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Extended eigenvalues of 2×2 block operator matrices

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Abstract. In this work, the notion of extended eigenvalues of a 2×2 lower triangular operator matrix has been researched. More precisely, the relations between the extended spectrum of a 2×2 lower triangular operator matrix with the spectrum, the point spectrum, and the extended spectrum of its diagonal entries have been investigated. The obtained results have been supplemented by examples. In addition, some properties of the extended spectrum of 2×2 block operator matrices have been displayed.

1. Introduction

Operators which have a block operator matrix representation arise in different areas of mathematical physics like ordinary differential equations [9, 12], theory of elasticity [15], quantum mechanics [13], and optimal control [14]. The spectral properties of the corresponding block operator matrices are crucial, as it opens up a new line of attack for various problems by describing the solvability and stability of the underlying physical systems.

One of modern approaches that deals with spectral analysis is the extended spectrum of operators. Recall that a complex scalar λ is an extended eigenvalue of a bounded linear operator A on a Banach Space E, if there exists a nonzero bounded linear operator X acting on E, called extended eigenoperator associated to A, and satisfying the following equation:

$$AX = \lambda XA. \tag{1}$$

The family of all the extended eigenvalues of an operator A is called the extended spectrum of A, and it is denoted by $\sigma_{ext}(A)$. This notion is related to the simultaneous and independent works of Brown in [4] and Kim, Moore and Pearcy in [7], as a mean of generalizing the well-known Lomonosov theorem on the existence of nontrivial hyperinvariant subspace for the compact operators on Banach spaces. Particularly, they asserted that if the non-zero operator X is a compact operator, then A has a nontrivial hyperinvariant subspace for any number $\lambda \in \mathbb{C}$. The special case, when $\lambda = 1$ in Eq. (1) for which A commutes with a compact operator X, refers to Lomonosov's theorem [8] that is the algebra $\{A\}'$ of the commutant of A possesses a common nontrivial invariant subspace.

The structure of the set of extended eigenvalues in the complex plane for bounded linear operators has various forms. One of the fundamental problems in the theory of extended spectrum is to represent the

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structure of this set. On this way, Biswas, Lambert and Petrovic computed the set of extended eigenvalues of the integral Volterra operator on the space $L^2(0,1)$ (see [2]). In [3], Biswas and Petrovic applied the Rosenblum theorem [10] leading to derive the following important inclusion:

$$\sigma_{ext}(A) \subset \left\{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset\right\},$$

where $\sigma(A)$ is the spectrum of A. In the same paper, it was proved that this inclusion is an equality on the finite dimensional spaces. It was revealed in [11] that there are compact quasinilpotent operators, for which the set of extended eigenvalues is the one point set $\{1\}$. Important results in this subject was obtained by Gürdal in [6], which gave extended eigenvalues and extended eigenoperator of integration operators on the Winner algebra. In [1], Ammar, Boutaf and Jeribi generalized some obtained results of extended eigenvalues of a bounded linear operator in Banach space to the closed case and investigated some results of extended eigenvalues of a 2×2 upper triangular operator matrix.

The intrinsic objective of this work is to investigate some results of extended eigenvalues of a 2×2 lower triangular operator matrix.

The rest of this paper is organized as follows: In section 2, we display some notations and establish some results from the theory of the extended spectrum of linear operators. Such results will be used in the sequel. In section 3, we introduce and study the notion of extended eigenvalues of a 2×2 lower triangular operator matrix. The basic goal of this section is to provide characterizations that describe the relationships between the extended spectrum of a 2×2 lower triangular operator matrix with the spectrum, the point spectrum, and the extended spectrum of its diagonal entries. The obtained results are illustrated by several examples. We close this article by setting forward some properties of the extended eigenvalue of 2×2 block operator matrices.

2. Preliminaries

Throughout this paper, let E and F be complex Banach spaces and denote by $\mathcal{L}(E,F)$ the set of all bounded linear operators from E to F. We let $\mathcal{L}(E)$ denote $\mathcal{L}(E,E)$. The symbols $\mathcal{D}(A)$, $R(A)$ and $N(A)$ stand for the domain, the range and the kernel of a linear operator A, respectively. We will use the notation A^* for the adjoint of A.

Definition 2.1. *Let E and F be two Banach spaces and A be a linear operator from E into F.*
(*i*) *The point spectrum, $\sigma_p(A)$, of A is the set of all eigenvalues of A. That is,*

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : (\lambda I - A) \text{ is not injective}\}.$$

(*ii*) *The spectrum, $\sigma(A)$, of A is defined by*

$$\sigma(A) = \{\lambda \in \mathbb{C} \ : A - \lambda I \ \text{ has not a bounded inverse}\}.$$

(*iii*)*The resolvent set, $\rho(A)$, is the complement of the set $\sigma(A)$ in $\mathbb{C}$.*
(*iv*) *The Schechter essential spectrum is defined by*

$$\sigma_s(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K),$$

where $\mathcal{K}(X)$ stands for the ideal of all compact operators on X. ◊

Lemma 2.2. *[2] Let $V \in \mathcal{L}(L^2(0,1))$ be the integral Volterra operator. Hence, $\sigma_{ext}(V) =]0, \infty[$.* ◊

Proposition 2.3. *Let E be a Banach space and $A \in \mathcal{L}(E)$. Hence, A is injective if, and only if, $0 \notin \sigma_{ext}(A)$.*

◊

Proof. To prove the "if "part, let A be injective and suppose that $0 \in \sigma_{ext}(A)$. Then, there exists $0 \neq X \in \mathcal{L}(E)$ such that

$$AX = 0. \tag{2}$$

The fact that A is injective and $X \neq 0$, we infer that

$$AX \neq 0,$$

which contradicts Eq. (2). To prove the "only if "part, claim that $0 \notin \sigma_{ext}(A)$. It follows that

$$AX \neq 0, \quad \text{for all } \ 0 \neq X \in \mathcal{L}(E). \tag{3}$$

Let $X_1 \in \mathcal{L}(E)$ and $X_2 \in \mathcal{L}(E)$ such that $X_1 \neq X_2$. We have $X_1 - X_2 \neq 0$, thus from Eq. (3), we deduce that $A(X_1 - X_2) \neq 0$, and hence $AX_1 \neq AX_2$. That is, A is injective. □

Proposition 2.4. *[3] Let A be a bounded linear operator on a Hilbert space such that A and A^* have nontrivial kernels. Then, $\sigma_{ext}(A) = \mathbb{C}$.* ◊

Lemma 2.5. *[3] Suppose that $A \in \mathcal{L}(E)$. If $\sigma(A) = \{\lambda\}$, with $\lambda \neq 0$, then $\sigma_{ext}(A) = \{1\}$.* ◊

Lemma 2.6. *[1] Let A be a closed linear operator on E. If $0 \in \rho(A)$, then we have*

$$\lambda \in \sigma_{ext}(A) \text{ if, and only if, } \frac{1}{\lambda} \in \sigma_{ext}(A^{-1}).$$ ◊

Following the same reasoning of Biswas and Petrovic [3], in which it was demonstrated that the extended spectrum is invariant under a quasisimilarity, we can set forward the next Lemma:

Lemma 2.7. *Let $R, S \in \mathcal{L}(E)$ such that R has a dense range and S is injective. Then, we have*
(i) $\sigma_{ext}(RS) \subset \sigma_{ext}(SR)$.
(ii) If, further R is injective and S has a dense range, then $\sigma_{ext}(RS) = \sigma_{ext}(SR)$. ◊

Proof. (*i*) Let us assume that $\lambda \in \sigma_{ext}(RS)$, then there exists a nonzero operator X such that

$$RSX = \lambda XRS. \tag{4}$$

Multiplying Eq. (4) by R on the left and by S on the right, we obtain

$$SRSXR = \lambda SXRSR. \tag{5}$$

In Eq. (5), we have $X \neq 0$. The fact that S is injective implies that $SX \neq 0$. Since R has a dense range, it follows that $SXR \neq 0$, which assures that $\lambda \in \sigma_{ext}(SR)$.
(*ii*) The inverse inclusion follows by symmetry. □

Using similar methods of Cvetković-Ilić [5], we can set forward the following Theorems

Theorem 2.8. *Suppose that H and K are two Hilbert spaces. Let $A \in \mathcal{L}(H)$ such that A and A^* have nontrivial kernels and let $B \in \mathcal{L}(K)$ be an injective operator. Then, there exists $D \in \mathcal{L}(H, K)$ such that the operator matrix M_D is injective if, and only if, $R(B)$ is not closed.* ◊

Theorem 2.9. *Let H, K be two Hilbert spaces. Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$ be given operators. There exists $D \in \mathcal{L}(H, K)$ such that $R(M_D)$ is not dense in $H \times K$ if, and only if, one of the following conditions is satisfied:*
(i) $R(B)$ is not dense in K.
(ii) $R(A)$ is not dense in H. ◊

3. Main results

Let E and F be two Banach spaces and consider the 2×2 lower triangular operator matrices defined on $E \times F$ by

$$M_D = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \tag{6}$$

and

$$M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{7}$$

where $A \in \mathcal{L}(E)$, $B \in \mathcal{L}(F)$ and $D \in \mathcal{L}(E, F)$.

Definition 3.1. *Let M_D be the 2×2 lower triangular operator matrix defined in Eq. (6). A complex number λ is an extended eigenvalue of M_D if there exists a nonzero 2×2 lower triangular operator matrix,*

$$X = \begin{pmatrix} X_1 & 0 \\ X_3 & X_2 \end{pmatrix}, \tag{8}$$

where $X_1 \in \mathcal{L}(E)$, $X_3 \in \mathcal{L}(E, F)$ and $X_2 \in \mathcal{L}(F)$ such that

$$M_D X = \lambda X M_D. \tag{9}$$

The operator X is called eigenoperator corresponding to λ. The set of extended eigenvalues and the set of eigenoperators corresponding to λ are represented, respectively, by $\sigma_{ext}(M_D)$ and $E_{ext}(M_D, \lambda)$ ◊

Remark 3.2. (*i*) *$\sigma_{ext}(M_D) \neq \emptyset$. Certainly, $1 \in \sigma_{ext}(M_D)$ due to*

$$M_D X = X M_D,$$

for which $X = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}$, where the I_1 and I_2 are identity operators on E and F, respectively.

(*ii*) *If $A = B = 0$, then $\sigma_{ext}(M_D) = \mathbb{C}$. In fact, we have for all $X_3 \in \mathcal{L}(E, F) \setminus \{0\}$*

$$\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ X_3 & 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ X_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix},$$

for any $\lambda \in \mathbb{C}$.

(*iii*) *$\sigma_{ext}(M_0) = \sigma_{ext}(A) \cup \sigma_{ext}(B) \cup \{\lambda \in \mathbb{C} :$ there exists $0 \neq X_3 \in \mathcal{L}(E, F),\ BX_3 = \lambda X_3 A\}$.*

(*iv*)*If X_1, X_2 and X_3 are non zeros, then*

$$\sigma_{ext}(M_D) = \sigma_{ext}(A) \cap \sigma_{ext}(B) \cap \{\lambda \in \mathbb{C} : DX_1 + BX_3 = \lambda X_3 A + \lambda X_2 D\}.$$ ◊

For an arbitrary 2×2 lower triangular operator matrix M_D, a relation between $\sigma_{ext}(M_D)$ and the spectrum of its diagonal entries can be established as follows:

Proposition 3.3. *Let M_D be the 2×2 lower triangular operator matrix defined in Eq. (6). We have*

$$\sigma_{ext}(M_D) \subset \Big\{\lambda \in \mathbb{C} : \{\sigma(A) \cap \sigma(\lambda A)\} \cup \{\sigma(A) \cap \sigma(\lambda B)\} \cup \{\sigma(B) \cap \sigma(\lambda B)\} \cup \{\sigma(B) \cap \sigma(\lambda A)\} \neq \emptyset\Big\}.$$ ◊

Proof. From the factorization formula:

$$M_D = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ D & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, \tag{10}$$

it is clear that for every $D \in \mathcal{L}(E,F)$, we have

$$\sigma(M_D) \subseteq \sigma(A) \cup \sigma(B).$$

By using [3, Proposition 2.2], it follows that

$$\sigma_{ext}(M_D) \subset \Big\{\lambda \in \mathbb{C} : \sigma(M_D) \cap \sigma(\lambda M_D) \neq \emptyset\Big\}.$$

Hence,

$$\begin{aligned}\sigma_{ext}(M_D) &\subset \Big\{\lambda \in \mathbb{C} : \{\sigma(A) \cup \sigma(B)\} \cap \{\sigma(\lambda A) \cup \sigma(\lambda B)\} \neq \emptyset\Big\}\\ &= \Big\{\lambda \in \mathbb{C} : \{\sigma(A) \cap \{\sigma(\lambda A) \cup \sigma(\lambda B)\}\} \cup \{\sigma(B) \cap \{\sigma(\lambda A) \cup \sigma(\lambda B)\}\} \neq \emptyset\Big\}\\ &= \Big\{\lambda \in \mathbb{C} : \{\sigma(A) \cap \sigma(\lambda A)\} \cup \{\sigma(A) \cap \sigma(\lambda B)\} \cup \{\sigma(B) \cap \sigma(\lambda B)\} \cup \{\sigma(B) \cap \sigma(\lambda A)\} \neq \emptyset\Big\},\end{aligned}$$

then we reach the desired result. □

Remark 3.4. (*i*) *The inclusion in Proposition 3.3 can be strict. Evidently, let $E = F = L^2(0,1)$, $A = B = V$ be the Volterra operator on $L^2(0,1)$ and $D = 0$ be the zero operator on $L^2(0,1)$. We have that*

$$\Big\{\lambda \in \mathbb{C} : \sigma(V) \cap \sigma(\lambda V) \neq \emptyset\Big\} = \mathbb{C},$$

besides
$$\begin{aligned}\sigma_{ext}\Big(\begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}\Big) = &\Big\{\lambda \in \mathbb{C} : \text{ there exists } X_1 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_1 = \lambda X_1 V\Big\}\\ &\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exists } X_2 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_2 = \lambda X_2 V\Big\}\\ &\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exists } X_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_3 = \lambda X_3 V\Big\}\\ &\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } X_1, X_2 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_1 = \lambda X_1 V\\ &\qquad \text{and } VX_2 = \lambda X_2 V\Big\}\\ &\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } X_1, X_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_1 = \lambda X_1 V\\ &\qquad \text{and } VX_3 = \lambda X_3 V\Big\}\\ &\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } X_2, X_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_3 = \lambda X_3 V\\ &\qquad \text{and } VX_2 = \lambda X_2 V\Big\}\\ &\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } X_1, X_2, X_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_1 = \lambda X_1 V,\\ &\qquad VX_2 = \lambda X_2 V, \text{ and } VX_3 = \lambda X_3 V\Big\}.\end{aligned}$$

It follows that $\sigma_{ext}\Big(\begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}\Big) = \sigma_{ext}(V)$. Using Lemma 2.2, we obtain

$$\sigma_{ext}\Big(\begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}\Big) =]0, \infty[.$$

(*ii*)*If $\sigma(A) = \sigma(B) = \{\alpha\}$, with $\alpha \neq 0$, then $\sigma_{ext}(M_D) = \{1\}$. Indeed, we have $\{1\} \subset \sigma_{ext}(M_D)$ (see Remark 3.2 (i)). Now,it remains to prove the converse inclusion. Accordingly, let us assume that $\sigma(A) = \sigma(B) = \{\alpha\}$. There are four possible cases: $\sigma(A) \cap \sigma(\lambda A) \neq \emptyset$, $\sigma(A) \cap \sigma(\lambda B) \neq \emptyset$, $\sigma(B) \cap \sigma(\lambda B) \neq \emptyset$ or $\sigma(B) \cap \sigma(\lambda A)\} \neq \emptyset$, which imply that $\alpha \in \sigma(\lambda A)$ and $\alpha \in \sigma(\lambda B)$. Thus, $\frac{\alpha}{\lambda} \in \sigma(A)$ and $\frac{\alpha}{\lambda} \in \sigma(B)$. That is, $\lambda = 1$. Therefore,*

$$\Big\{\lambda \in \mathbb{C} : \{\sigma(A) \cap \sigma(\lambda A)\} \cup \{\sigma(A) \cap \sigma(\lambda B)\} \cup \{\sigma(B) \cap \sigma(\lambda B)\} \cup \{\sigma(B) \cap \sigma(\lambda A)\} \neq \emptyset\Big\} = \{1\}.$$

As a consequence, $\sigma_{ext}(M_D) \subset \{1\}$. ◊

Proposition 3.5. *Let $E \times F$ be a finite dimensional Banach space and M_D be the 2×2 lower triangular operator matrix defined in Eq.* (6). *We have*

$$\sigma_{ext}(M_D) = \Big\{\lambda \in \mathbb{C} : \{\sigma(A) \cap \sigma(\lambda A)\} \cup \{\sigma(A) \cap \sigma(\lambda B)\} \cup \{\sigma(B) \cap \sigma(\lambda B)\} \cup \{\sigma(B) \cap \sigma(\lambda A)\} \neq \emptyset\Big\}.$$

Further, if M_D is invertible, then

$$\sigma_{ext}(M_D) = \Big\{\frac{\lambda}{\mu} : \lambda, \mu \in \sigma(A) \cup \sigma(B)\Big\}. \qquad \diamond$$

Proof. Taking into account [3, Theorem 2.5], we obtain

$$\sigma_{ext}(M_D) = \Big\{\lambda \in \mathbb{C} : \sigma(M_D) \cap \sigma(\lambda M_D) \neq \emptyset\Big\}.$$

The fact that $E \times F$ is a finite dimensional Banach space, we infer that

$$\sigma(M_D) = \sigma(A) \cup \sigma(B),$$

for every $D \in \mathcal{L}(E, F)$. So,

$$\begin{aligned} \sigma_{ext}(M_D) &= \Big\{\lambda \in \mathbb{C} : \{\sigma(A) \cup \sigma(B)\} \cap \{\sigma(A) \cup \sigma(B)\} \neq \emptyset\Big\} \\ &= \Big\{\lambda \in \mathbb{C} : \{\sigma(A) \cap \sigma(\lambda A)\} \cup \{\sigma(A) \cap \sigma(\lambda B)\} \cup \{\sigma(B) \cap \sigma(\lambda B)\} \cup \{\sigma(B) \cap \sigma(\lambda A)\} \neq \emptyset\Big\}. \end{aligned}$$

Furthermore, if M_D is invertible, we deduce that

$$\sigma_{ext}(M_D) = \Big\{\frac{\lambda}{\mu} : \lambda, \mu \in \sigma(M_D)\Big\}.$$

Hence,

$$\sigma_{ext}(M_D) = \Big\{\frac{\lambda}{\mu} : \lambda, \mu \in \sigma(A) \cup \sigma(B)\Big\}. \qquad \square$$

The next proposition sets a connection between the extended spectrum of a 2×2 lower triangular operator matrix and the point spectrum of its diagonal elements.

Proposition 3.6. *Let M_D be the 2×2 lower triangular operator matrix defined in Eq.* (6). *Then,*

$$\Big\{\frac{\alpha}{\overline{\beta}} : \ \alpha \in \sigma_p(A) \cup \sigma_p(B) \ \text{ and } \ 0 \neq \beta \in \sigma_p(A^*) \cup \sigma_p(B^*)\Big\} \subset \sigma_{ext}(M_D). \qquad \diamond$$

Proof. Using [1, Theorem 3.5],

$$\Big\{\frac{\alpha}{\overline{\beta}} : \ \alpha \in \sigma_p(M_D), \ \text{ and } \ 0 \neq \beta \in \sigma_p({M_D}^*)\Big\} \subset \sigma_{ext}(M_D)$$

holds. Since $\sigma_p(M_D) = \sigma_p(A) \cup \sigma_p(B)$, it follows that

$$\Big\{\frac{\alpha}{\overline{\beta}} : \ \alpha \in \sigma_p(A) \cup \sigma_p(B) \ \text{ and } \ 0 \neq \beta \in \sigma_p(A^*) \cup \sigma_p(B^*)\Big\} \subset \sigma_{ext}(M_D). \qquad \square$$

As a direct consequence of Proposition 3.6, we infer the following result:

Corollary 3.7. *Let M_D be a 2×2 lower triangular operator matrix defined in Eq.* (6).
(*i*) *If $1 \in \sigma_p(A^*) \cup \sigma_p(B^*)$, then $\sigma_p(A) \cup \sigma_p(B) \subset \sigma_{ext}(M_D)$.*
(*ii*) *If A and A^* (or B and B^*) have nontrivial kernels, then $\sigma_{ext}(M_D) = \mathbb{C}$.*
(*iii*) *Let $\lambda \in \mathbb{R}$. If $\lambda \in \sigma_p(A) \cap \sigma_p(A^*)$ or $\lambda \in \sigma_p(B) \cap \sigma_p(B^*)$, then $\sigma_{ext}(\lambda I - M_D) = \mathbb{C}$.* $\diamond$

Proof. (*i*) It is clear.
(*ii*) If A and A^* (or B and B^*) have nontrivial kernels, then there exist $0 \neq x \in E$ and $0 \neq y \in E^*$ (or $0 \neq u \in F$ and $0 \neq v \in F^*$, respectively) such that $Ax = A^*y = 0$ (or $Bu = B^*v = 0$, respectively). Hence, the operator

$$X = \begin{pmatrix} x \otimes y & 0 \\ 0 & 0 \end{pmatrix} \ \text{(or } X = \begin{pmatrix} 0 & 0 \\ 0 & u \otimes v \end{pmatrix}, \text{ respectively)}$$

holds for all $\lambda \in \mathbb{C}$

$$M_D X = \lambda X M_D = 0.$$

Therefore, $\sigma_{ext}(M_D) = \mathbb{C}$.
(*iii*) Supposing that $\lambda \in \mathbb{R}$ such that $\lambda \in \sigma_p(A) \cap \sigma_p(A^*)$ (or $\lambda \in \sigma_p(B) \cap \sigma_p(B^*)$). It follows that $0 \in \sigma_p(\lambda I - A) \cap \sigma_p((\bar{\lambda} I - A)^*)$ (or $0 \in \sigma_p(\lambda I - B) \cap \sigma_p((\bar{\lambda} I - B)^*)$, respectively). As λ is a real number, $0 \in \sigma_p(\lambda I - A) \cap \sigma_p((\lambda I - A)^*)$ (or $0 \in \sigma_p(\lambda I - B) \cap \sigma_p((\lambda I - B)^*)$, respectively). Departing from (*ii*), we conclude that $\sigma_{ext}(\lambda I - M_D) = \mathbb{C}$. □

The reader could ask if there exists an inclusion between $\sigma_{ext}(M_D)$ and $\sigma_{ext}(A) \cup \sigma_{ext}(B)$. In general case, this question has a negative answer as shown by examples below.

Example 3.8. *Let $E = F = \mathbb{R}^2$ and M_D be the 2×2 lower triangular operator matrix defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by*

$$M_D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 9 & 7 & 4 & 0 \\ 8 & 10 & 11 & 3 \end{pmatrix}.$$

Put $A = \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 0 \\ 11 & 3 \end{pmatrix}$ *and* $D = \begin{pmatrix} 9 & 7 \\ 8 & 10 \end{pmatrix}$. *We have that $\sigma(A) = \{1, 5\}$ and $\sigma(B) = \{3, 4\}$. Then,*

$$\Big\{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset\Big\} = \Big\{1, 5, \frac{1}{5}\Big\},$$

$$\Big\{\lambda \in \mathbb{C} : \sigma(B) \cap \sigma(\lambda B) \neq \emptyset\Big\} = \Big\{1, \frac{4}{3}, \frac{3}{4}\Big\},$$

$$\Big\{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda B) \neq \emptyset\Big\} = \Big\{\frac{1}{3}, \frac{1}{4}, \frac{5}{3}, \frac{5}{4}\Big\},$$

and

$$\Big\{\lambda \in \mathbb{C} : \sigma(B) \cap \sigma(\lambda A) \neq \emptyset\Big\} = \Big\{3, 4, \frac{3}{5}, \frac{4}{5}\Big\}.$$

Applying Proposition 3.5, it follows that

$$\sigma_{ext}(M_D) = \Big\{1, 3, 4, 5, \frac{1}{5}, \frac{4}{3}, \frac{3}{4}, \frac{1}{3}, \frac{1}{4}, \frac{5}{3}, \frac{5}{4}, \frac{3}{5}, \frac{4}{5}\Big\}.$$

Besides,

$$\sigma_{ext}(A) \cup \sigma_{ext}(B) = \Big\{1, 5, \frac{1}{5}, \frac{4}{3}, \frac{3}{4}\Big\}.$$

That is,

$$\sigma_{ext}(M_D) \not\subset \sigma_{ext}(A) \cup \sigma_{ext}(B).$$ ◇

Example 3.9. *Let H and K be two Hilbert spaces. Let $A \in \mathcal{L}(H)$ such that A and A^* have nontrivial kernels and let $B \in \mathcal{L}(K)$ be an injective operator with $R(B)$ is not closed. Then, there exists $D \in \mathcal{L}(H, K)$ such that*

$$\sigma_{ext}(A) \cup \sigma_{ext}(B) \not\subset \sigma_{ext}(M_D).$$

Absolutely, using Theorem 2.8 leads to M_D being injective. According Proposition 2.3, we get

$$0 \notin \sigma_{ext}(M_D).$$

On the other side, relying on the fact that A and A^* have nontrivial kernels together with Proposition 2.4, allows us to deduce that

$$\sigma_{ext}(A) = \mathbb{C}.$$

Consequently,

$$\sigma_{ext}(A) \cup \sigma_{ext}(B) = \mathbb{C}. \qquad \diamond$$

In the following lines, we show relations between the extended spectrum of a 2×2 lower triangular operator matrix and the extended spectrum of its diagonal entries, respect to some conditions.

Theorem 3.10. *Suppose that M_D is the 2×2 lower triangular operator matrix defined in Eq.* (6) *and consider the 2×2 lower triangular operator matrix, X, defined in Eq.* (8)*. We have the following assertions:*
(*i*) *Suppose that $DX_1 = \lambda X_2 D$, for any $\lambda \in \mathbb{C}$.*

$$\text{If } \lambda \in \sigma_{ext}(A) \cup \sigma_{ext}(B), \text{ then } \lambda \in \sigma_{ext}(M_D). \tag{11}$$

(*ii*) *If $X_1 \neq 0$ or $X_2 \neq 0$, then*

$$\sigma_{ext}(M_D) \subseteq \sigma_{ext}(A) \cup \sigma_{ext}(B). \tag{12}$$

$\diamond$

Proof. (*i*) Assume that $DX_1 = \lambda X_2 D$, for any $\lambda \in \mathbb{C}$ and suppose that $\lambda \in \sigma_{ext}(A) \cup \sigma_{ext}(B)$. There are two cases. If $\lambda \in \sigma_{ext}(A)$, then there exists $X_1 \in \mathcal{L}(E) \setminus \{0\}$ such that $AX_1 = \lambda X_1 A$. In this case, we have

$$M_D X = \lambda X M_D,$$

where

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, $\lambda \in \sigma_{ext}(M_D)$. If $\lambda \in \sigma_{ext}(B)$, then there exists $X_2 \in \mathcal{L}(F) \setminus \{0\}$ satisfying $BX_2 = \lambda X_2 B$. In this case, X in Eq. (8) can be chosen as

$$X = \begin{pmatrix} 0 & 0 \\ 0 & X_2 \end{pmatrix},$$

in such a way that

$$M_D X = \lambda X M_D,$$

As a consequence, $\lambda \in \sigma_{ext}(M_D)$.
(*ii*) Note that $\sigma_{ext}(A) \cup \sigma_{ext}(B) \neq \emptyset$ as $1 \in \sigma_{ext}(A) \cup \sigma_{ext}(B)$. Now, let $\lambda \in \sigma_{ext}(M_D)$. Consider the following cases: First case, if $X_1 \neq 0$, then Eq. (9) implies, in particular, the existence of $X_1 \in \mathcal{L}(E) \setminus \{0\}$ such that

$$AX_1 = \lambda X_1 A.$$

That is,

$$\lambda \in \sigma_{ext}(A).$$

Therefore,

$$\lambda \in \sigma_{ext}(A) \cup \sigma_{ext}(B).$$

Second case, if $X_2 \neq 0$, the use of Eq. (9) leads, in particular, to the existence of $X_2 \in \mathcal{L}(F) \setminus \{0\}$ such that

$$BX_2 = \lambda X_2 B.$$

In other words,

$$\lambda \in \sigma_{ext}(B).$$

Consequently,

$$\lambda \in \sigma_{ext}(A) \cup \sigma_{ext}(B). \qquad \square$$

Remark 3.11. (*i*) *The converse of implication* (11) *in Theorem 3.10 is not always true. In fact, let $A = I$ be the identity operator on $E = L^2(0,1)$, $B = V$ be the Volterra integral operator on $F = L^2(0,1)$, and let $D = 0$ be the zero operator on $L^2(0,1)$. Obviously, we have that $DX_1 = \lambda X_2 D$, for any $\lambda \in \mathbb{C}$. On the one hand, we have that*

$$\sigma_{ext}\Big(\begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}\Big) = \Big\{\lambda \in \mathbb{C} : \text{ there exists } \begin{pmatrix} X_1 & 0 \\ X_3 & X_2 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ such that } \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}\begin{pmatrix} X_1 & 0 \\ X_3 & X_2 \end{pmatrix} = \lambda \begin{pmatrix} X_1 & 0 \\ X_3 & X_2 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}\Big\},$$

which implies that

$$\begin{aligned} \sigma_{ext}\Big(\begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}\Big) = & \Big\{\lambda \in \mathbb{C} : \text{ there exists } X_1 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } X_1 = \lambda X_1\Big\} \\ & \bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exists } X_2 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_2 = \lambda X_2 V\Big\} \\ & \bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exists } X_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_3 = \lambda X_3\Big\} \\ & \bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } X_1, X_2 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } X_1 = \lambda X_1 \\ & \qquad \text{and } VX_2 = \lambda X_2 V\Big\} \\ & \bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } X_1, X_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } X_1 = \lambda X_1 \\ & \qquad \text{and } VX_3 = \lambda X_3\Big\} \\ & \bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } X_2, X_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_3 = \lambda X_3 \\ & \qquad \text{and } VX_2 = \lambda X_2 V\Big\} \\ & \bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } X_1, X_2, X_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } X_1 = \lambda X_1, \\ & \qquad VX_2 = \lambda X_2 V, \text{ and } VX_3 = \lambda X_3\Big\}. \end{aligned}$$

We have that

$$\Big\{\lambda \in \mathbb{C} : \text{ there exists } X_2 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_2 = \lambda X_2 V\Big\} = \sigma_{ext}(V),$$

and

$$\Big\{\lambda \in \mathbb{C} : \text{ there exists } X_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VX_3 = \lambda X_3\Big\} = \sigma(V),$$

Since, $\sigma_{ext}(V) =]0,\infty[$ (see Lemma 2.2) and $\sigma(V) = \{0\}$, these allow us to deduce that

$$\begin{aligned} \sigma_{ext}\Big(\begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}\Big) &= \{1\} \cup]0,\infty[\cup \{0\} \cup \{1\} \cup \emptyset \cup \emptyset \cup \emptyset \\ &= [0,\infty[. \end{aligned}$$

On the other hand, it is easy to check that $\sigma_{ext}(I) = \{1\}$, so we get

$$\sigma_{ext}(I) \cup \sigma_{ext}(V) =]0,\infty[.$$

As a result, there exists $\lambda = 0 \in \sigma_{ext}\Big(\begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}\Big)$. However, $\lambda = 0 \notin \sigma_{ext}(I) \cup \sigma_{ext}(V)$.

(*ii*) *If $X_1 = X_2 = 0$, then there is no inclusion relation among $\sigma_{ext}(A) \cup \sigma_{ext}(B)$ and $\sigma_{ext}(M_D)$. Indeed, let H be a Hilbert space, $A \in \mathcal{L}(H)$ such that $\sigma(A) = \{\lambda\}$ with $\lambda \neq 0$, $B = I$ be the identity operator on H and $D \in \mathcal{L}(H)$. Then, the use of Lemma 2.5 implies $\sigma_{ext}(A) = \{1\}$. Thus, we obtain*

$$\sigma_{ext}(A) \cup \sigma_{ext}(I) = \{1\}.$$

On the other side, we have

$$\begin{aligned}\sigma_{ext}\Big(\begin{pmatrix} A & 0 \\ D & I \end{pmatrix}\Big) &= \Big\{\lambda \in \mathbb{C} : \text{ there exists } X_3 \in \mathcal{L}(H)\setminus\{0\} \text{ such that} \\ &\qquad \begin{pmatrix} A & 0 \\ D & I \end{pmatrix}\begin{pmatrix} 0 & 0 \\ X_3 & 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ X_3 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ D & I \end{pmatrix}\Big\} \\ &= \Big\{\lambda \in \mathbb{C} : \text{ there exists } Y_3 \in \mathcal{L}(H)\setminus\{0\} \text{ such that } (A - \frac{1}{\lambda}I)Y_3 = 0\Big\} \\ &= \{\frac{1}{\lambda}\}.\end{aligned}$$

That would be obvious if $\lambda \notin \{-1, 1\}$. ◇

Corollary 3.12. *Suppose that $A = B$ and $DX_1 = \lambda X_2 D$, for any $\lambda \in \mathbb{C}$. Then,*

$$\lambda \in \sigma_{ext}(M_D) \text{ if, and only if, } \lambda \in \sigma_{ext}(A).$$ ◇

Proof. Let $A = B$ and $DX_1 = \lambda X_2 D$, for every $\lambda \in \mathbb{C}$. Assume that $\lambda \in \sigma_{ext}(A)$. From Theorem 3.10 (*i*), we get $\lambda \in \sigma_{ext}(M_D)$. Conversely, let $\lambda \in \sigma_{ext}(M_D)$. Based on Theorem 3.10 (*ii*), it is sufficient to prove $\lambda \in \sigma_{ext}(A)$ if X of Eq. (8) is equal to $\begin{pmatrix} 0 & 0 \\ X_3 & 0 \end{pmatrix}$. In this case, Eq. (9) implies that there exists $X_3 \in \mathcal{L}(E)\setminus\{0\}$ satisfying

$$AX_3 = \lambda X_3 A.$$

Consequently, we obtain $\lambda \in \sigma_{ext}(A)$. □

The following theorem extends results obtained in Theorem 3.10 from bounded 2×2 lower triangular block operator matrices to invertible closed ones.

Theorem 3.13. *Let E and F two Banach spaces, we consider an unbounded 2×2 lower triangular block operator matrix defined on $\mathcal{D}(M_D) = \mathcal{D}(A) \times \mathcal{D}(B) \subset E \times F$ by*

$$M_D = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix}, \tag{13}$$

where A and B are, respectively, two closed linear operators on E and F and $D \in \mathcal{L}(E, F)$ such that $0 \in \rho(A) \cap \rho(B)$. Consider the 2×2 lower triangular block operator matrices, X, defined in Eq. (8). Then, we have the following assertions:

(*i*) *Suppose that $DX_1 = \lambda X_2 D$, for any $\lambda \in \mathbb{C}$. Hence,*

$$\text{if } \lambda \in \{\sigma_{ext}(A) \cup \sigma_{ext}(B)\} \setminus \{0\}, \text{ then } \lambda \in \sigma_{ext}(M_D) \setminus \{0\}. \tag{14}$$

(*ii*) *If we have $X_1 \neq 0$ or $X_2 \neq 0$, then*

$$\sigma_{ext}(M_D) \setminus \{0\} \subseteq \{\sigma_{ext}(A) \cup \sigma_{ext}(B)\} \setminus \{0\}. \tag{15}$$

Proof. First, if we claim that A and B are closed linear operators and D is a bounded linear operator, then M_D with its domain $\mathcal{D}(A) \times \mathcal{D}(B)$ is closed since it is the sum of a closed and a bounded operator. Since $0 \in \rho(A) \cap \rho(B)$, we infer that $0 \in \rho(M_D)$ such that

$$M_D^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -B^{-1}DA^{-1} & B^{-1} \end{pmatrix}.$$

(i) Again, relying on the fact that $0 \in \rho(A) \cap \rho(B)$ together with Lemma 2.6 allows us to deduce that

$$\lambda \in \{\sigma_{ext}(A) \cup \sigma_{ext}(B)\} \setminus \{0\} \text{ if, and only if, } \frac{1}{\lambda} \in \{\sigma_{ext}(A^{-1}) \cup \sigma_{ext}(B^{-1})\} \setminus \{0\}.$$

Based on Theorem 3.10 (i), we get

$$\frac{1}{\lambda} \in \sigma_{ext}(M_D^{-1}) \setminus \{0\}.$$

Again, regarding to Lemma 2.6, we have

$$\lambda \in \sigma_{ext}(M_D) \setminus \{0\}.$$

(ii) The fact that $0 \in \rho(M_D)$ together with Lemma 2.6 leads to

$$\lambda \in \sigma_{ext}(M_D) \setminus \{0\} \text{ if, and only if, } \frac{1}{\lambda} \in \sigma_{ext}(M_D^{-1}) \setminus \{0\}.$$

Theorem 3.10 (ii) allows us to conclude that

$$\frac{1}{\lambda} \in \{\sigma_{ext}(A^{-1}) \cup \sigma_{ext}(B^{-1})\} \setminus \{0\}.$$

Using again Lemma 2.6, it follows that

$$\lambda \in \{\sigma_{ext}(A) \cup \sigma_{ext}(B)\} \setminus \{0\}.$$ □

In the last part of this section, we give some sufficient conditions to obtain $\sigma_{ext}(M_D) = \sigma_{ext}(M_0)$.

Theorem 3.14. *Let $A \in \mathcal{L}(E)$ and $B \in \mathcal{L}(F)$. If $\sigma_s(A) \cap \sigma_s(B) = \emptyset$, then for every $D \in \mathcal{L}(E, F)$*
$$\sigma_{ext}(M_D) = \sigma_{ext}(M_0).$$ ◊

Proof. Let $A \in \mathcal{L}(E)$ and $B \in \mathcal{L}(F)$. If we suppose that $\sigma_s(A) \cap \sigma_s(B) = \emptyset$, then the same reasoning as in the proof of [1, Lemma 3.8] allows us to deduce that for every $D \in \mathcal{L}(E, F)$, the equation $YA - BY = D$ has a solution $-Y$. Hence,

$$M_D = \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} M_0 \begin{pmatrix} I & 0 \\ -Y & I \end{pmatrix},$$

where $\begin{pmatrix} I & 0 \\ -Y & I \end{pmatrix}$ is the inverse of $\begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}$. Now, if we claim that $\lambda \in \sigma_{ext}(M_D)$, then there exists a nonzero 2×2 lower triangular operator matrix, X, such that

$$M_D X = \lambda X M_D.$$

It follows that,

$$\begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} M_0 \begin{pmatrix} I & 0 \\ -Y & I \end{pmatrix} X = \lambda X \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} M_0 \begin{pmatrix} I & 0 \\ -Y & I \end{pmatrix}.$$

Thus, we obtain

$$M_0 \begin{pmatrix} I & 0 \\ -Y & I \end{pmatrix} X \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} = \lambda \begin{pmatrix} I & 0 \\ -Y & I \end{pmatrix} X \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} M_0.$$

In other words, there exists a nonzero block operator matrix

$$\begin{aligned} Z &= \begin{pmatrix} I & 0 \\ -Y & I \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ X_3 & X_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} \\ &= \begin{pmatrix} X_1 & 0 \\ -YX_1 + X_3 + X_2 Y & X_2 \end{pmatrix}, \end{aligned}$$

satisfying

$$M_0 Z = \lambda Z\, M_0.$$

So, $\lambda \in \sigma_{ext}(M_0)$. The proof of the opposite inclusion follows the same way. □

Theorem 3.15. *Let M_D be the 2×2 lower triangular operator matrix defined in Eq. (6). If one of the following conditions is satisfied:*
(i) $0 \in \rho(B)$ and $\begin{pmatrix} I & 0 \\ B^{-1}D & I \end{pmatrix}$ commutes with every $X \in E_{ext}(M_D, \lambda)$.
(ii) $0 \in \rho(A)$ and $\begin{pmatrix} I & 0 \\ DA^{-1} & I \end{pmatrix}$ commutes with every $X \in E_{ext}(M_D, \lambda)$.
Then,

$$\sigma_{ext}(M_D) = \sigma_{ext}(M_0),$$

for every $D \in \mathcal{L}(E, F)$.

Proof. (*i*) The fact that $0 \in \rho(B)$ implies that

$$M_D = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & 0 \\ B^{-1}D & I \end{pmatrix}.$$

Observe that $\begin{pmatrix} I & 0 \\ B^{-1}D & I \end{pmatrix}$ is invertible with $\begin{pmatrix} I & 0 \\ -B^{-1}D & I \end{pmatrix}$ is its inverse. Actually, let $\lambda \in \sigma_{ext}(M_D)$ hence there is a nonzero 2×2 lower triangular operator matrix, X, such that

$$M_D X = \lambda X M_D.$$

Hence,

$$M_0\begin{pmatrix} I & 0 \\ B^{-1}D & I \end{pmatrix} X = \lambda X M_0 \begin{pmatrix} I & 0 \\ B^{-1}D & I \end{pmatrix}.$$

So, we have

$$M_0\begin{pmatrix} I & 0 \\ B^{-1}D & I \end{pmatrix} X \begin{pmatrix} I & 0 \\ -B^{-1}D & I \end{pmatrix} = \lambda X M_0.$$

Since $\begin{pmatrix} I & 0 \\ B^{-1}D & I \end{pmatrix}$ commutes with every $X \in E_{ext}(M_D, \lambda)$. It follows that

$$M_0 X = \lambda X M_0.$$

The proof of the inverse inclusion can be checked in a similar way.
(*ii*) The proof of the item (*ii*) follows by the same reasoning as (*i*). □

Theorem 3.16. *Let $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$ be given injective operators such that $R(A)$ dense in H and $R(B)$ dense in K. Then,*

$$\sigma_{ext}(M_D) = \sigma_{ext}(M_0),$$

for every $D \in \mathcal{L}(H, K)$. ◊

Proof. We have the following formula

$$M_D = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ D & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}.$$

Put $R = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ and $S = \begin{pmatrix} I & 0 \\ D & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ D & B \end{pmatrix}$. Since A, B and I are injective, one can check easily that both R and S are injective. Moreover, A, B and I have dense ranges. According to Theorem 2.9, we infer that both R and S have dense ranges. Now, applying Lemma 2.7 leads to

$$\sigma_{ext}(M_D) = \sigma_{ext}(SR) = \sigma_{ext}\left(\begin{pmatrix} A & 0 \\ DA & B \end{pmatrix}\right), \tag{16}$$

for all $D \in \mathcal{L}(H, K)$. So, if $D = 0$, we get

$$\sigma_{ext}(M_D) = \sigma_{ext}(M_0),$$

which ends the proof. □

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