p \leq \infty$. If $f \in C^{1u}(U, Y)$, then the following assertions are equivalent:*
(i) $f \in C^p_{rsc}(U, Y)$;
(ii) *For every U-bounded p-Right Cauchy sequence (x_n) and every p-Right Cauchy sequence $(h_n) \subset X$, the sequence $(f'(x_n)(h_n))_n$ is norm converges in Y;*
(iii) *For every U-bounded p-Right Cauchy sequence $(x_n)_n$ and every p-Right null sequence $(h_n)_n \subset X$, we have*

$$\lim_n \sup_m \| f'(x_m)(h_n) \| = 0;$$

(iv) *For every U-bounded p-Right Cauchy sequence $(x_n)_n$ and every p-Right null sequence $(h_n)_n \subset X$, we have*

$$\lim_n f'(x_n)(h_n) = 0;$$

(v) *f' takes U-bounded, Dunford-Pettis and weakly p-precompact subsets of U into p-(DPL) subsets of $L(X, Y)$.*

Proof. (i) ⇒ (ii) Let $(x_n)_n$ be a U-bounded p-Right Cauchy sequence and let $(h_n)_n$ be a p-Right Cauchy sequence in X. Without loss of generality, we assume that $\sup_n \| h_n \| < 1$. Consider $B := \{x_n : n \in \mathbb{N}\}$ and let $d := min\{1, dist(B, \partial U)\}$. It is easy to show that the set

$$B' := B + \frac{d}{2} B_X \subset U$$

is also U-bounded. Since $f \in C^{1u}(U, Y)$, f' is uniformly continuous on B'. Hence, for given $\varepsilon > 0$, there exists $0 < \delta < \frac{d}{4}$ such that if $t_1, t_2 \in B'$ satisfy $\| t_1 - t_2 \| < 2\delta$, then

$$\| f'(t_1) - f'(t_2) \| < \frac{\varepsilon}{4}. \tag{3}$$

If $c \in I(x_n, x_n + \delta h_n)$ for some $n \in \mathbb{N}$, then

$$\| c - x_n \| \leq \delta \| h_n \| < \delta < 2\delta < \frac{d}{2}$$

and so,

$$c = x_n + (c - x_n) \in B' = B + \frac{d}{2} B_X$$

As an immediate consequence of the Mean Value Theorem ([7, Theorem 6.4]), and formula (3), we get

$$\| f'(x_n)(\delta h_n) - f(x_n + \delta h_n) + f(x_n) \|$$

$$\leq \sup_{c \in I(x_n, x_n + \delta h_n)} \| f'(c) - f'(x_n) \| \| \delta h_n \| \leq \frac{\varepsilon\delta}{4}.$$

Similarly,

$$\| f(x_m + \delta h_m) - f(x_m) - f'(x_m)(h_m) \|$$

$$\leq \sup_{c \in I(x_m, x_m + \delta h_m)} \| f'(c) - f'(x_m) \| \| \delta h_m \| \leq \frac{\varepsilon\delta}{4}.$$

On the other hand, the sequences $(x_n + \delta h_n)_n$ and $(x_n)_n$ are U-bounded and p-Right Cauchy in U. Hence, by the hypothesis the sequences $(f(x_n + \delta h_n))_n$ and $(f(x_n))_n$ are norm convergent in Y. Hence, we can find $n_0 \in \mathbb{N}$ so that for $n, m > n_0$:

$\| f(x_n + \delta h_n) - f(x_m + \delta h_m) \| < \frac{\varepsilon\delta}{4}$, $\quad \| f(x_n) - f(x_m) \| < \frac{\varepsilon\delta}{4}$

So, for $n, m > n_0$, we have

$$\| f'(x_n)(h_n) - f'(x_m)(h_m) \| < \varepsilon.$$

(ii) ⇒ (iii) Let $(x_n)_n$ be a U-bounded p-Right Cauchy sequence and let $(h_n)_n$ be a p-Right null sequence in X. By the part (ii), for every $h \in X$, the set $\{f'(x_n)(h) : n \in \mathbb{N}\}$ is bounded in Y. On the other hand, there exists a subsequence $(x_{m_k})_k$ of $(x_m)_m$ in U such that

$$\| f'(x_{m_k})(h_k) \| \geq \sup_m \| f'(x_m)(h_k) \| - \frac{1}{k} \quad (k \in \mathbb{N}).$$

Since the sequences $(x_{m_k})_k$ in U and $(h_1, 0, h_2, 0, h_3, 0, \cdots)$ in X are p-Right Cauchy, the sequence

$$(f'(x_{m_1})(h_1), 0, f'(x_{m_2})(h_2), 0, f'(x_{m_3})(h_3), 0, \cdots)$$

norm convergent in Y. Therefore, $\lim_k \| f'(x_{m_k})(h_k) \| = 0$. Hence, we have

$$\lim_k \sup_m \| f'(x_m)(h_k) \| = 0.$$

(iii) ⇒ (iv) is obvious.
(iv) ⇒ (v) Let K be a Dunford-Pettis, weakly p-precompact and U-bounded set. It is clear that, for every $h \in X$, the set $f'(K)(h)$ is bounded in Y. Let $(h_n)_n$ be a p-Right null sequence in X. If $(h_{n_k})_k$ is a subsequence of $(h_n)_n$, then for every $k \in \mathbb{N}$, there exists $a_k \in K$ such that

$$\sup_{a \in K} \| f'(a)(h_{n_k}) \| < \| f'(a_k)(h_{n_k}) \| + \frac{1}{k}.$$

Since K is a Dunford-Pettis and weakly p-precompact set, the sequence $(a_k)_k$ admits a p-Right Cauchy subsequence $(a_{k_r})_r$. Hence, by the hypothesis we have

$$\lim_r \| f'(a_{k_r})(h_{n_{k_r}}) \| = 0.$$

So, every subsequence of $(\sup_{a\in K} \| f'(a)(h_n) \|)_n$ has a subsequence converging to 0. Hence, the sequence itself converges to 0, that is, $\lim_n \sup_{a\in K} \| f'(a)(h_n) \|= 0$.
(v) ⇒ (i) Since the proof is similar to the proof of Proposition 3.15, its proof is omitted. □

Let us recall from [2], that a bounded subset K of X is a p-Right* set, if $\lim_{n\to\infty} \sup_{x\in K} |x_n^*(x)| = 0$, for every p-Right null sequence $(x_n^*)_n$ in X^*.

Proposition 3.13. *Let $K \subset L(X,Y)$ be a p-(DPL) set and $X^* \in (DPP_p)$, whenever $2 < p \leq \infty$. If $S \in L(G,X)$ is a bounded linear operator with Dunford-Pettis p-convergent adjoint, then the set $\{S^* \circ T^*(B_{Y^*}) : T \in K\}$ is relatively compact in G^*.*

Proof. Take a p-Right null sequence $(x_n)_n$ in X. Since K is a p-(DPL) set in $L(X,Y)$, we have

$$|\langle x_n, T^*(y^*)\rangle| \leq |\langle T(x_n), y^*\rangle| \leq\| T(x_n) \|\to 0$$

uniformly for $T \in K$ and $y^* \in B_{Y^*}$. So, $\{T^*(B_{Y^*}) : T \in K\}$ is a p-(DPL) set. Adapting of ([15, Proposition 3.5]), there are a Banach space Z and an operator L, that takes Right Cauchy sequences into norm convergent sequences, such that

$$\{T^*(B_{Y^*}) : T \in K\} \subset L^*(B_{Z^*}).$$

Therefore, we have

$$\{(S^* \circ T^*)(B_{Y^*}) : T \in K\} = S^*(\{T^*(B_{Y^*}) : T \in K\}) \subset S^*(L^*(B_{Z^*})).$$

Since S^* is Dunford-Pettis p-convergent, the part (iii) of ([2, Lemma 3.4]) implies that $S(B_G)$ is a p-Right* set in X and so, an application of Proposition 3.5 of [2] shows that $S(B_G)$ is a p-(V^*) set in X. Thus, it is Rosenthal set (see,([13, Corollary 17])). Hence, $L \circ S$ is compact and so, $(S^* \circ L^*)$ is compact and we are done. □

Recall from [14], that a Banach space X has the p-Dunford-Pettis relatively compact property (in short, p-$(DPrcP)$) if every p-Right null sequence $(x_n)_n$ in X is norm null.

Corollary 3.14. *Let $2 < p \leq \infty$ and $K \subset L(X,Y)$ be a p-(DPL) set. If X^* has both properties (DPP_p) and p-$(DPrcP)$, then the set $\{T^*(B_{Y^*}) : T \in K\}$ is relatively compact in X^*.*

A Banach space X has the p-(SR) property if every p-Right subset of X^* is relatively weakly compact; see [14].

Proposition 3.15. *Let X be a Banach space and let U be an open convex subset of X. If for every Banach space Y, every mapping $f \in C^{1u}(U,Y)$ whose derivative f' takes U-bounded sets into p-(DPL) sets, is weakly compact, then X has the p-(SR) property.*

Proof. Let $T : X \to c_0$ be a Dunford-Pettis p-convergent operator. We proved that T is weakly compact. Since

$$T'(x) = T, \quad \forall x \in X,$$

for every U-bounded set B and for every p-Right null sequence $(x_n)_n$, it follows

$$\lim_{n\to\infty} \sup_{x\in B} \| T'(x)(x_n) \|= \lim_{n\to\infty} \| T(x_n) \|= 0.$$

So, T' takes U-bounded sets into p-(DPL) sets. So, by the hypothesis, T is weakly compact. Hence, Theorem 3.10 of [14] implies that X has the p-(SR) property. □

4. Factorization theorem through a Dunford-Pettis p-convergent operator

Results on factorization through bounded linear operators of polynomials, holomorphic mappings and differentiable mappings between Banach spaces obtained in recent years by several authors. For instance, a factorization result for differentiable mappings through compact operators was obtained by Cilia et al.[11]. For more information in this area, we refer to [4, 5, 10, 16] and references therein.
In this section, for given a mapping $f : X \to Y$, we show that f is differentiable so that f' takes bounded sets into p-(DPL) sets if and only if it happens $f = g \circ S$, where S is a Dunford-Pettis p-convergent operator from X into a suitable normed space Z and $g : Z \to Y$ is a Gâteaux differentiable mapping with some additional properties.

Theorem 4.1. *Let $f : X \to Y$ be a mapping between real Banach spaces. Then the following assertions are equivalent:*
(a) *f is differentiable, f' takes U-bounded sets into p-(DPL) sets and f is p-Right sequentially continuous.*
(b) *There exist a normed space Z, a surjective operator $S : X \to Z$, and a mapping $g : Z \to Y$ such that:*
(i) *$f(x) = g(S(x))$ for all $x \in X$.*
(ii) *S is a Dunford-Pettis p-convergent.*
(iii) *$g \in D_{\mathcal{M}}(S(x), Y)$ for every $x \in X$, where*

$$\mathcal{M} := \{S(B) : B \text{ is a bounded subset of } X\}.$$

(iv) *g' is bounded on $S(B)$ for every bounded subset $B \subset X$.*

Proof. (a) $\Rightarrow$ (b) Let $K := \bigcup_{r=1}^{\infty} \frac{f'(rB_X)}{r\|f'\|_{rB_X}}$. By hypothesis, for every $r \in \mathbb{N}$, $f'(rB_X)$ is a p-(DPL) set. First of all we cliam that K is a p-(DPL) set. For this purpose, for a fixed natural number N, we define $A_N := \bigcup_{r\le N} \frac{f'(rB_X)}{r\|f'\|_{rB_X}}$ and $B_N := \bigcup_{r>N} \frac{f'(rB_X)}{r\|f'\|_{rB_X}}$. Proposition 3.2(i) implies that A_N is a p-(DPL) set. Now let $(x_n)_n$ be a p-Right null sequence in X and $M = \sup_n \|x_n\|$. Hence

$$\begin{aligned}
\limsup_{n\to\infty} \sup_{T\in K} \|T(x_n)\| &= \inf_{N\in\mathbb{N}} max\{\limsup_{n\to\infty} \sup_{T\in A_N} \|T(x_n)\|, \limsup_{n\to\infty} \sup_{T\in B_N} \|T(x_n)\|\} \\
&\le \inf_{N\in\mathbb{N}} max\{0, \limsup_{n\to\infty} \sup_{T\in B_N} \|T(x_n)\|\} \\
&\le \inf_{N\in\mathbb{N}} \limsup_{n\to\infty} \sup_{T\in B_N} \|T\|\|x_n\| \le \inf_{N\in\mathbb{N}} \frac{M}{N} = 0.
\end{aligned}$$

So, $\lim_{n\to\infty} \sup_{T\in K} \|T(x_n)\| = 0$ and hence, K is a p-(DPL) set. As in [10], we define a continuous seminorm on X by $\|x\|_K := \sup_{\phi\in K} \|\phi(x)\|$ for all $x \in X$. It is clear that the set $V_K := \{x \in X : \|x\|_K = 0\}$ is a closed linear subspace of X. Let π be the canonical quotient map of X onto the quotient space $\frac{X}{V_K}$. We define a norm on $\frac{X}{V_K}$ by

$$\|\pi(x)\| := \|x\|_K \quad (x \in X). \tag{4}$$

Let $Z := \frac{X}{V_K}$ be endowed with the norm introduced in (4), and denote by $S : X \to Z$ the quotient map π. An easy verification shows that $S : X \to Z$ is a Dunford-Pettis p-convergent operator. Indeed, let $(x_n)_n$ be a p-Right null sequence in X. Since K is a p-(DPL) set, $\| S(x_n) \| = \sup_{\phi\in K} \| \phi(x_n) \| \to 0$. Hence S is Dunford-Pettis p-convergent, which proves (ii). Now we define $g : Z \to Y$ by $g(S(x)) = f(x), \quad x \in X$. We proved that g is well defined. Suppose that $\| S(x - y) \| = 0$. Since the span of K contains the range of f', we have

$$\| f'(c)(x - y) \| = 0 \qquad (c \in X).$$

By using the Mean Value Theorem ([7, Theorem 6.4]),

$$\| f(x) - f(y) \| \leq \sup_{c \in I(x,y)} \| f(x) - f(y) \| \leq \sup_{c \in I(x,y)} \| f'(c)(x - y) \| = 0,$$

and so $f(x) = f(y)$. Therefore g is well defined. Now, we show that g is Gáteaux differentiable. For given $x, y \in X$, the following limit exists:

$$\lim_{t \to 0} \frac{g(S(x) + tS(y)) - g(S(x))}{t} = \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} = f'(x)(y). \tag{5}$$

For $x \in X$ fixed, the mapping $g'(S(x)) : Z \to Y$ given by $g'(S(x))(S(y)) = f'(x)(y) \quad (y \in X)$ is linear. Moreover, choosing $r \in \mathbb{N}$ so that $x \in rB_X$, we have

$$\|g'(S(x))(S(y))\| = \|f'(x)(y)\| \quad \leq \quad r\|f'\|_{rB_X} \sup_{\phi \in K} \|\phi(y)\| = r\|f'\|_{rB_X} \|S(y)\|.$$

Consequently, $g'(S(x))$ is continuous. Hence g is Gáteaux differentiable. Since f is Fréchet differentiable, for every bounded set B, the limit in (5) exists uniformly for $S(y) \in S(B)$. So, $g \in D_{\mathcal{M}}(S(x), Y)$ for every $x \in X$, where $\mathcal{M} = \{S(B) : \text{B is a bounded subset of } X\}$ and this implies (iii). On the other hand, we have $\|g'(S(x))\| = \sup_{\|S(y)\| \leq 1} \|g'(S(x))(S(y))\| \leq r\|f'\|_{rB_X}, (x \in rB_X)$ and this yields (iv).

(b) ⇒ (a). Assume that there exist a normed space Z, an operator S from X onto Z, and a mapping $g : Z \to Y$ satisfying conditions (i)-(iv) of (b). It is clear that f is differentiable. We claim that f' takes bounded sets into p-(DPL) sets. For this purpose, suppose that B is a bounded set and $(x_n)_n$ is a p-Right null sequence in X. Since $S \in DPC_p(X, Z)$, we obtain

$$\sup_{x \in B} \|f'(x)(x_n)\| = \sup_{x \in B} \|g'(S(x))(S(x_n))\| \leq \sup_{x \in B} \|g'(S(x))\| \|S(x_n)\|.$$

But the right-hand side of the above inequality approaches zero whenever $n \to \infty$, since $S \in DPC_p(X, Z)$. So, $f'(B)$ is a p-(DPL) subset of $L(X, Y)$. □

Finally, we conclude this paper by an application of Theorem 4.1.

Example 4.2. *Let $h \in C^1(\mathbb{R})$. Define $f : c_0 \to \mathbb{R}$ by $f((x_n)_n) = \sum_{n=1}^{\infty} \frac{h(x_n)}{2^n}$. By using the same argument as in the* ([11, Example 2.4])*, one can show that f is differentiable such that $f'((x_n)_n) = (\frac{h'(x_n)}{2^n})_n \in \ell_1$. It is easy to verify that $f' : c_0 \to L(c_0, \mathbb{R})$ is compact. So, $f'(B_{c_0})$ is a relatively compact set in $DPC_p(c_0, \mathbb{R})$. Hence the part* (i) *of Proposition* 3.2*, implies that $f'(B_{c_0})$ is a p-(DPL) set. Now, let K be an arbitrary U-bounded Dunford-Pettis set in B_{c_0}. Clearly, $f'(K)$ is a p-(DPL) set in $L(c_0, \mathbb{R})$. Hence, Theorem* 3.10*, implies that f is p-Right sequentially continuous. An application of Theorem* 4.1*, shows that there exists a Banach space Z, an operator $S \in DPC_p(c_0, Z)$ and a Gâteaux differentiable mapping $g : Z \to \mathbb{R}$ such that $f = g \circ S$ with some additional properties.*

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